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ON A MODIFICATION OF RELATIONAL AXIOMS

Jiří Karásek

Dedicated to Professor M. Novotný on the occasion of his seventieth birthday

0. Introduction

While axioms of binary relations are already stable, it cannot be said about axioms of ternary relations. So, in [1] and [3], the reflexivity of ternary relations is defined in the following way:

$$x, y, z \in G, \ \text{card}\, \{x, y, z\} \leq 2 \Rightarrow (x, y, z) \in R$$

while in [7] it is defined more weakly:

$$x \in G \Rightarrow (x, x, x) \in R.$$

Similarly, the transitivity of ternary relations is defined in [1], [3], and [7] by the condition:

$$(x, y, z) \in R, \quad (x, z, u) \in R \Rightarrow (x, y, u) \in R,$$

while in [11] by the requirement:

$$(x, z, y) \in R, \quad (y, z, u) \in R \Rightarrow (x, z, u) \in R.$$

In [13], the author has presented a general scheme of relational axioms for relations of any arity (not necessarily finite). The aim of this paper is to give a modification of that scheme yielding richer possibilities.

Let G, H be everywhere nonempty sets. By a *relation* (with the *carrier* G and the *index set* H) we understand a set $R \subseteq G^H$ where G^H denotes (as usually) the set of all mappings of the set H into the set G. N will denote the set of all positive integers. For any $n \in N$ we denote $(n] = \{m \in N;\ m \leq n\}$. In the case of a finite set H of cardinality k we shall not distinguish between mappings of the set H into the set G and k-tuples of elements of the set G. For any $n \in N$ we denote by S_n the set of all permutations of the set $(n]$, by $S_n(1)$ the set of all permutations of the set $(n]$ mapping 1 onto itself (preserving 1). For any $\varphi \in S_n$ and $m \in N$, φ^m denotes the m-th iteration of the permutation φ, for any φ, $\psi \in S_n$, $\varphi\psi$ denotes the composition of the permutations φ and ψ. id denotes the identical permutation of the set $(n]$.

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1. Operations with relations

1.1. Definition. Let $n \in N$. Then the sequence of $n+1$ sets $\mathcal{K} = \{K_i\}_{i=1}^{n+1}$ is called an *n-decomposition* of the set H if

(1) $\bigcup\limits_{i=1}^{n+1} K_i = H$,

(2) $K_i \cap K_j = \emptyset$ for all $\quad i, j \in (n+1], \quad i \neq j$.
If, moreover,

(3) card $K_i =$ card K_j for all $\quad i, j \in (n]$,
the n-decomposition $\mathcal{K}$ is called *regular*.

1.2. Definition. Let $\mathcal{K} = \{K_i\}_{i=1}^{n+1}$ be an n-decomposition of the set H. Then the relation

$$E_{\mathcal{K}} = \{f \in G^H;\ f(K_i) = f(K_j) \text{ for all } \quad i, j \in (n]\}$$

is called the *diagonal* with regard to $\mathcal{K}$.

1.3. Remark. Let $\mathcal{K} = \{K_i\}_{i=1}^{n+1}$ be an n-decomposition of the set H. Obviously, then:

(1) If $K_{n+1} = H$ or $n = 1$, then $E_{\mathcal{K}} = G^H$.

(2) If there exist $i, j \in (n]$ such that $K_i \neq \emptyset = K_j$, then $E_{\mathcal{K}} = \emptyset$.

1.4. Definition. Let $R \subseteq G^H$ be a relation, $\mathcal{K} = \{K_i\}_{i=1}^{n+1}$ an n-decomposition of the set H, $\varphi \in S_n$. Then we define the relation $R_{\mathcal{K},\varphi} \subseteq G^H$ by

$$\begin{aligned} R_{\mathcal{K},\varphi} = \big\{f \in G^H; \exists g \in R : f(K_i) = \\ = g(K_{\varphi(i)}) \text{ for all } \ i \in (n], \quad f(K_{n+1}) = g(K_{n+1})\big\}\,. \end{aligned}$$

$R_{\mathcal{K},\varphi}$ is called the $(\mathcal{K},\varphi)$-*modification* of the relation R.

1.5. Remark. Let $R \subseteq G^H$ be a relation, $\mathcal{K} = \{K_i\}_{i=1}^{n+1}$ an n-decomposition of the set H. Clearly, then:

(1) $R \subseteq R_{\mathcal{K},\mathrm{id}}$; if, moreover, card $K_i \leq 1$ for all $i \in (n+1]$,
then $R = R_{\mathcal{K},\mathrm{id}}$.

(2) $\emptyset_{\mathcal{K},\varphi} = \emptyset$.

(3) If there exist $i, j \in (n]$ such that $K_i \neq \emptyset = K_j$, $\varphi(i) = j$
or $\varphi(j) = i$, then $R_{\mathcal{K},\varphi} = \emptyset$.

1.6. Example. Let $R \subseteq G^H$ be a relation, $H = \{1, 2\}$ (i.e. R be binary), $\mathcal{K} = \{K_i\}_{i=1}^{3}$, $K_1 = \{1\}$, $K_2 = \{2\}$, and let φ be the permutation of the set $(2]$ defined by: $\varphi(1) = 2$, $\varphi(2) = 1$. Then $R_{\mathcal{K},\varphi} = R^{-1}$. Hence, in this case, the $(\mathcal{K},\varphi)$-modification of a binary relation coincides with its standard inverse.

1.7. Definition. Let $R_1, \ldots, R_n \subseteq G^H$ be relations, $\mathcal{K} = \{K_i\}_{i=1}^{n+1}$ an n-decomposition of the set H. Then we define the relation $(R_1 \ldots R_n)_{\mathcal{K}} \subseteq G^H$ by

$$\begin{aligned}(R_1 \ldots R_n)_{\mathcal{K}} =\{f \in G^H;\, &\exists f_i \in R_i \text{ for all } i \in (n] \text{ such that}\\ &f(K_i) = f_i(K_i) \text{ for all } i \in (n],\\ &f(K_{n+1}) = f_i(K_{n+1}) \text{ for all } i \in (n],\\ &f_i(K_j) = f_j(K_i) \text{ for all } i, j \in (n]\}.\end{aligned}$$

$(R_1 \ldots R_n)_{\mathcal{K}}$ is called the *composition* of $R_1, \ldots, R_n$ with regard to $\mathcal{K}$.

1.8. Definition. Let $R \subseteq G^H$ be a relation, $\mathcal{K}$ an n-decomposition of the set H. Then we put

$$\begin{aligned}R^1_{\mathcal{K}} &= R,\\ R^2_{\mathcal{K}} &= (R \ldots R)_{\mathcal{K}},\\ R^m_{\mathcal{K}} &= (R^{m-1}_{\mathcal{K}} R \ldots R)_{\mathcal{K}} \cup (R R^{m-1}_{\mathcal{K}} R \ldots R)_{\mathcal{K}} \cup \cdots \cup\\ &\quad \cup (R \ldots R R^{m-1}_{\mathcal{K}})_{\mathcal{K}}\end{aligned}$$

for any $M \in N$, $m \geq 3$. $R^m_{\mathcal{K}}$ is called the *m-th power* of R with regard to $\mathcal{K}$.

1.9. Example. Let R_1, $R_2 \subseteq G^H$ be relations, $H = \{1, 2\}$ (i.e. R_1, R_2 be binary), $\mathcal{K} = \{K_i\}_{i=1}^3$, $K_1 = \{1\}$, $K_2 = \{2\}$. Then $(R_1R_2)_{\mathcal{K}} = R_1R_2$. Hence, in this case, the composition with regard to $\mathcal{K}$ coincides with the standard composition of binary relations.

1.10. Remark. Let $R_1, \ldots, R_n \subseteq G^H$ be relations, $\mathcal{K} = \{K_i\}_{i=1}^{n+1}$ an n-decomposition of the set H, let $\varphi \in S_n$. Evidently, then:

(1) If $K_i = \emptyset$ for some $i \in (n]$, $(R_1 \ldots R_n)_{\mathcal{K}} \neq \emptyset$, then $\mathcal{K}$ is regular and there exist $f_j \in R_j$ for all $j \in (n]$ such that $f_j(H) = f_k(H)$ for all $j, k \in (n]$.
(2) If $K_i = \emptyset$ for some $i \in (n]$ and there exist $j, k \in (n]$ such that for each $f \in R_j$, $g \in R_k$ we have $f(H) \neq g(H)$, then $(R_1 \ldots R_n)_{\mathcal{K}} = \emptyset$.

1.11. Notation. Let $\mathcal{K} = \{K_i\}_{i=1}^{n+1}$ be an n-decomposition of the set H. Then $\mathcal{K}^*$ denotes the n-decomposition of the set H defined by $\mathcal{K}^* = \{K_i^*\}_{i=1}^{n+1}$ where

$$K_i^* = \begin{cases} K_{i+1} & \text{for all} \quad i \in (n-1] \\ K_1 & \text{for} \quad i = n \\ K_{n+1} & \text{for} \quad i = n+1 \end{cases}$$

Further, $*$ denotes the mapping of the set S_n into itself assigning to any permutation $\varphi \in S_n$ the permutation $\varphi^* \in S_n$ defined by:

$$\varphi^*(i) = \begin{cases} \varphi(i+1) - 1 & \text{if} \quad i \in (n-1], \varphi(i+1) \neq 1 \\ \varphi(1) - 1 & \text{if} \quad i = n, \varphi(1) \neq 1 \\ n & \text{otherwise} \end{cases}$$

1.12. Lemma. *Let $R, R_1, \dots, R_n \subseteq G^H$ be relations, $\mathcal{K}$ an n-decomposition of the set H, let $\varphi \in S_n$, $m \in N$. Then:*

(1) $\mathcal{K} \underbrace{^{*\cdots *}}_{n\text{times}} = \mathcal{K}$.
(2) $E_{\mathcal{K}} = E_{\mathcal{K}^*}$.
(3) $R_{\mathcal{K},\varphi} = R_{\mathcal{K}^*,\varphi^*}$.
(4) $(R_1 \dots R_n)_{\mathcal{K}} = (R_2 \dots R_n R_1)_{\mathcal{K}^*}$.
(5) $R^m_{\mathcal{K}} = R^m_{\mathcal{K}^*}$.

Proof is obvious.

1.13. Definition. Let $R \subseteq G^H$ be a relation, $\mathcal{K}$ an n-decomposition of the set H, $\varphi \in S_n$. Then we put

$$R^1_{\mathcal{K},\varphi} = R_{\mathcal{K},\varphi},$$
$$R^m_{\mathcal{K},\varphi} = (R^{m-1}_{\mathcal{K},\varphi})_{\mathcal{K},\varphi}$$

for any $m \in N$, $m \geq 2$.

1.14. Lemma. *Let $R \subseteq G^H$ be a relation, $\mathcal{K}$ an n-decomposition of the set H, let φ, $\psi \in S_n$, $m \in N$. Then:*

(1) $(R_{\mathcal{K},\varphi})_{\mathcal{K},\psi} = R_{\mathcal{K},\varphi\psi}$.
(2) $R^m_{\mathcal{K},\varphi} = R_{\mathcal{K},\varphi^m}$.

Proof. (1) Let $\mathcal{K} = \{K_i\}_{i=1}^{n+1}$, let $f \in (R_{\mathcal{K},\varphi})_{\mathcal{K},\psi}$. Then there exists $g \in R_{\mathcal{K},\varphi}$ such that $f(K_i) = g(K_{\psi(i)})$ for all $i \in (n]$, $f(K_{n+1}) = g(K_{n+1})$. As $g \in R_{\mathcal{K},\varphi}$, there exists $h \in R$ such that $g(K_i) = h(K_{\varphi(i)})$ for all $i \in (n]$, $g(K_{n+1}) = h(K_{n+1})$. Thus, $f(K_i) = h(K_{(\varphi\psi)(i)})$ for all $i \in (n]$, $f(K_{n+1}) = h(K_{n+1})$, and $f \in R_{\mathcal{K},\varphi\psi}$. The converse inclusion can be shown similarly.

(2) It follows from (1).

1.15. Lemma. *Let J be a nonempty set, $R, R_1, \dots, R_n$, $R'_1, \dots, R'_n, T, T_j$ for all $j \in J$ relations with the carrier G and the index set H. Let $\mathcal{K}$ be an n-decomposition of the set H, $\varphi \in S_n$, $r \in N$ such that $\varphi^r =$id. Let $k \in (n]$, $m \in N$. Then:*

(1) $E_{\mathcal{K}} = (E_{\mathcal{K}})_{\mathcal{K},\varphi} = (E_{\mathcal{K}} \dots E_{\mathcal{K}})_{\mathcal{K}}$.
(2) $R \subseteq R^r_{\mathcal{K},\varphi} = R_{\mathcal{K},\mathrm{id}}$.
(3) $(E_{\mathcal{K}} \dots E_{\mathcal{K}} \underset{k\text{-th place}}{\underset{\uparrow}{R}} E_{\mathcal{K}} \dots E_{\mathcal{K}})_{\mathcal{K}} \subseteq R_{\mathcal{K},\mathrm{id}}$.
(4) *$R \subseteq T$ implies $R_{\mathcal{K},\varphi} \subseteq T_{\mathcal{K},\varphi}$.*
(5) $(\bigcup_{j\in J} T_j)_{\mathcal{K},\varphi} = \bigcup_{j\in J} (T_j)_{\mathcal{K},\varphi}$.
(6) $(\bigcap_{j\in J} T_j)_{\mathcal{K},\varphi} \subseteq \bigcap_{j\in J} (T_j)_{\mathcal{K},\varphi}$.
(7) $((R_1 \dots R_n)_{\mathcal{K}})_{\mathcal{K},\varphi} \supseteq ((R_{\varphi(1)})_{\mathcal{K},\varphi} \dots (R_{\varphi(n)})_{\mathcal{K},\varphi})_{\mathcal{K}}$.
(8) *If $n \leq 2$ or $\mathcal{K}$ is regular, then (7) becomes the equality.*

(9) $R_i \subseteq R'_i$ *for all* $i \in (n]$ *imply* $(R_1 \dots R_n)_{\mathcal{K}} \subseteq (R'_1 \dots R'_n)_{\mathcal{K}}$.
(10) $R \subseteq T$ *implies* $R^m_{\mathcal{K}} \subseteq T^m_{\mathcal{K}}$.
(11) $(R_{\mathcal{K},\varphi})^m_{\mathcal{K}} \subseteq (R^m_{\mathcal{K}})_{\mathcal{K},\varphi}$.
(12) *If* $n \leq 2$ *or* $\mathcal{K}$ *is regular, then (11) becomes the equality.*

Proof. The assertion folows directly from the definitions of the operations. For example, let us prove (2) and (7).

(2) By 1.14 (2), we have $R^r_{\mathcal{K},\varphi} = R_{\mathcal{K},\varphi^r} = R_{\mathcal{K},\mathrm{id}}$. By 1.5 (1), $R \subseteq R_{\mathcal{K},\mathrm{id}}$.

(7) Let $\mathcal{K} = \{K_i\}_{i=1}^{n+1}$. Let $f \in ((R_{\varphi(1)})_{\mathcal{K},\varphi} \dots (R_{\varphi(n)})_{\mathcal{K},\varphi})_{\mathcal{K}}$. Then there exist $f_i \in (R_{\varphi(i)})_{\mathcal{K},\varphi}$ for all $i \in (n]$ such that $f(K_i) = f_i(K_i)$ for all $i \in (n]$, $f(K_{n+1}) = f_i(K_{n+1})$ for all $i \in (n]$, and $f_i(K_j) = f_j(K_i)$ for all $i,j \in (n]$. Hence, there exist $g_i \in R_{\varphi(i)}$ for all $i \in (n]$ such that $f_i(K_j) = g(K_{\varphi(j)})$ for all $i,j \in (n]$, $f_i(K_{n+1}) = g_i(K_{n+1})$ for all $i \in (n]$. Now, let us conctruct a mapping $g \in G^H$ such that $g(K_{\varphi(i)}) = g_i(K_{\varphi(i)})$ for all $i \in (n], g(K_{n+1}) = f(K_{n+1})$. Then $g \in (R_1 \dots R_n)_{\mathcal{K}}$, for $g_{\varphi^{-1}(i)} \in R_i$ for all $i \in (n]$, $g(K_i) = g_{\varphi^{-1}(i)}(K_i)$ for all $i \in (n]$, $g(K_{n+1}) = f(K_{n+1}) = f_{\varphi^{-1}(i)}(K_{n+1}) = g_{\varphi^{-1}(i)}(K_{n+1})$ for all $i \in (n], g_{\varphi^{-1}(i)}(K_j) = f_{\varphi^{-1}(i)}(K_{\varphi^{-1}(j)}) = f_{\varphi^{-1}(j)}(K_{\varphi^{-1}(i)}) = g_{\varphi^{-1}(j)}(K_i)$ for all $i,j \in (n]$. From this it follows that $f \in ((R_1 \dots R_n)_{\mathcal{K}})_{\mathcal{K},\varphi}$, for we have $f(K_i) = f_i(K_i) = g_i(K_{\varphi(i)}) = g(\mathcal{K}_{\varphi(i)})$ for all $i \in (n]$, $f(K_{n+1}) = g(K_{n+1})$.

1.16. Remark. The inclusions in (2), (3), (6), and (7) cannot, in general, be replaced by the equalities. For example, let $G = \{a,b\}, H = \{1,2,3,4,5\}, \mathcal{K} = \{K_j\}_{i=1}^4$, $K_1 = \{1,2\}$, $K_2 = \{3\}$, $K_3 = \{4,5\}$, let $R_1 = \{(a,b,a,a,b)\}$, $R_2 = \{(a,a,a,a,a)\}$, $R_3 = \{(a,b,a,a,a)\}$ and let $\varphi \in S_3$ be such that $\varphi(1) = 2$, $\varphi(2) = 3$, $\varphi(3) = 1$. Then $((R_1R_2R_3)_{\mathcal{K}})_{\mathcal{K},\varphi} = \{(a,a,a,a,b),(a,a,a,b,a)\}$, but $((R_2)_{\mathcal{K},\varphi}(R_3)_{\mathcal{K},\varphi}(R_1)_{\mathcal{K},\varphi}) = \emptyset$.

1.17. Definition. Let $R \subseteq G^H$ be a relation, $\mathcal{K}$ an n-decomposition of the set H, let $\pi \in S_n$ be the permutation defined by:

$$\pi(i) = \begin{cases} i+1 & \text{for all } i \in (n-1] \\ 1 & \text{for } i = n \end{cases}.$$

Then we define:

$$\begin{aligned} {}^1R_{\mathcal{K}} &= R_{\mathcal{K},\pi}, \\ {}^mR_{\mathcal{K}} &= {}^1({}^{m-1}R_{\mathcal{K}})_{\mathcal{K}} \end{aligned}$$

for any $m \in N$, $m \geq 2$. ${}^mR_{\mathcal{K}}$ is called the *m-th cyclic transposition* of R with regard to $\mathcal{K}$.

1.18. Lemma. *Let* $R, R_1, \dots, R_n \subseteq G^H$ *be relations,* $\mathcal{K}$ *an* n*-decomposition of the set* H*. Then:*

(1) ${}^1R_{\mathcal{K}} = {}^1R_{\mathcal{K}^*}$.

(2) $E_{\mathcal{K}} =^1 (E_{\mathcal{K}})_{\mathcal{K}}$.

(3) ${}^1((R_1 \dots R_n)_{\mathcal{K}})_{\mathcal{K}} \supseteq ({}^1(R_2)_{\mathcal{K}} \dots {}^1(R_n)_{\mathcal{K}}{}^1(R_1)_{\mathcal{K}})_{\mathcal{K}}$.

(4) *If $n \leq 2$ or $\mathcal{K}$ is regular, then (3) becomes the equality.*

Proof. (1) follows from the fact that $\pi^* = \pi$.
(2), (3), and (4) follow from 1.15 (1), (7), and (8).

1.19. Lemma. *Let J be a nonempty set, R, T, T_j for all $j \in J$ be relations with the carrier G and the index set H. Let $\mathcal{K}$ be an n-decomposition of the set H. Then:*

(1) $R \subseteq {}^n R_{\mathcal{K}}$.
(2) $R \subseteq T$ *implies* ${}^1R_{\mathcal{K}} \subseteq^1 T_{\mathcal{K}}$.
(3) ${}^1(\bigcup\limits_{j \in J} T_j)_{\mathcal{K}} = \bigcup\limits_{j \in J} {}^1(T_j)_{\mathcal{K}}$.
(4) ${}^1(\bigcap\limits_{j \in J} T_j)_{\mathcal{K}} \subseteq \bigcap\limits_{j \in J} {}^1(T_j)_{\mathcal{K}}$.

Proof. The assertion follows from 1.15 (2), (4), (5), and (6).

2. Properties of relations

2.1. Definition. Let $R \subseteq G^H$ be a relation, $\mathcal{K} = \{K_i\}_{i=1}^{n+1}$ an n-decomposition of the set H, $\varphi \in S_n$. Then R is called

(1) *reflexive (irreflexive)* with regard to $\mathcal{K}$ if $E_{\mathcal{K}} \subseteq R$ $(R \cap E_{\mathcal{K}} = \emptyset)$,
(2) *symmetric (asymmetric, antisymmetric)* with regard to $\mathcal{K}$ and φ if $R_{\mathcal{K},\varphi} \subseteq R (R \cap R_{\mathcal{K},\varphi} = \emptyset,\ R \cap R_{\mathcal{K},\varphi} \subseteq E_{\mathcal{K}})$,
(3) *transitive (atransitive)* with regard to $\mathcal{K}$ if $R^2_{\mathcal{K}} \subseteq R$ $(R \cap R^m_{\mathcal{K}} = \emptyset$ for any $m \in N, m \geq 2)$,
(4) *complete* with regard to $\mathcal{K}$ if $f \in G^H$, $f(K_i) \neq f(K_j)$ for all $i, j \in (n]$, $i \neq j$ imply the existence of $\psi \in S_n$ such that $f \in R_{\mathcal{K},\psi}$,
(5) *regular* with regard to $\mathcal{K}$ if $f \in R$, $g \in G^H$, $f(K_i) = g(K_i)$ for all $i \in (n+1]$ imply $g \in R$.

2.2. Remark. Let $R \subseteq G^H$ be a relation, $\mathcal{K} = \{K_i\}_{i=1}^{n+1}$ an n-decomposition of the set H. If card $K_i \leq 1$ for all $i \in (n+1]$, then R is clearly regular with regard to $\mathcal{K}$.

2.3. Lemma. *Let $R, R_1, \dots, R_n \subseteq G^H$ be relations, $\mathcal{K}$ an n-decomposition of the set H, let $\varphi \in S_n$, $m \in N$, $m \geq 2$. Then:*

(1) *$E_{\mathcal{K}}, R_{\mathcal{K},\varphi}, (R_1 \dots R_n)_{\mathcal{K}}$, and $R^m_{\mathcal{K}}$ are regular with regard to $\mathcal{K}$*
(2) *If R is symmetric with regard to $\mathcal{K}$ and φ, then $R_{\mathcal{K},\varphi} = R$ and R is regular with regard to $\mathcal{K}$.*

Proof. (1) is evident.

(2) Let R be symmetric with regard to $\mathcal{K}$ and φ. By 1.15 (2) and (4), $R \subseteq R^r_{\mathcal{K},\varphi} \subseteq R^{r-1}_{\mathcal{K},\varphi} \subseteq \cdots \subseteq R_{\mathcal{K},\varphi} \subseteq R$ for any $r \in N$ such that φ^r =id. Thus $R_{\mathcal{K},\varphi} = R$. The rest follows from (1).

2.4. Lemma. *Let J be a nonempty set, $j_0 \in J, R, T_j$ for all $j \in J$ relations with the carrier G and the index set H. Let $\mathcal{K}$ be an n-decomposition of the set H, $\varphi \in S_n$, $r \in N$ be such that φ^r =id. Then:*

(1) *If R is regular with regard to $\mathcal{K}$, then $R = R^r_{\mathcal{K},\varphi}$.*
(2) *If T_j is regular with regard to $\mathcal{K}$ for each $j \in J - \{j_0\}$, then* $(\bigcap_{j\in J} T_j)_{\mathcal{K},\varphi} = \bigcap_{j\in J} (T_j)_{\mathcal{K},\varphi}$.

Proof. The inclusions $\subseteq$ result from 1.15 (2) and (6). Using regularity one can easily prove the converse ones.

2.5. Theorem. *Let J be a nonempty set, $j_0 \in J$. Let $R, R_1, \ldots, R_n, T_j$ for all $j \in J$ be relations with the carrier G and the index set H. Let $\mathcal{K}$ be an n-decomposition of the set H, $\varphi \in S_n$. Then:*

(1) *If T_{j_0} is reflexive with regard to $\mathcal{K}$, then $\bigcup_{j\in J} T_j$ is reflexive with regard to $\mathcal{K}$.*
(2) *If $R, R_1, \ldots, R_n$, and T_j for all $j \in J$ are reflexive with regard to $\mathcal{K}$, then $\bigcap_{j\in J} T_j$, $R_{\mathcal{K},\varphi}$, and $(R_1 \ldots R_n)_{\mathcal{K}}$ are reflexive with regard to $\mathcal{K}$.*
(3) *If T_j for all $j \in J$ are regular with regard to $\mathcal{K}$, then $\bigcup_{j\in J} T_j$ and $\bigcap_{j\in J} T_j$ are regular with regard to $\mathcal{K}$.*
(4) *If R and T_j for all $j \in J$ are irreflexive (symmetric) with regard to $\mathcal{K}$ (and φ), then $\bigcup_{j\in J} T_j$, $\bigcap_{j\in J} T_j$, and $R_{\mathcal{K},\varphi}$ have the same property.*
(5) *If R and T_j for all $j \in J$ are transitive with regard to $\mathcal{K}$, then $\bigcap_{j\in J} T_j$ and $R_{\mathcal{K},\varphi}$ are transitive with regard to $\mathcal{K}$.*
(6) *If R and T_{j_0} are atransitive (asymmetric, antisymmetric) with regard to $\mathcal{K}$ (and φ), then $\bigcap_{j\in J} T_j$ and $R_{\mathcal{K},\varphi}$ have the same property.*
(7) *If R and T_{j_0} are complete with regard to $\mathcal{K}$, then $\bigcup_{j\in J} T_j$ and $R_{\mathcal{K},\varphi}$ are complete with regard to $\mathcal{K}$.*

Proof. The assertions (1) and (3) are evident, the others follow from 1.5 (2), 1.14 (1), 1.15 (1), (4) - (6), (9) - (11), 2.3 (1), and 2.4 (2).

2.6. Theorem. *Let $R_1, \ldots, R_n \subseteq G^H$ be relations, let $\mathcal{K}$ be an n-decomposition, $\varphi \in S_n$. Let $R_1, \ldots, R_n$ be symmetric with regard to $\mathcal{K}$ and φ.*

(1) *If $(R_1 \ldots R_n)_{\mathcal{K}}$ is symmetric with regard to $\mathcal{K}$ and φ, then $(R_1 \ldots R_n)_{\mathcal{K}} \supseteq (R_{\varphi(1)} \ldots R_{\varphi(n)})_{\mathcal{K}}$.*

(2) *If $n \leq 2$ or $\mathcal{K}$ is regular, then $(R_1 \dots R_n)_{\mathcal{K}}$ is symmetric with regard to $\mathcal{K}$ and φ if and only if $(R_1 \dots R_n)_{\mathcal{K}} \supseteq (R_{\varphi(1)} \dots R_{\varphi(n)})_{\mathcal{K}}$.*

Proof. The statements result from 1.15 (7) - (9) and 2.3 (2).

2.7. Lemma. *Let $R \subseteq G^H$ be a relation, $\mathcal{K}$ an n-decomposition of the set H, $\varphi \in S_n$. Then:*

(1) *If R is reflexive (irreflexive, transitive, atransitive, complete, regular) with regard to $\mathcal{K}$, then it has the same property with regard to $\mathcal{K}^*$.*

(2) *If R is symmetric (asymmetric, antisymmetric) with regard to $\mathcal{K}$ and φ, then it has the same property with regard to $\mathcal{K}^*$ and φ^*.*

Proof. For regularity the assertion is obvious, and for the other properties it follows from 1.12 (2), (3), and (5).

2.8. Definition. Let $R \subseteq G^H$ be a relation, $\mathcal{K}$ an n-decomposition of the set H. Then R is called:

(1) *cyclic (acyclic, anticyclic)* with regard to $\mathcal{K}$ if it is symmetric (asymmetric, antisymmetric) with regard to $\mathcal{K}$ and π,

(2) *symmetric (asymmetric, antisymmetric)* with regard to $\mathcal{K}$ if it is symmetric with regard to $\mathcal{K}$ and φ for any $\varphi \in S_n$ (asymmetric, antisymmetric with regard to $\mathcal{K}$ and φ for any odd permutation $\varphi \in S_n$).

2.9. Lemma. *Let $R \subseteq G^H$ be a relation, $\mathcal{K}$ an n-decomposition of the set H. Then:*

(1) *${}^1R_{\mathcal{K}}$ is regular with regard to $\mathcal{K}$.*

(2) *If R is cyclic or symmetric with regard to $\mathcal{K}$, then it is regular with regard to $\mathcal{K}$.*

Proof. (1) follows from 2.3 (1).
(2) follows from 2.3 (2).

2.10. Lemma. *Let J be a nonempty set, $j_0 \in J$. Let R and T_j for all $j \in J$ be relations with the carrier G and the index set H, $\mathcal{K}$ an n-decomposition of the set H. Then:*

(1) *If R is regular with regard to $\mathcal{K}$, then $R = {}^nR_{\mathcal{K}}$.*

(2) *If T_j is regular with regard to $\mathcal{K}$ for all $j \in J - \{j_0\}$, then ${}^1(\bigcap\limits_{j \in J} T_j)_{\mathcal{K}} = \bigcap\limits_{j \in J} {}^1(T_j)_{\mathcal{K}}$.*

Proof. (1) follows from 2.4 (1).
(2) follows from 2.4 (2).

2.11. Remark. Let $R \subseteq G^H$ be a relation, $\mathcal{K}$ an n-decomposition of the set H. Evidently, then ${}^1R_{\mathcal{K}} = {}^{n+1}R_{\mathcal{K}}$.

2.12. Theorem. *Let $R \subseteq G^H$ be a relation, $\mathcal{K}$ an n-decomposition of the set H. Then:*

(1) *R is a symmetric with regard to $\mathcal{K}$ if and only if it is cyclic with regard to $\mathcal{K}$ and symmetric with regard to $\mathcal{K}$ and φ for any $\varphi \in S_n(1)$.*
(2) *If n is odd and R is cyclic with regard to $\mathcal{K}$, then R is asymmetric (antisymmetric) with regard to $\mathcal{K}$ if and only if it is asymmetric (antisymmetric) with regard to $\mathcal{K}$ and φ for any odd permutation $\varphi \in S_n(1)$.*

Proof. (1) " $\Rightarrow$ ": Let R be symmetric with regard to $\mathcal{K}$. Then it is symmetric with regard to $\mathcal{K}$ and φ for any $\varphi \in S_n$, consequently for any $\varphi \in S_n(1)$ and for $\varphi = \pi$.
" $\Leftarrow$ ": Let R be cyclic with regard to $\mathcal{K}$ and symmetric with regard to $\mathcal{K}$ and φ for any $\varphi \in S_n(1)$. Let $\psi \in S_n$. Denote $\psi(1) = k$, $\varphi = \pi^{n-k+1}\psi$. Then $\varphi \in S_n(1)$ and $\psi = \pi^{k-1}\varphi$. By 1.14 (1) and 1.15 (4), we have $R_{\mathcal{K},\psi} = R_{\mathcal{K},\pi^{k-1}\varphi} = (R_{\mathcal{K},\pi})_{\mathcal{K},\pi^{k-2}\varphi} \subseteq R_{\mathcal{K},\pi^{k-2}\varphi} = (R_{\mathcal{K},\pi})_{\mathcal{K},\pi^{k-3}\varphi} \subseteq R_{\mathcal{K},\pi^{k-3}\varphi} = \cdots \subseteq R_{\mathcal{K},\varphi} \subseteq R$. Thus R is symmetric with regard to $\mathcal{K}$ and ψ for any $\psi \in S_n$.

(2)" $\Rightarrow$ ": Let R be asymmetric (antisymmetric) with regard to $\mathcal{K}$. Then it is asymmetric (antisymmetric) with regard to $\mathcal{K}$ and φ for any odd permutation $\varphi \in S_n$, consequently for any odd permutation $\varphi \in S_n(1)$.
" $\Leftarrow$ ": Let R be asymmetric (antisymmetric) with regard to $\mathcal{K}$ and φ for any odd permutation $\varphi \in S_n(1)$. Let $\psi \in S_n$ be odd. If we again denote $\psi(1) = k$, $\varphi = \pi^{n-k+1}\psi$, we have $\varphi \in S_n(1)$ and $\psi = \pi^{k-1}\varphi$. Clearly π is even, thus φ is odd. By 2.3 (2), $R_{\mathcal{K},\pi} = R$, so that, by 1.14 (1), $R_{\mathcal{K},\psi} = R_{\mathcal{K},\pi^{k-1}\varphi} = (R_{\mathcal{K},\pi})_{\mathcal{K},\pi^{k-2}\varphi} = R_{\mathcal{K},\pi^{k-2}\varphi} = (R_{\mathcal{K},\pi})_{\mathcal{K},\pi^{k-3}\varphi} = R_{\mathcal{K},\pi^{k-3}\varphi} = \cdots = R_{\mathcal{K}}, \varphi$. From this it follows that $R \cap R_{\mathcal{K}}, \psi = R \cap R_{\mathcal{K}}, \varphi = \emptyset (\subseteq E_{\mathcal{K}})$. Hence R is asymmetric (antisymmetric) with regard to $\mathcal{K}$ and ψ for any odd permutation $\psi \in S_n$.

2.13. Theorem. *Let J be a nonempty set, $j_0 \in J$. Let R, T_j for all $j \in J$ be relations with the carrier G and the index set H. Let $\mathcal{K}$ be an n-decomposition of the set H, $\varphi \in S_n$. Then:*

(1) *If R and T_j for all $j \in J$ are cyclic with regard to $\mathcal{K}$, then $\bigcup_{j\in J} T_j$, $\bigcap_{j\in J} T_j$, and ${}^1R_{\mathcal{K}}$ are cyclic with regard to $\mathcal{K}$.*
If, moreover, $\pi\varphi\pi = \varphi$, then $R_{\mathcal{K},\varphi}$ is cyclic with regard to $\mathcal{K}$, too.
(2) *If R and T_j for all $j \in J$ are symmetric with regard to $\mathcal{K}$, then $\bigcup_{j\in J} T_j$, $\bigcap_{j\in J} T_j$, $R_{\mathcal{K},\varphi}$, and ${}^1R_{\mathcal{K}}$ are symmetric with regard to $\mathcal{K}$.*
(3) *If R and T_{j_0} are acyclic (anticyclic) with regard to $\mathcal{K}$, then $\bigcap_{j\in J} T_j$ and ${}^1R_{\mathcal{K}}$ have the same property. If, moreover, $\pi\varphi = \varphi\pi$, then $R_{\mathcal{K},\varphi}$ has the same property, too.*
(4) *If T_{j_0} is asymmetric (antisymmetric) with regard to $\mathcal{K}$, then $\bigcap_{j\in J} T_j$ has the same property.*
(5) *If R is asymmetric (antisymmetric) with regard to $\mathcal{K}$ and $n \leq 2$ or $\varphi =$id, then $R_{\mathcal{K},\varphi}$ has the same property.*

(6) *If R is complete with regard to $\mathcal{K}$, then ${}^1R_{\mathcal{K}}$ is complete with regard to $\mathcal{K}$.*

Proof. (1) By 2.5 (4), $\bigcup_{j\in J} T_j$, $\bigcap_{j\in J} T_j$, and ${}^1R_{\mathcal{K}}$ are cyclic with regard to $\mathcal{K}$. Let R be cyclic with regard to $\mathcal{K}$, $\pi\varphi\pi = \varphi$, and let $f \in {}^1(R_{\mathcal{K},\varphi})_{\mathcal{K}}$. Let $\mathcal{K} = \{K_i\}_{i=1}^{n+1}$. By 1.14 (1), we have ${}^1(R_{\mathcal{K},\varphi})_{\mathcal{K}} = R_{\mathcal{K},\varphi\pi}$. Consequently, $f \in R_{\mathcal{K},\varphi\pi}$, thus there exists $g \in R$ such that $f(K_i) = g(K_{(\varphi\pi)(i)})$ for all $i \in (n]$, $f(K_{n+1}) = g(K_{n+1})$. As R is cyclic with regard to $\mathcal{K}$, we have, by 2.3 (2), ${}^1R_{\mathcal{K}} = R$. Hence $g \in {}^1R_{\mathcal{K}}$, consequently there exists $h \in R$ such that $g(K_i) = h(K_{\pi(i)})$ for all $i \in (n]$, $g(K_{n+1}) = h(K_{n+1})$. Thus, there exists $h \in R$ such that $f(K_i) = h(K_{(\pi\varphi\pi)(i)}) = h(K_{\varphi(i)})$ for all $i \in (n]$, $f(K_{n+1}) = h(K_{n+1})$, and $f \in R_{\mathcal{K},\varphi}$. Hence ${}^1(R_{\mathcal{K},\varphi})_{\mathcal{K}} \subseteq R_{\mathcal{K},\varphi}$ and $R_{\mathcal{K},\varphi}$ is cyclic with regard to $\mathcal{K}$.

(2) follows from 2.3 (2) and 2.5 (4).

(3) By 2.5 (6), $\bigcap_{j\in J} T_j$ and ${}^1R_{\mathcal{K}}$ are acyclic (anticyclic) with regard to $\mathcal{K}$. Let R be acyclic (anticyclic) with regard to $\mathcal{K}$, $\pi\varphi = \varphi\pi$. Admit that $R_{\mathcal{K},\varphi}$ is not acyclic (anticyclic) with regard to $\mathcal{K}$. Then there exists $f \in R_{\mathcal{K},\varphi} \cap {}^1(R_{\mathcal{K},\varphi})_{\mathcal{K}}$ ($f \in R_{\mathcal{K},\varphi} \cap {}^1(R_{\mathcal{K},\varphi})_{\mathcal{K}} - E_{\mathcal{K}}$). Let $\mathcal{K} = \{K_i\}_{i=1}^{n+1}$. By 1.14 (1), we have ${}^1(R_{\mathcal{K},\varphi})_{\mathcal{K}} = R_{\mathcal{K},\varphi\pi}$, consequently there exist $g, h \in R$ such that $f(K_i) = g(K_{\varphi(i)}) = h(K_{(\varphi\pi)(i)})$ for all $i \in (n]$, $f(K_{n+1}) = g(K_{n+1}) = h(K_{n+1})$. As $\pi\varphi = \varphi\pi$, we have $g(K_{\varphi(i)}) = h(K_{(\pi\varphi)(i)})$ for all $i \in (n]$, $g(K_{n+1}) = h(K_{n+1})$, thus $g(K_i) = h(K_{\pi(i)})$for all $i \in (n]$, $g(K_{n+1}) = h(K_{n+1})$. Hence $g \in {}^1R_{\mathcal{K}}$. We obtain $g \in R \cap {}^1R_{\mathcal{K}}$. In the case of acyclicity we get a contradiction. In the case of anticyclicity there exist $i, j \in (n]$ such that $f(K_i) \neq f(K_j)$, consequently $g(K_{\varphi(i)}) \neq g(K_{\varphi(j)})$, so that $g \in R \cap {}^1R_{\mathcal{K}} - E_{\mathcal{K}}$ and we again obtain a contradiction.

The reamining assertions can be proved similarly with the use of 1.14 (1), 2.5 (6), and (7).

2.14. Remark. (1) The condition $\pi\varphi\pi = \varphi$ is obviously satisfied exactly by the permutations $\varphi_1, \dots, \varphi_n \in S_n$ given by

$$\varphi_k(i) = \begin{cases} k - i + 1 & \text{for all } \; i \in (k] \\ k + n - i + 1 & \text{for all } \; i \in (n] - (k] \end{cases}$$

for any $k \in (n]$.

(2) The condition $\pi\varphi = \varphi\pi$ is obviously satisfied exactly by all the iterations of π.

2.15. Lemma. *Let $n \in N$. Then:*

(1) *The mapping $*$ of the set S_n into itself is a bijection.*

(2) *If $\varphi \in S_n$, then the permutations φ, φ^* have the same sign.*

Proof is evident.

2.16. Theorem. *Let $R \subseteq G^H$ be a relation, $\mathcal{K}$ an n-decomposition of the set H. If R has any of the properties defined in 2.8 with regard to $\mathcal{K}$, then it has the same property with regard to $\mathcal{K}^*$.*

Proof. Let R be cyclic (acyclic, anticyclic) with regard to $\mathcal{K}$. Then it is symmetric (asymmetric, antisymmetric) with regard to $\mathcal{K}$ and π, consequently, by 2.7 (2), it has the same property with regard to $\mathcal{K}^*$ and $\pi^* = \pi$. Thus, R is cyclic (acyclic, anticyclic) with regard to $\mathcal{K}^*$.

Let R be symmetric with regard to $\mathcal{K}$. Then it is symmetric with regard to $\mathcal{K}$ and φ for any $\varphi \in S_n$. By 2.7 (2), it is symmetric with regard to $\mathcal{K}^*$ and φ^* for any $\varphi \in S_n$ as well. By 2.15 (1), it is symmetric with regard to $\mathcal{K}^*$ and φ for any $\varphi \in S_n$, thus it is symmetric with regard to $\mathcal{K}^*$.

Let R be asymmetric (antisymmetric) with regard to $\mathcal{K}$. Then it is asymmetric (antisymmetric) with regard to $\mathcal{K}$ and φ for any odd permutation $\varphi \in S_n$. By 2.7 (2), it is asymmetric (antisymmetric) with regard to $\mathcal{K}^*$ and φ^* for any odd permutation $\varphi \in S_n$ as well. By 2.15 (1) and (2), it is asymmetric (antisymmetric) with regard to $\mathcal{K}^*$ and φ for any odd permutation $\varphi \in S_n$, hence it is asymmetric (antisymmetric) with regard to $\mathcal{K}^*$.

3. Hulls of relations

3.1. Definition. Let $R \subseteq G^H$ be a relation, $\mathcal{K}$ an n-decomposition of the set H, $\varphi \in S_n$. Let (p) be any of the properties defined in 2.1 or 2.8. A relation $Q \subseteq G^H$ is called the (p)-*hull* of R with regard to $\mathcal{K}$ (and φ) if

(1) $R \subseteq Q$,
(2) Q has the property (p),
(3) if $T \subseteq G^H$ is any relation having the property (p) and such that $R \subseteq T$, then $Q \subseteq T$.

3.2. Remark. Let $R \subseteq G^H$ be a relation, $\mathcal{K}$ an n-decomposition of the set H, $\varphi \in S_n$. Let (p) be any of the properties defined in 2.1 or 2.8. Obviously, then R has the property (p) if and only if there exists the (p)-hull Q of R with regard to $\mathcal{K}$ (and φ) and $R = Q$.

3.3. Lemma. *Let $R, T \subseteq G^H$ be relations, $\mathcal{K}$ an n-decomposition of the set H, $\varphi \in S_n$. Let (p) be any of the properties defined in 2.1 or 2.8, $R^{(p)}$ $(T^{(p)})$ the (p)-hull of R (T) with regard to $\mathcal{K}$ (and φ). Then $R \subseteq T$ implies $R^{(p)} \subseteq T^{(p)}$.*

Proof. Let $R \subseteq T$. We have $T \subseteq T^{(p)}$. Thus $R \subseteq T^{(p)}$. As $T^{(p)}$ has the property (p), we obtain $R^{(p)} \subseteq T^{(p)}$.

3.4. Definition. Let $R \subseteq G^H$ be a relation, $\mathcal{K}$ an n-decomposition of the set H. Then we define

$$\begin{aligned} {}_1R_{\mathcal{K}} &= R, \\ {}_mR_{\mathcal{K}} &= {}_{m-1}R_{\mathcal{K}} \cup ({}_{m-1}R_{\mathcal{K}})^2_{\mathcal{K}} \end{aligned}$$

for any $m \in N$, $m \geq 2$.

3.5. Remark. Let $R \subseteq G^H$ be a relation, $\mathcal{K}$ an n-decomposition of the set H. Clearly, then

$$_mR_{\mathcal{K}} \subseteq {}_{m+1}R_{\mathcal{K}}$$

for any $m \in N$.

3.6. Theorem. *Let $R \subseteq G^H$ be a relation, $\mathcal{K}$ an n-decomposition of the set H. Let $\varphi \in S_n$, $r \in N$ be such that φ^r =id. Then the following relations exist:*

(1) *the reflexive hull $R^{(r)}_{\mathcal{K}}$ of R with regard to $\mathcal{K}$ and we have $R^{(r)}_{\mathcal{K}} = R \cup E_{\mathcal{K}}$,*

(2) *the symmetric hull $R^{(s)}_{\mathcal{K},\varphi}$ of R with regard to $\mathcal{K}$ and φ and we have*

$$R^{(s)}_{\mathcal{K},\varphi} = \bigcup_{i=1}^{r} R^i_{\mathcal{K},\varphi},$$

(3) *the transitive hull $R^{(t)}_{\mathcal{K}}$ of R with regard to $\mathcal{K}$ and we have*

$$R^{(t)}_{\mathcal{K}} = \bigcup_{i=1}^{\infty} {}_iR_{\mathcal{K}},$$

(4) *the regular hull $R^{(g)}_{\mathcal{K}}$ of R with regard to $\mathcal{K}$ and we have*

$$R^{(g)}_{\mathcal{K}} = R^r_{\mathcal{K},\varphi}.$$

Proof. (1) is evident.

(2) Put $Q = \bigcup\limits_{i=1}^{r} R^i_{\mathcal{K},\varphi}$. By 1.15 (2), we have $R \subseteq R^r_{\mathcal{K},\varphi}$, consequently $R \subseteq Q$. By 1.15 (5), we get $Q_{\mathcal{K},\varphi} = (\bigcup\limits_{i=1}^{r} R^i_{\mathcal{K},\varphi})_{\mathcal{K},\varphi} = \bigcup\limits_{i=1}^{r} R^{i+1}_{\mathcal{K},\varphi} = \bigcup\limits_{i=2}^{r} R^i_{\mathcal{K},\varphi} \cup (R_{\mathcal{K},\varphi})^r_{\mathcal{K},\varphi}$. By 2.3 (1), $R_{\mathcal{K},\varphi}$ is regular with regard to $\mathcal{K}$, thus, by 2.4 (1), $R_{\mathcal{K},\varphi} = (R_{\mathcal{K},\varphi})^r_{\mathcal{K},\varphi}$. Hence, we obtain $Q_{\mathcal{K},\varphi} = \bigcup\limits_{i=1}^{r} R^i_{\mathcal{K},\varphi} = Q$ and Q is symmetric with regard to $\mathcal{K}$ and φ. Let $T \subseteq G^H$ be a relation symmetric with regard to $\mathcal{K}$ and φ and such that $R \subseteq T$. Then, by 1.15 (4), $Q = \bigcup\limits_{i=1}^{r} R^i_{\mathcal{K},\varphi} \subseteq \bigcup\limits_{i=1}^{r} T^i_{\mathcal{K},\varphi} \subseteq T$, for T is symmetric with regard to $\mathcal{K}$ and φ, and we have $R^{(s)}_{\mathcal{K},\varphi} = Q$.

(3) Put $Q = \bigcup\limits_{i=1}^{\infty} {}_iR_{\mathcal{K}}$. Clearly $R = {}_1R_{\mathcal{K}} \subseteq Q$. Let $f \in Q^2_{\mathcal{K}}$. Let $\mathcal{K} = \{K_i\}_{i=1}^{n+1}$. Then there exists $f_i \in Q$ for each $i \in (n]$ such that $f(K_i) = f_i(K_i)$ for all $i \in (n]$, $f(K_{n+1}) = f_i(K_{n+1})$ for all $i \in (n]$, $f_i(K_j) = f_j(K_i)$ for all $i, j \in (n]$. There exists $j_i \in N$ for each $i \in (n]$ such that $f_i \in {}_{j_i}R_{\mathcal{K}}$ for all $i \in (n]$.
From this it follows that $f \in ({}_{j_1}R_{\mathcal{K}} \dots {}_{j_n}R_{\mathcal{K}})_{\mathcal{K}}$. Denote $j_0 = \max\{j_1, \dots, j_n\}$. By

3.5, we have ${}_{j_i}R_{\mathcal{K}} \subseteq {}_{j_0}R_{\mathcal{K}}$ for all $i \in (n]$. By 1.15 (9), $f \in ({}_{j_0}R_{\mathcal{K}} \dots {}_{j_0}R_{\mathcal{K}})_{\mathcal{K}} = {}_{j_0}R^2_{\mathcal{K}} \subseteq {}_{j_0+1}R_{\mathcal{K}} \subseteq \bigcup_{i=1}^{\infty} {}_iR_{\mathcal{K}} = Q$. Thus $Q^2_{\mathcal{K}} \subseteq Q$ and Q is transitive with regard to $\mathcal{K}$. Let T be transitive with regard to $\mathcal{K}$ and such that $R \subseteq T$. Using 1.15 (10), it is easy to prove by induction that ${}_iR_{\mathcal{K}} \subseteq T$ for any $i \in N$. Hence $Q = \bigcup_{i=1}^{\infty} {}_iR_{\mathcal{K}} \subseteq \bigcup_{i=1}^{\infty} T = T$, and we have $R^{(t)}_{\mathcal{K}} = Q$.

(4) The statement follows from 1.15 (2), (4), 2.3 (1), and 2.4 (1).

3.7. Remark. Let $R \subseteq G^H$ be a relation, $\mathcal{K}$ an n-decomposition of the set H. Choosing φ =id in 3.6 (4), we obtain

$$R^{(g)}_{\mathcal{K}} = R_{\mathcal{K},\mathrm{id}}.$$

3.8. Theorem. *Let $R \subseteq G^H$ be a relation, $\mathcal{K}$ an n-decomposition of the set H, $\varphi \in S_n$. Then:*

(1) *If R is complete (regular, symmetric, antisymmetric) with regard to $\mathcal{K}$ (and φ), then $R^{(r)}_{\mathcal{K}}$ has the same property.*

(2) *If $n \leq 2$ and R is both transitive and regular with regard to $\mathcal{K}$, then $R^{(r)}_{\mathcal{K}}$ is transitive with regard to $\mathcal{K}$.*

(3) *$R^{(s)}_{\mathcal{K},\varphi}$ is regular with regard to $\mathcal{K}$.*

(4) *If R is reflexive (irreflexive, complete) with regard to $\mathcal{K}$, then $R^{(s)}_{\mathcal{K},\varphi}$ has the same property.*

(5) *If R is reflexive (complete, regular) with regard to $\mathcal{K}$, then $R^{(t)}_{\mathcal{K}}$ has the same property.*

(6) *If R is symmetric with regard to $\mathcal{K}$ and φ and $n \leq 2$ or $\mathcal{K}$ is regular, then $R^{(t)}_{\mathcal{K}}$ is symmetric with regard to $\mathcal{K}$ and φ.*

(7) *If R has any of the properties defined in 2.1, then $R^{(g)}_{\mathcal{K}}$ has the same property.*

Proof. (1) follows from 1.15 (1), (5), 2.3 (1), 2.5 (3), (7), and 3.6 (1).

(2) Let $n \leq 2$ and R be both transitive and regular with regard to $\mathcal{K}$. The case of $n = 1$ is trivial. Let $n = 2$. Let $f \in (R^{(r)}_{\mathcal{K}})^2_{\mathcal{K}} = (R \cup E_{\mathcal{K}})^2_{\mathcal{K}}$. Let $\mathcal{K} = \{K_i\}_{i=1}^3$. Then there exist $f_1, f_2 \in R \cup E_{\mathcal{K}}$ such that $f(K_1) = f_1(K_1)$, $f(K_2) = f_2(K_2)$, $f(K_3) = f_1(K_3) = f_2(K_3)$, $f_1(K_2) = f_2(K_1)$. If $f_1, f_2 \in R$, then $f \in (R\,R)_{\mathcal{K}} = R^2_{\mathcal{K}} \subseteq R \subseteq R^{(r)}_{\mathcal{K}}$. If $f_1, f_2 \in E_{\mathcal{K}}$, then, by 1.15 (1), $f \in (E_{\mathcal{K}}\,E_{\mathcal{K}})_{\mathcal{K}} = E^2_{\mathcal{K}} = E_{\mathcal{K}} \subseteq R^{(r)}_{\mathcal{K}}$. If $f_1 \in R$, $f_2 \in E_{\mathcal{K}}$, then $f(K_1) = f_1(K_1)$, $f(K_2) = f_2(K_2) = f_2(K_1) = f_1(K_2)$, $f(K_3) = f_1(K_3)$. As R is regular with regard to $\mathcal{K}$ and $f_1 \in R$, we have $f \in R \subseteq R^{(r)}_{\mathcal{K}}$. The case of $f_1 \in E_{\mathcal{K}}$, $f_2 \in R$ is analogous. Thus $(R^{(r)}_{\mathcal{K}})^2_{\mathcal{K}} \subseteq R^{(r)}_{\mathcal{K}}$, and $R^{(r)}_{\mathcal{K}}$ is transitive with regard to $\mathcal{K}$.

(3), (4), and (5) follow from 1.5 (2), 1.15 (1), (2), 2.3 (1), 2.4 (2), 2.5 (3), (7), 3.5, and 3.6 (2) and (3).

(6) Let R be symmetric with regard to $\mathcal{K}$ and φ and let $n \leq 2$ or $\mathcal{K}$ be regular. We shall prove by induction that ${}_iR_{\mathcal{K}}$ is symmetric with regard to $\mathcal{K}$ and φ for any $i \in N$. For $i = 1$ it is true, for ${}_1R_{\mathcal{K}} = R$. Let ${}_{i-1}R_{\mathcal{K}}$ be symmetric with regard to $\mathcal{K}$ and φ for some $i \in N$, $i \geq 2$. By 2.6 (2), $({}_{i-1}R_{\mathcal{K}})^2_{\mathcal{K}}$ is symmetric with regard to $\mathcal{K}$ and φ, consequently, by 2.5 (4), ${}_iR_{\mathcal{K}} = {}_{i-1}R_{\mathcal{K}} \cup ({}_{i-1}R_{\mathcal{K}})^2_{\mathcal{K}}$ is symmetric with regard to $\mathcal{K}$ and φ. Thus, again by 2.5(4), $R^{(t)}_{\mathcal{K}} = \bigcup\limits_{i=1}^{\infty} {}_iR_{\mathcal{K}}$ is symmetric with regard to $\mathcal{K}$ and φ as well.

(7) follows from 2.5 (2), (4)-(7), 3.6 (4), and 3.7.

3.9. Corollary. *Let $R \subseteq G^H$ be a relation, $\mathcal{K}$ an n-decomposition of the set H, $\varphi \in S_n$. Then:*

(1) $(R^{(r)}_{\mathcal{K}})^{(s)}_{\mathcal{K},\varphi} = (R^{(s)}_{\mathcal{K},\varphi})^{(r)}_{\mathcal{K}}$.

(2) $(R^{(r)}_{\mathcal{K}})^{(g)}_{\mathcal{K}} = (R^{(g)}_{\mathcal{K}})^{(r)}_{\mathcal{K}}$.

(3) $(R^{(s)}_{\mathcal{K},\varphi})^{(g)}_{\mathcal{K}} = (R^{(g)}_{\mathcal{K}})^{(s)}_{\mathcal{K},\varphi}$.

(4) $(R^{(t)}_{\mathcal{K}})^{(g)}_{\mathcal{K}} = (R^{(g)}_{\mathcal{K}})^{(t)}_{\mathcal{K}}$.

(5) $(R^{(t)}_{\mathcal{K}})^{(r)}_{\mathcal{K}} \subseteq (R^{(r)}_{\mathcal{K}})^{(t)}_{\mathcal{K}}$.

(6) *If $n \leq 2$ or $\mathcal{K}$ is regular, then $(R^{(t)}_{\mathcal{K}})^{(s)}_{\mathcal{K},\varphi} \subseteq (R^{(s)}_{\mathcal{K},\varphi})^{(t)}_{\mathcal{K}}$.*

(7) *If $n \leq 2$ and R is regular, then $(R^{(t)}_{\mathcal{K}})^{(r)}_{\mathcal{K}} = (R^{(r)}_{\mathcal{K}})^{(t)}_{\mathcal{K}}$.*

Proof. As $R \subseteq R^{(s)}_{\mathcal{K},\varphi}$, we have, by 3.3, $R^{(r)}_{\mathcal{K}} \subseteq (R^{(s)}_{\mathcal{K},\varphi})^{(r)}_{\mathcal{K}}$, and again by 3.3, $(R^{(r)}_{\mathcal{K}})^{(s)}_{\mathcal{K},\varphi} \subseteq ((R^{(s)}_{\mathcal{K},\varphi})^{(r)}_{\mathcal{K}})^{(s)}_{\mathcal{K},\varphi}$. By 3.8 (1), $(R^{(s)}_{\mathcal{K},\varphi})^{(r)}_{\mathcal{K}}$ is symmetric with regard to $\mathcal{K}$ and φ, consequently, by 3.2, $((R^{(s)}_{\mathcal{K},\varphi})^{(r)}_{\mathcal{K}})^{(s)}_{\mathcal{K},\varphi} = (R^{(s)}_{\mathcal{K},\varphi})^{(r)}_{\mathcal{K}}$. Thus, $(R^{(r)}_{\mathcal{K}})^{(s)}_{\mathcal{K},\varphi} \subseteq (R^{(s)}_{\mathcal{K},\varphi})^{(r)}_{\mathcal{K}}$. Similarly we can prove the converse inclusion as well as the other inclusions.

3.10. Remark. The inclusions in 3.9 (5) and (6) cannot, in general, be replaced by the equalities (see [13], 3.7).

3.11. Corollary. *Let $R \subseteq G^H$ be a relation, $\mathcal{K}$ an n-decomposition of the set H, $\varphi \in S_n$. Then:*

(1) $(R^{(r)}_{\mathcal{K}})^{(t)}_{\mathcal{K}} = ((R^{(t)}_{\mathcal{K}})^{(r)}_{\mathcal{K}})^{(t)}_{\mathcal{K}}$.

(2) *If $n \leq 2$ or $\mathcal{K}$ is regular, then $(R^{(s)}_{\mathcal{K},\varphi})^{(t)}_{\mathcal{K}} = ((R^{(t)}_{\mathcal{K}})^{(s)}_{\mathcal{K},\varphi})^{(t)}_{\mathcal{K}}$.*

Proof. (1) Similarly as in the proof of 3.9 we get $(R^{(r)}_{\mathcal{K}})^{(t)}_{\mathcal{K}} \subseteq ((R^{(t)}_{\mathcal{K}})^{(r)}_{\mathcal{K}})^{(t)}_{\mathcal{K}}$. By 3.9 (5), $(R^{(t)}_{\mathcal{K}})^{(r)}_{\mathcal{K}} \subseteq (R^{(r)}_{\mathcal{K}})^{(t)}_{\mathcal{K}}$, consequently, by 3.3 and 3.2, $((R^{(t)}_{\mathcal{K}})^{(r)}_{\mathcal{K}})^{(t)}_{\mathcal{K}} \subseteq$ $\subseteq ((R^{(r)}_{\mathcal{K}})^{(t)}_{\mathcal{K}})^{(t)}_{\mathcal{K}} = (R^{(r)}_{\mathcal{K}})^{(t)}_{\mathcal{K}}$. Thus, $(R^{(r)}_{\mathcal{K}})^{(t)}_{\mathcal{K}} = ((R^{(t)}_{\mathcal{K}})^{(r)}_{\mathcal{K}})^{(t)}_{\mathcal{K}}$.

(2) can be proved analogously.

3.12. Theorem. *Let $R \subseteq G^H$ be a relation, $\mathcal{K}$ an n-decomposition of the set H. Then the following relations exist:*

(1) *the cyclic hull $R_{\mathcal{K}}^{(c)}$ of R with regard to $\mathcal{K}$ and we have*

$$R_{\mathcal{K}}^{(c)} = \bigcup_{i=1}^{n} {}^{i}R_{\mathcal{K}},$$

(2) *the symmetric hull $R_{\mathcal{K}}^{(d)}$ of R with regard to $\mathcal{K}$ and we have*

$$R_{\mathcal{K}}^{(d)} = \bigcup_{\varphi \in S_n} R_{\mathcal{K},\varphi}.$$

Proof. (1) As $R_{\mathcal{K}}^{(c)} = R_{\mathcal{K},\pi}^{(s)}$, we have, by 3.6 (2), $R_{\mathcal{K}}^{(c)} = \bigcup\limits_{i=1}^{n} R_{\mathcal{K},\pi}^{i} = \bigcup\limits_{i=1}^{n} {}^{i}R_{\mathcal{K}}$.

(2) Put $Q = \bigcup\limits_{\varphi \in S_n} R_{\mathcal{K},\varphi}$. By 1.5 (1), we have $R \subseteq R_{\mathcal{K},\mathrm{id}} \subseteq \bigcup\limits_{\varphi \in S_n} R_{\mathcal{K},\varphi} = Q$. Let $\psi \in S_n$. By 1.15 (5) and 1.14 (1), $Q_{\mathcal{K},\psi} = (\bigcup\limits_{\varphi \in S_n} R_{\mathcal{K},\varphi})_{\mathcal{K},\psi} = \bigcup\limits_{\varphi \in S_n} (R_{\mathcal{K},\varphi})_{\mathcal{K},\psi} = \bigcup\limits_{\varphi \in S_n} R_{\mathcal{K},\varphi\psi}$. But $R_{\mathcal{K},\varphi\psi} \subseteq \bigcup\limits_{\chi \in S_n} R_{\mathcal{K},\chi}$ for each $\varphi \in S_n$, so that we get $Q_{\mathcal{K},\psi} = \bigcup\limits_{\varphi \in S_n} R_{\mathcal{K},\varphi\psi} \subseteq \bigcup\limits_{\chi \in S_n} R_{\mathcal{K},\chi} = Q$, and Q is symmetric with regard to $\mathcal{K}$ and ψ for any $\psi \in S_n$, thus symmetric with regard to $\mathcal{K}$. Now, let $R \subseteq T$ where T is symmetric with regard to $\mathcal{K}$. Then, by 1.15 (4), $Q = \bigcup\limits_{\varphi \in S_n} R_{\mathcal{K},\varphi} \subseteq \bigcup\limits_{\varphi \in S_n} T_{\mathcal{K},\varphi} \subseteq T$. Hence Q is the symmetric hull of R with regard to $\mathcal{K}$.

3.13. Remark. Let $R \subseteq G^H$ be a relation, $\mathcal{K} = \{K_i\}_{i=1}^{n+1}$ an n-decomposition of the set H. Obviously, then:

$$\begin{aligned} R_{\mathcal{K}}^{(c)} &= \{f \in G^H; \exists k \in (n], g \in R : f(K_i) = g(K_{\pi^k(i)}) \\ &\qquad \text{for all } \ i \in (n], f(K_{n+1}) = g(K_{n+1})\}, \\ R_{\mathcal{K}}^{(d)} &= \{f \in G^H; \exists \varphi \in S_n, g \in R : f(K_i) = g(K_{\varphi(i)}) \\ &\qquad \text{for all } \ i \in (n], f(K_{n+1}) = g(K_{n+1})\}. \end{aligned}$$

3.14. Theorem. *Let $R \subseteq G^H$ be a relation, $\mathcal{K}$ an n-decomposition of the set H, $\varphi \in S_n$. Then:*

(1) *If R is cyclic with regard to $\mathcal{K}$ and $\pi\varphi\pi = \varphi$, then $R_{\mathcal{K},\varphi}^{(s)}$ is cyclic with regard to $\mathcal{K}$.*
(2) *If R has any of the properties defined in 2.8, then $R_{\mathcal{K}}^{(g)}$ has the same property.*
(3) *$R_{\mathcal{K}}^{(c)}$ and $R_{\mathcal{K}}^{(d)}$ are regular with regard to $\mathcal{K}$.*

(4) *If R is reflexive (irreflexive, complete, symmetric) with regard to $\mathcal{K}$, then $R_{\mathcal{K}}^{(c)}$ has the same property.*

(5) *If R is symmetric with regard to $\mathcal{K}$ and φ and $\varphi\pi\varphi = \pi$, then $R_{\mathcal{K}}^{(c)}$ is symmetric with regard to $\mathcal{K}$ and φ.*

(6) *If n is odd and R is asymmetric (antisymmetric) with regard to $\mathcal{K}$, then $R_{\mathcal{K}}^{(c)}$ has the same property.*

(7) *If R is reflexive (irreflexive, complete) with regard to $\mathcal{K}$, then $R_{\mathcal{K}}^{(d)}$ has the same property.*

Proof. (1), (2), (3), and (4) follow from 2.5 (3), 2.13 (1) - (4), 2.14 (1), 3.6 (2), (4), 3.7, 3.12 (1), and (2).

(5) Let R be a symmetric with regard to $\mathcal{K}$ and φ, let $\varphi\pi\varphi = \pi$. Let $f \in ({}^1R_{\mathcal{K}})_{\mathcal{K},\varphi}$. Let $\mathcal{K} = \{K_i\}_{i=1}^{n+1}$. By 1.14 (1), we have $({}^1R_{\mathcal{K}})_{\mathcal{K},\varphi} = R_{\mathcal{K},\pi\varphi}$. Consequently, there exists $g \in R$ such that $f(K_i) = g(K_{(\pi\varphi)(i)})$ for all $i \in (n]$, $f(K_{n+1}) = g(K_{n+1})$. By 2.3 (2), $R_{\mathcal{K},\varphi} = R$, thus $g \in R_{\mathcal{K},\varphi}$. Hence, there exists $h \in R$ such that $g(K_i) = h(K_{\varphi(i)})$ for all $i \in (n]$, $g(K_{n+1}) = h(K_{n+1})$. Summarizing, we get $f(K_i) = h(K_{(\varphi\pi\varphi)(i)}) = h(K_{\pi(i)})$ for all $i \in (n]$, $f(K_{n+1}) = h(K_{n+1})$, thus $f \in R_{\mathcal{K},\pi} = {}^1R_{\mathcal{K}}$. Hence $({}^1R_{\mathcal{K}})_{\mathcal{K},\varphi} \subseteq {}^1R_{\mathcal{K}}$ and ${}^1R_{\mathcal{K}}$ is symmetric with regard to $\mathcal{K}$ and φ. It is easy to show by induction that ${}^iR_{\mathcal{K}}$ is symmetric with regard to $\mathcal{K}$ and φ for any $i \in N$. Now, by 3.12 (1) and 2.5 (4), we obtain that $R_{\mathcal{K}}^{(c)} = \bigcup_{i=1}^{n} {}^1R_{\mathcal{K}}$ is symmetric with regard to $\mathcal{K}$ and φ.

(6) Let R be asymmetric (antisymmetric) with regard to $\mathcal{K}$. Admit that $R_{\mathcal{K}}^{(c)}$ does not have the same property. There exists an odd permutation $\psi \in S_n$ such that $R_{\mathcal{K}}^{(c)} \cap (R_{\mathcal{K}}^{(c)})_{\mathcal{K},\psi} \neq \emptyset$ ($R_{\mathcal{K}}^{(c)} \cap (R_{\mathcal{K}}^{(c)})_{\mathcal{K},\psi} \nsubseteq E_{\mathcal{K}}$). Let $\mathcal{K} = \{K_i\}_{i=1}^{n+1}$. Let $f \in R_{\mathcal{K}}^{(c)} \cap (R_{\mathcal{K}}^{(c)})_{\mathcal{K},\psi}$ ($f \in R_{\mathcal{K}}^{(c)} \cap (R_{\mathcal{K}}^{(c)})_{\mathcal{K},\psi} - E_{\mathcal{K}}$). Then there exists $g \in R_{\mathcal{K}}^{(c)}$ such that $f(K_i) = g(K_{\psi(i)})$ for all $i \in (n]$, $f(K_{n+1}) = g(K_{n+1})$. As $f, g \in R_{\mathcal{K}}^{(c)}$ there exist, by 3.13, $k, l \in (n]$ and $h, m \in R$ such that $f(K_i) = h(K_{\pi^k(i)})$ for all $i \in (n]$, $f(K_{n+1}) = h(K_{n+1}), g(K_i) = m(K_{\pi^l(i)})$ for all $i \in (n]$, $g(K_{n+1}) = m(K_{n+1})$. Since n is odd, π is even and also π^k, π^l are even. As ψ is odd, $\chi = \pi^l\psi(\pi^k)^{-1}$ is odd, too. Thus, we have $h(K_i) = m(K_{(\pi^l\psi(\pi^k)^{-1})(i)}) = m(K_{\chi(i)})$ for all $i \in (n]$, $h(K_{n+1}) = m(K_{n+1})$. Hence $h \in R \cap R_{\mathcal{K},\chi}$ for an odd permutation $\chi \in S_n$. In the case of asymmetry we obtain a contradiction. In the case of antisymmetry we have $f \notin E_{\mathcal{K}}$, thus there exist $i, j \in (n]$ such that $f(K_i) \neq f(K_j)$, so that $h(K_{\pi^k(i)}) = f(K_i) \neq f(K_j) = h(K_{\pi^k(j)})$ and $h \notin E_{\mathcal{K}}$. Hence $h \in R \cap R_{\mathcal{K},\chi} - E_E$, which is a contradiction, too.

(7) follows from 2.5 (1), (4), (7), and 3.12 (2).

3.15. Corollary. *Let $R \subseteq G^H$ be a relation, $\mathcal{K}$ an n-decomposition of the set H, $\varphi \in S_n$. Then:*

(1) $(R_{\mathcal{K}}^{(c)})_{\mathcal{K}}^{(r)} \subseteq (R_{\mathcal{K}}^{(r)})_{\mathcal{K}}^{(c)}$.

(2) $(R_{\mathcal{K}}^{(c)})_{\mathcal{K}}^{(g)} = (R_{\mathcal{K}}^{(g)})_{\mathcal{K}}^{(c)}$.

(3) $(R_{\mathcal{K}}^{(d)})_{\mathcal{K}}^{(r)} \subseteq (R_{\mathcal{K}}^{(r)})_{\mathcal{K}}^{(d)}$.

(4) $(R_{\mathcal{K}}^{(d)})_{\mathcal{K},\varphi}^{(s)} = (R_{\mathcal{K},\varphi}^{(s)})_{\mathcal{K}}^{(d)} = R_{\mathcal{K}}^{(d)}$.
(5) $(R_{\mathcal{K}}^{(d)})_{\mathcal{K}}^{(g)} = (R_{\mathcal{K}}^{(g)})_{\mathcal{K}}^{(d)}$.
(6) $(R_{\mathcal{K}}^{(c)})_{\mathcal{K}}^{(d)} = (R_{\mathcal{K}}^{(d)})_{\mathcal{K}}^{(c)} = R_{\mathcal{K}}^{(d)}$.
(7) *If φ is such that $\varphi\pi\varphi = \pi$, then*

$$(R_{\mathcal{K}}^{(c)})_{\mathcal{K},\varphi}^{(s)} \subseteq (R_{\mathcal{K},\varphi}^{(s)})_{\mathcal{K}}^{(c)}.$$

(8) *If φ is such that $\pi\varphi\pi = \varphi$, then*

$$(R_{\mathcal{K}\varphi}^{(s)})_{\mathcal{K}}^{(c)} \subseteq (R_{\mathcal{K}}^{(c)})_{\mathcal{K},\varphi}^{(s)}.$$

Proof. The statement follows from 3.14 analogously as 3.9 follows from 3.8.

3.16. Corollary. *Let $R \subseteq G^H$ be a relation, $\mathcal{K}$ an n-decomposition of the set H. Then:*

(1) $((R_{\mathcal{K}}^{(c)})_{\mathcal{K}}^{(r)})_{\mathcal{K}}^{(c)} = (R_{\mathcal{K}}^{(r)})_{\mathcal{K}}^{(c)}$.
(2) $((R_{\mathcal{K}}^{(d)})_{\mathcal{K}}^{(r)})_{\mathcal{K}}^{(d)} = (R_{\mathcal{K}}^{(r)})_{\mathcal{K}}^{(d)}$.

Proof is analogous to that of 3.11.

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