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Abstract. A polyunary algebra is an ordered pair $\mathfrak{A} = (A, \mathscr{S})$, where A is a non-empty set and $\mathscr{S}$ is a monoid consisting of unary operations on A. A tolerance on a polyunary algebra or on a monoid is defined similarly as a congruence, only the requirement of transitivity is omitted. In the paper the properties of tolerances on $\mathfrak{A} = (A, \mathscr{S})$ and on $\mathscr{S}$ are studied.

Key words. Polyunary algebra, monoid, tolerance, stable relation, congruence.

Ms Classification. 08 A 60, 08 A 30.

In [2] tolerances on monounary algebras, i.e. algebras with one unary operation, were studied. Here we shall investigate tolerances on polyunary algebras.

A polyunary algebra $\mathfrak{A}$ is an ordered pair $(A, \mathscr{F})$, where A is a non-empty set (the support of $\mathfrak{A}$) and $\mathscr{F}$ is a non-empty set of unary operations on $\mathfrak{A}$, i.e. of mappings of A into A. If we speak about polyunary algebras, we suppose usually that $\mathscr{F}$ contains at least two elements (otherwise the algebra would be monounary).

The operations from $\mathscr{F}$ will be called fundamental operations on $\mathfrak{A}$. By their superpositions new operations are obtained; this superposition is associative. Thus the set $\mathscr{F}$ generates a semigroup whose elements are unary operations acting on A. To the operations from this semigroup we may add the identity mapping e on A; in this way a monoid $\mathscr{S}$ is obtained. For our purposes it is reasonable to consider whole the monoid $\mathscr{S}$ instead of the set $\mathscr{F}$ of fundamental operations. Thus we shall consider a polyunary algebra $\mathfrak{A}$ as an ordered pair $(A, \mathscr{S})$, where A is a non-empty set and $\mathscr{S}$ is a monoid consisting of unary operations acting on A; the operation on $\mathscr{S}$ is the superposition of mappings.

We shall consider tolerances on $\mathfrak{A}$ and on $\mathscr{S}$.

Let $\mathfrak{A} = (A, \mathscr{S})$ be a polyunary algebra, let T be a reflexive and symmetric binary relation on A. If T has the property that $(x, y) \in T$ implies $(f(x), f(y)) \in T$ for any $x \in A$, $y \in A$, $f \in \mathscr{S}$, then T is called a tolerance on $\mathfrak{A}$.

Now let T be a reflexive and symmetric binary relation on $\mathscr{S}$. If T has the property that $(x_1, x_2) \in T$, $(y_1, y_2) \in T$ imply $(x_1y_1, x_2y_2) \in T$ for any elements x_1, x_2, y_1, y_2 of $\mathscr{S}$, then T is called a tolerance on $\mathscr{S}$.

All tolerances on $\mathfrak{A}$ form the lattice $LT(\mathfrak{A})$, all tolerances on $\mathscr{S}$ form the lattice $LT(\mathscr{S})$ (both with respect to set inclusion). In both $LT(\mathfrak{A})$ and $LT(\mathscr{S})$ the meet is the set intersection. In $LT(\mathfrak{A})$ the join is the set union; this can be proved in the same way as in [1] for monounary algebras. In $LT(\mathscr{S})$ the join is not equal to the set union in general.

First we shall investigate a simple case – the monogeneous polyunary algebras. A polyunary algebra $\mathfrak{A} = (A, \mathscr{S})$ is called monogeneous, if there exists an element $a \in A$ with the property that for each $x \in A$ there exists an operation $f_x \in \mathscr{S}$ such that $x = f_x(a)$. The element a will be called a generator of a.

We shall use also the concept of the left stable binary relation on a monoid. A binary relation R on a monoid $\mathscr{S}$ is called left stable, if $(x, y) \in R$ implies $(zx, zy) \in \in R$ for any elements x, y, z of $\mathscr{S}$. The binary relations on $\mathscr{S}$ which are reflexive, symmetric and left stable form a lattice which will be denoted by $LS(\mathscr{S})$.

Theorem 1. *Let $\mathfrak{A} = (A, \mathscr{S})$ be a monogeneous polyunary algebra, let $\mathscr{S}$ be a commutative monoid. Then there exists a one-to-one mapping φ of A onto $\mathscr{S}$ such that $\varphi(x) = f_x$ for each $x \in A$.*

Proof. For each $x \in A$ there exists at least one operation from $\mathscr{S}$ such that x is the image of a in this operation; namely the operation f_x. Let f, g be two operations from $\mathscr{S}$ such that $f(a) = g(a)$. Let b be an arbitrary element of A; as A is monogeneous, there exists $h \in \mathscr{S}$ such that $h(a) = b$. Then we have:

$$f(b) = fh(a) = hf(a) = hg(a) = gh(a) = g(b).$$

As b was chosen arbitrarily, this implies $f = g$. Hence to each $x \in A$ there exists exactly one operation $f_x \in \mathscr{S}$ such that $f_x(a) = x$ and the mapping φ is one-to-one.

Theorem 2. *Let $\mathfrak{A} = (A, \mathscr{S})$ be a monogeneous polyunary algebra, let $\mathscr{S}$ be a commutative monoid. Then $LT(\mathfrak{A}) \cong LS(\mathscr{S})$.*

Proof. Let φ be the mapping defined in Theorem 1. Let $\bar{\varphi}$ be the mapping defined so that $\bar{\varphi}(T)$ for $T \in LT(\mathfrak{A})$ is the set of all pairs $(\varphi(x), \varphi(y))$ for such pairs x, y that $(x, y) \in T$. Let $(x, y) \in T$. We have $x = f_x(a)$, $y = f_y(a)$, $f_x = \varphi(x)$, $f_y = \varphi(y)$. As T is a tolerance on $\mathfrak{A}$, it contains all pairs $(g(x), g(y))$ for $g \in \mathscr{S}$. We have $(g(x), g(y)) = = (gf_x(a), gf_y(a))$. Hence for each $g \in \mathscr{S}$ the relation $\bar{\varphi}(T)$ contains the pair (gf_x, gf_y); as x, y were chosen arbitrarily, we see that $\bar{\varphi}(T) \in LS(\mathscr{S})$ and $\bar{\varphi}$ is a mapping of $LT(\mathfrak{A})$ onto $LS(\mathscr{S})$. On the other hand, let $U \in LS(\mathscr{S})$, $(h_1, h_2) \in U$. As U is a left stable relation on $\mathscr{S}$, it contains all pairs (gh_1, gh_2) for $g \in \mathscr{S}$. Let $u = h_1(a)$, $v = = h_2(a)$. Then $h_1 = \varphi(u)$, $h_2 = \varphi(v)$, $gh_1 = \varphi(g(u))$, $gh_2 = \varphi(g(v))$ and thus (gh_1, gh_2) is the image of $(g(u), g(v))$ in the mapping φ. Hence $\varphi^{-1}(U)$ with $(u, v) = (h_1(a), h_2(a))$ contains also $(g(u), g(v)) = (gh_1(a), gh_2(a))$; as u, v, g were chosen arbitrarily, $\bar{\varphi}^{-1}(U) \in LT(\mathfrak{A})$ and $\bar{\varphi}^{-1}$ is a mapping of $LS(\mathscr{S})$ into $LT(\mathfrak{A})$. Combining these two results, we obtain that $\bar{\varphi}$ is a one-to-one mapping of $LT(\mathfrak{A})$ onto $LS(\mathscr{S})$. It is easy to see that $\bar{\varphi}$ preserves the lattice operations and thus it is an isomorphism of $LT(\mathfrak{A})$ onto $LS(\mathscr{S})$.

If a left stable binary relation on a monoid $\mathscr{S}$ is an equivalence, we shall call it a left stable equivalence. If $\mathscr{S}$ is commutative, we may omit the word "left" and, according to [1], any stable equivalence is a congruence; hence $LS(\mathscr{S}) = Con(\mathscr{S})$.

Theorem 3. *Let $\mathfrak{A} = (A, \mathscr{S})$ be a monogeneous polyunary algebra with the generator a. Then there exists a left stable equivalence R on S such that $f(a) = g(a)$ if and only if $(f, g) \in R$.*

Proof. Let R be the relation on $\mathscr{S}$ such that $(f, g) \in R$ if and only if $f(a) = g(a)$; it is evidently an equivalence. Let $h \in \mathscr{S}$. If $(f, g) \in R$, then $f(a) = g(a)$ and thus also $hf(a) = hg(a)$ and $(hf, hg) \in R$. Hence R is left stable, which was to be proved.

Theorem 4. *Let $\mathfrak{A} = (A, \mathscr{S})$ be a monogeneous polyunary algebra. Then $LT(\mathfrak{A})$ is isomorphic to a sublattice of $LS(\mathscr{S})$.*

Proof. Let R be the relation from Theorem 4. We have $R \in LS(\mathscr{S})$. Consider the set L^* of the products UR for $U \in LS(\mathscr{S})$. (We have $(x, y) \in UR$, if and only if there exists z such that $(x, z) \in U$, $(z, y) \in R$). Each element of L^* is evidently a left stable relation on $\mathscr{S}$; the join and the meet of any two elements of L^* in $LS(\mathscr{S})$ belongs to L^*. Thus L^* is a sublattice of $LS(\mathscr{S})$. Now we define a mapping χ of the set of equivalence classes of R onto A such that if C is an equivalence class of R, then $\chi(C) = f(a)$, where $f \in C$; obviously this does not depend on the choice of f. Now we define the mapping $\bar{\chi}$ of L^* onto $LT(A)$ in such a way that $\bar{\chi}(U)$ for $U \in L^*$ is the set of all pairs $(\chi(C_1), \chi(C_2))$, where C_1, C_2 are classes of R with the property that $(g_1, g_2) \in U$ for any $g_1 \in C_1$ and $g_2 \in C_2$. The mapping $\bar{\chi}$ is evidently an isomorphism of L^* onto $LT(A)$.

By the symbol $Con(\mathfrak{A})$ (or $Con(\mathscr{S})$) we denote the lattice of all congruences on $\mathfrak{A}$ (or on $\mathscr{S}$ respectively).

Theorem 5. *Let $\mathfrak{A} = (A, \mathscr{S})$ be a monogeneous polyunary algebra, let $\mathscr{S}$ be a commutative monoid. Then $Con(\mathfrak{A}) \cong Con(\mathscr{S})$.*

Proof. If we consider the mapping $\bar{\varphi}$ from the proof of Theorem 2, we see that $\bar{\varphi}$ maps transitive relations again onto transitive relations. Thus the restriction of $\bar{\varphi}$ onto $Con(\mathfrak{A})$ maps $Con(\mathfrak{A})$ onto the set of stable equivalences of $\mathscr{S}$ which is $Con(\mathscr{S})$.

We have described the case of monogeneous polyunary algebras. For polyunary algebras which are not monogeneous the situation is more complicated. We shall not describe the properties of $LT(\mathfrak{A})$ in that case; we shall only point out the difference between monogeneous polyunary algebras and non-monogeneous ones. We limit our considerations to algebras with finitely many generators. Then evidently there exists a finite set $G(\mathfrak{A})$ of independent generators of $\mathfrak{A}$, i.e. the subset of A with the property that to each $x \in \mathfrak{A}$ there exists $a \in G(\mathfrak{A})$ and $f \in \mathscr{S}$ such that $x = f(a)$ and if $a \in G(\mathfrak{A})$, $b \in G(\mathfrak{A})$, $a \neq b$, then $a \neq f(b)$ for any $f \in \mathcal{S}$. For each $a \in G(\mathfrak{A})$ we denote $A(a) = \{f(a) \mid f \in \mathcal{S}\}$ and we call this set the orbit of a.

Orbits of different generators may have a non-empty intersection. But the following theorem holds.

Theorem 6. *Let a, b be two different independent generators of $\mathfrak{A} = (A, \mathcal{S})$. If $A(a) \cap A(b) \neq \emptyset$, then $A(a) \cap A(b)$ contains only elements of the form $f(a)$, where f does not divide the unit e of $\mathcal{S}$.*

Proof. Let $x \in A(a) \cap A(b)$; then $x = f(a) = g(b)$ for some elements f, g of $\mathcal{S}$. Suppose that f is a divisor of e; then there exists an element $f^{-1} \in \mathcal{S}$ such that $f^{-1}f = e$. But then $a = f^{-1}(x) = f^{-1}g(b)$; as $f^{-1}g \in \mathcal{S}$, the generators a, b are not independent.

Corollary. *Let a, b be two different independent generators of $\mathfrak{A} = (A, \mathcal{S})$, let $\mathcal{S}$ be a group. Then $A(a) \cap A(b) = \emptyset$.*

Note that any monounary algebra $\mathfrak{A}$ can be considered as a polyunary algebra $(A, \mathcal{S})$, where $\mathcal{S}$ is a monogeneous monoid. If the graph of $\mathfrak{A}$ is a cycle or a two-way infinite directed path, $\mathcal{S}$ may be considered as a cyclic group.

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