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# PSEUDOCOMPLEMENTED ORDERED SETS

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Abstract. The aim of this paper is to transfer the concept of pseudocomplement from lattices to ordered sets and to prove some basic results holding for pseudocomplemented ordered sets.

For some well-known results on pseudocomplemented lattices see e.g. T. Katriňák [1].

Let $(A, \leq)$ be an ordered set $A$ with the order relation $\leq$. Let $B \subseteq A$ be a subset of $A$. The *lower cone* $L(B)$ of the set $B$ is the set:

$$L(B) = \{x \in A : x \leq b \text{ for all } b \in B\};$$

the *upper cone* $U(B)$ is defined analogously:

$$U(B) = \{x \in A : b \leq x \text{ for all } b \in B\}\,.$$

If there is not danger of misunderstanding, we shall write briefly $LU(B)$ (or $UL(B)$) instead of $L(U(B))$ (or $U(L(B))$). If $B = \{b_1, \dots, b_n\}$, we shall write $L(b_1, \dots, b_n)$ instead of $L(B)$ and dually for the upper cone of $B$.

J. Rachůnek and I. Chajda studied in [2] the concept of modular and distributive ordered sets. Recall that an ordered set $(A, \leq)$ is
*modular* if $\forall a, b, c \in A : a \leq c \Rightarrow L(c, U(a, b)) = LU(a, L(b, c))$;
*distributive* if $\forall a, b, c \in A : L(U(a, b), c) = LU(L(a, c), L(b, c))$;
*w-distributive* if $\forall a, b, c \in A : L(U(a, b), c) \subseteq LU(a, L(b, c))$.

**Remark.** It can be shown that every distributive set is $w$-distributive. These two concepts are equivalent in lattices (see [5], §4, Lemma 10).

I. Chajda in [3] introduced and studied the concept of complemented ordered set: an ordered set $(A, \leq)$ is *complemented* if $\forall a \in A\, \exists b \in A : LU(a, b) = A$ and $UL(a, b) = A$. Then the element $b$ is called a *complement* of the element $a$. Every $(w)$- distributive and complemented ordered set is called $(w$-$)$ *boolean*.

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**Definition 1.** An ordered set $(A, \leq)$ with the least element 0 is called *pseudocomplemented* if for every $a \in A$ there exists an element $a^* \in A$ such that $L(a, a^*) = \{0\}$ and if $L(a, x) = \{0\}$ for some $x \in A$, then $x \leq a^*$. The element $a^*$ is called the *pseudocomplement* of the element $a$.

**Remark.** A pseudocomplemented ordered set has the greatest element 1. This element is the pseudocomplement of the element 0.

**Example 1.** (see [3]): The ordered set in Fig. 1 is complemented, pseudocomplemented and boolean, and it is not a lattice.

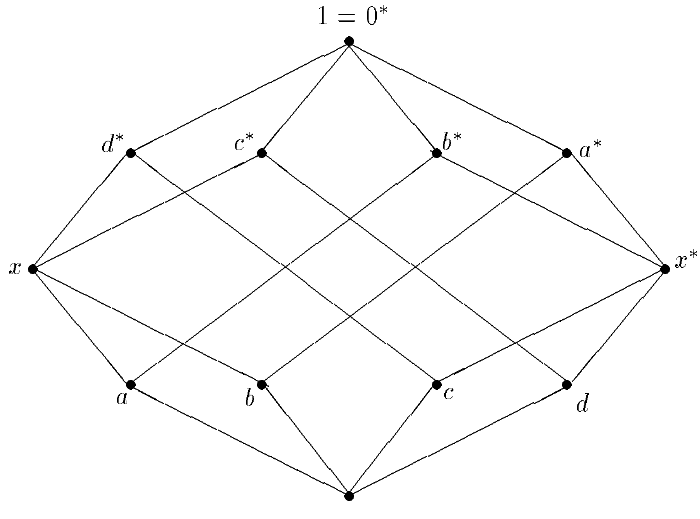


Fig. 1

**Example 2.** The ordered set in Fig. 2 is pseudocomplemented, but not complemented, not distributive and it is not a lattice.

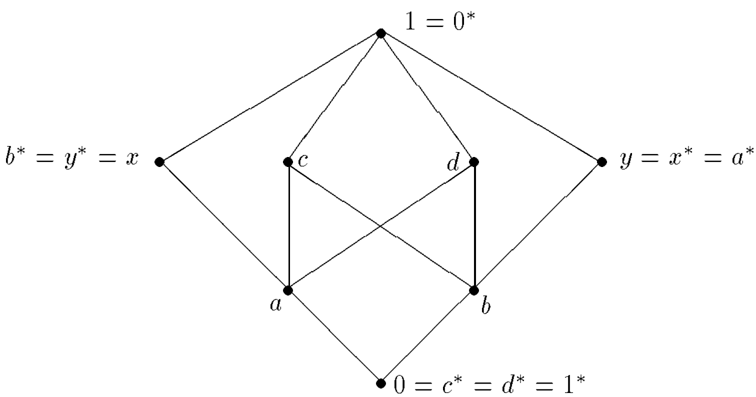


Fig. 2

**Proposition 1.** *Let $(A, \leq)$ be a pseudocomplemented ordered set. Then $\forall\, a, b \in A\colon a \leq b \Rightarrow b^* \leq a^*$, $a^{***} = a^*$.*

**Proof.** Let $a^*$ or $b^*$ be the pseudocomplements of the elements $a$ or $b$, respectively. Then $L(a, a^*) = L(b, b^*) = \{0\}$ and $a \leq b$ implies $L(a) = L(a, b)$. Further, $L(a, b^*) = L(a, b, b^*) = \{0\}$. According to Definition 1, the element $a^*$ is the greatest element with the property $L(a, a^*) = \{0\}$. Hence $b^* \leq a^*$. The proof of the second claim is analogous. $\square$

Let $(A, \leq)$ b a pseudocomplemented ordered set. Denote by $\mathcal{B}(A) = \{x \in A\colon x = x^{**}\}$. For $B \subseteq A$ let $L^*(B) = \{x^*\colon x \in L(B)\}$ and $U^*(B) = \{x^*\colon x \in U(B)\}$ and $B^* = \{x^*\colon x \in B\}$.

**Proposition 2.** $\forall\, a, b \in \mathcal{B}(A)\colon$ $U(a, b) \cap \mathcal{B}(A) = L^*(a^*, b^*)$,
$L(a, b) \cap \mathcal{B}(A) = U^*(a^*, b^*)$.

**Proof.** Let $x \in U(a, b) \cap \mathcal{B}(A)$. Then $x = x^{**}$, $a \leq x$, $b \leq x$. By Proposition 1, $x^* \leq a^*$, $x^* \leq b^*$, hence $x^* \in L(a^*, b^*)$ and $x = x^{**} \in L^*(a^*, b^*)$. Conversely, let $x \in L^*(a^*, b^*)$. Then $x = c^*$, where $c \leq a^*$, $c \leq b^*$. By Proposition 1, $a^{**} \leq c^*$, $b^{**} \leq c^*$, $x = c^* = c^{***} = x^{**}$. Hence $x \in \mathcal{B}(A)$. But $a, b \in \mathcal{B}(A)$, therefore $a \leq c^*$, $b \leq c^*$ and $x \in U(a, b)$. The proof of the second claim is dual. $\square$

**Proposition 3.** *Let $(A, \leq)$ be a pseudocomplemented ordered set. Then the set $\mathcal{B}(A)$ with the induced order is a uniquely complemented ordered set.*

**Proof.** First we prove that for every $a \in \mathcal{B}(A)$ the element $a^*$ is a complement of the element $a$:
$L(a, a^*) = \{0\}$, hence $UL(a, a^*) = U(0) = \mathcal{B}(A)$; according to Proposition 2, $LU(a, a^*) = LL^*(a^*, a^{**}) = L(0^*) = L(1) = \mathcal{B}(A))$. Hence $a^*$ is a complement of the element $a$ in the set $\mathcal{B}(A)$. By the definition of pseudocomplement, $a^*$ is the unique complement of $a$. $\square$

The proof of the next statement is derived from the following lemma:

**Lemma 1.** *Let $A$ be a pseudocomplemented ordered set, $x, y, z \in A$; $B$, $C \subseteq A$. Then the statements* (i), (ii), (v), (vii), (viii), (ix) *are valid in $A$ and* (iii), (iv), (vi) *are valid in $\mathcal{B}(A)$:*

(i) $L(x, z) \subseteq LU(x, L(y, z))$;
(ii) $L(x, y, U(x^*, y^*)) = \{0\}$;
(iii) $U^*(B) = L(B^*)$ *and* $L^*(B) = U(B^*)$;
(iv) $L(x, z, UU^*(x, L(y, z))) = \{0\}$;
(v) $L(y, z) \subseteq LU(x, L(y, z))$;
(vi) $L(y, z, UU^*(x, L(y, z))) = \{0\}$;
(vii) $L(x, B) = \{0\} \Rightarrow L(B) \subseteq L(x^*)$;
(viii) $L(B, C) = \{0\} \Rightarrow L(B) \subseteq LL^*(C)$.

**Proof.** (i) $U(x, L(y, z)) = U(x) \cap UL(y, z) \subseteq U(x)$, hence $LU(x, L(y, z)) \supseteq LU(x) = L(x) \supseteq L(x, z)$.

(ii) Let $y \in L(x, z, U(x^*, z^*))$. Then $y \leq x$, $y \leq z$, $y \in LU(x^*, z^*)$. Hence $y \leq k$ for every $k \in A$ such that $x^* \leq k$, $z^* \leq k$. According to Proposition 1 it holds $x^* \leq y^*$, $z^* \leq y^*$ and therefore $y \leq y^*$. Then $L(y, y^*) = L(y) = \{0\}$, hence $y = 0$.

(iii) By Proposition 2, we have

$$\forall\, a \in B : U^*(a, a) = U^*(a) = L(a^*, a^*) = L(a^*)\,.$$

But $U^*(B) = (\bigcap_i U(b_i))^* = (\bigcap_i U^*(b_i)) = \bigcap_i L(b_i^*) = L(\bigcup_i \{b_i^*\}) = L(B^*)$.
The proof of the second claim is dual.

(iv) By (i), $LU(x, L(y,z)) \supseteq L(x,z)$, hence $U(x, L(y,z)) \subseteq UL(x,z)$; then $U^*(x, L(y,z)) \subseteq U^*L(x,z)$. This implies the inclusion $UU^*(x, L(y,z)) \supseteq UU^*L(x,z)$ and then $LUU^*(x, L(y,z)) \subseteq LUU^*L(x,z)$. The right side of the last inclusion is equal to $LUU^*L(x,z) = LULL^*(x,z) = LL^*(x,z)$ according to (iii), and by Proposition 2 the set $LL^*(x,z)$ is equal to $LU(x^*, z^*)$. Hence, $L(x, z, UU^*(x, L(y,z)) \subseteq L(x,z) \cap LU(x^*, z^*) = L(x, z, U(x^*, z^*)) = \{0\}$ by (ii).

(v) $U(x, L(y,z)) = U(x) \cap UL(y,z) \subseteq UL(y,z)$, hence $LU(x, L(y,z)) \supseteq LUL(y,z) = L(y,z)$.

(vi) By (v) it holds: $L(y,z) \subseteq LU(x, L(y,z))$, hence $UL(y,z) \supseteq U(x, L(y,z))$; then $U^*L(y,z) \supseteq U^*(x, L(y,z))$ and therefore $UU^*L(y,z) \subseteq UU^*(x, L(y,z))$; it implies $LUU^*L(y,z) \supseteq LUU^*(x, L(y,z))$. But the left side of the last inclusion is equal to $LU(y^*, z^*)$ for the same reason as in the proof of (iv). Therefore $L(y, z, UU^*(x, L(y,z))) \subseteq L(y, z, U(y^*, z^*)) = \{0\}$.

(vii) If $y \in L(B)$, then $y \leq b$ for all $b \in B$. Hence $L(y) \subseteq L(B)$. Then $L(x,y) = L(x) \cap L(y) \subseteq L(x) \cap L(B) = L(x, B) = \{0\}$, therefore $L(x,y) = \{0\}$. But according to the definition of pseudocomplement $y \leq x^*$ and thus $y \in L(x^*)$.

(viii) Let $y \in L(C)$. Then $L(y) \subseteq L(C)$ and therefore $L(y, B) = L(y) \cap L(B) \subseteq L(B, C) = \{0\}$. According to (vii) $L(B) \subseteq L(y^*)$ for all $y \in L(C)$. This implies $L(B) \subseteq \bigcap_i L(y_i^*)$, where $y_i \in L(C)$. The right side of the last inclusion is equal to $L(\bigcup_i \{y_i^*\}) = LL^*(C)$.

Further, let $x, y, z \in \mathcal{B}(A)$ and lower and upper cones be in $\mathcal{B}(A)$.

Now, by (iv) and (vii) we obtain

$$L(z, UU^*(x, L(y,z))) \subseteq L(x^*)\,, \tag{1}$$

and by (vi) and (vii) we have

$$L(z, UU^*(x, L(y,z))) \subseteq L(y^*)\,. \tag{2}$$

From (1) and (2) it is clear, that $L(z, UU^*(x, L(y,z))) \subseteq L(x^*, y^*)$. This implies $L(z, UU^*(x, L(y,z)), L^*(x^*, y^*)) \subseteq L(x^*, y^*) \cap LL^*(x^*, y^*) = L(x^*, y^*) \cap LU(x,y) = L(x^*, y^*, U(x,y)) = \{0\}$, hence

$$L(z, UU^*(x, L(y,z)), L(x^*, y^*)) = \{0\}\,. \tag{3}$$

Further, we use (viii), where we put $B = \{z\} \cup L^*(x^*, y^*)$, $C = UU^*(x, L(y, z))$. We obtain inclusion $L(z, L^*(x^*, y^*)) \subseteq LL^*UU^*(x, L(y, z))$. Using (iii) and Proposition 2, we obtain $L(z, U(x, y)) = L(z, L^*(x^*, y^*)) \subseteq LL^*UU^*(x, L(y, z)) = LUU^*U^*(x, L(y, z)) = LULU(x, L(y, z)) = LU(x, L(y, z))$, hence

$$L(z, U(x, y)) \subseteq LU(x, L(y, z)) \tag{4}$$

so $\mathcal{B}(A)$ is a $w$-distributive ordered set.
By Proposition 3, $\mathcal{B}(A)$ is uniquely complemented, hence the following Theorem holds:

**Theorem 1.** *The set $\mathcal{B}(A)$ with the induced order is a $w$-boolean ordered set.*

**Definition 2.** A pseudocomplemented distributive ordered set $A$ is called a *Stone ordered set* if it satisfies the condition:

$$\forall\, a \in A : U(a^*, a^{**}) = \{1\}\,.$$

**Example 3.** The set visualized in Fig. 3 is a pseudocomplemented ordered set which is not a lattice and not a Stone ordered set, but it is a distributive ordered set.

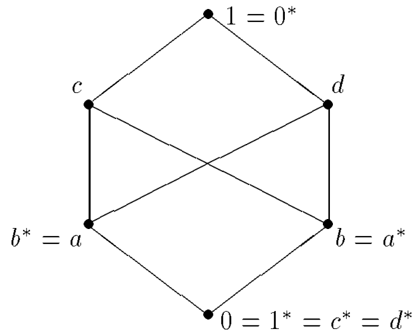


Fig. 3

**Example 4.** The ordered set in Fig. 4 is a Stone ordered set, which is not a lattice.

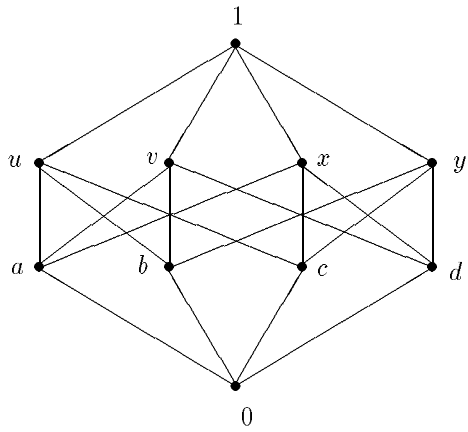


Fig. 4

**Theorem 2.** *Let $A$ be a pseudocomplemented distributive ordered set. Then the following conditions are equivalent:*
(i) *$A$ is a Stone ordered set*
(ii) *If $x, a, b \in A$, then $L(a, b, x) = \{0\}$ implies $L(x) \subseteq LU(a^*, b^*)$.*

**Proof.** (ii)$\Rightarrow$(i) Let $b = a^*$ in the condition (ii). Then $L(a, a^*, x) = \{0\}$ for all $x \in A$. Hence $L(x) \subseteq LU(a^*, a^{**})$ for arbitrary $x \in A$. Let $x = 1$. Then $A = L(1) \subseteq LU(a^*, a^{**})$. This implies $A = LU(a^*, a^{**})$ and $\{1\} = U(A) = U(a^*, a^{**})$.

(i)$\Rightarrow$(ii) Let $x \in A$, $L(a, b, x) = \{0\}$. according to (vii), it is valid: $L(b, x) \subseteq L(a^*)$. Then $L(b, x, a^{**}) \subseteq L(a^*, a^{**}) = \{0\}$ and hence $L(b, x, a^{**}) = \{0\}$. Now, using (vii) again, we'll obtain: $L(x, a^{**}) \subseteq L(b^*)$. Then $L(x) = L(x) \cap A = L(x) \cap LU(a^*, a^{**}) = L(x, U(a^*, a^{**}))$. By distributivity of $A$, $L(x, U(a^*, a^{**})) = LU(L(x, a^*), L(x, a^{**}))$ and $L(x, a^*) \subseteq L(a^*)$, $L(x, a^{**}) \subseteq L(b^*)$. Therefore

$$LU(L(x, a^*), L(x, a^{**})) \subseteq LU(L(a^*), L(b^*)) = LU(a^*, b^*),$$

hence $L(x) \subseteq LU(a^*, b^*)$. $\square$

**Definition 3.** Let $A$ be an ordered set. The set $B \subseteq A$ is called a *filter* in $A$ if the following condition holds:

$$a, b \in B \Rightarrow UL(a, b) \subseteq B\,.$$

Let $A$ be a Stone ordered set. Denote by $\mathcal{D}(A) = \{a \in A; a^* = 0\}$.

**Proposition 4.** *The set $\mathcal{D}(A)$ is a filter in a Stone ordered set $A$.*

**Proof.** If $a, b \in \mathcal{D}(A)$, then $a^* = b^* = 0$. Now, let $z \in UL(a, b)$ for some element $z \in A$; then we can prove:
$L(z) \supseteq L(a, b)$, thus $\{0\} = L(z, z^*) \supseteq L(a, b, z^*)$ and $L(a, b, z^*) = \{0\}$. By Theorem 2, $L(z^*) \subseteq LU(a^*, b^*) = L(0) = 0$, thus $z^* = 0$. $\square$

**Proposition 5.** *Let $A$ be a Stone ordered set. Then*
(i) $\forall a \in A : L(a^{**}, U(a, a^*)) = L(a)$;
(ii) $\forall a \in A : U(a, a^*) \subseteq \mathcal{D}(A)$.

**Proof.** (i) Using distributivity of the set $A$, we get:
$L(a^{**}, U(a, a^*)) = LU(L(a^{**}, a), L(a^{**}, a^*)) = L(a^{**}, a)$, but $a \leq a^{**}$ for all $a \in A$ and $L(a, a^{**}) = L(a)$.
(ii) If $z \in U(a, a^*)$, then $a \leq z$. Due to Proposition 1 it is valid: $z^* \leq a^*$, $z^* \leq a^{**}$ and $L(z^*) \subseteq L(a^*, a^{**}) = \{0\}$, thus $z^* = 0$. □

**Example 5.** Let's consider an ordered set visualized in Fig. 5. It is a Stone ordered set and it is not a lattice. The set $\mathcal{B}(A)$ of this ordered set is equal to
$\mathcal{B}(A) = \{0, 1, a, b, c, d, x, x^*, a^*, b^*, c^*, d^*\}, \mathcal{D}(A) = \{1, t\}$. For $A$ in Example 4, we have $\mathcal{B}(A) = A$ and $\mathcal{D}(A) = \{1\}$.

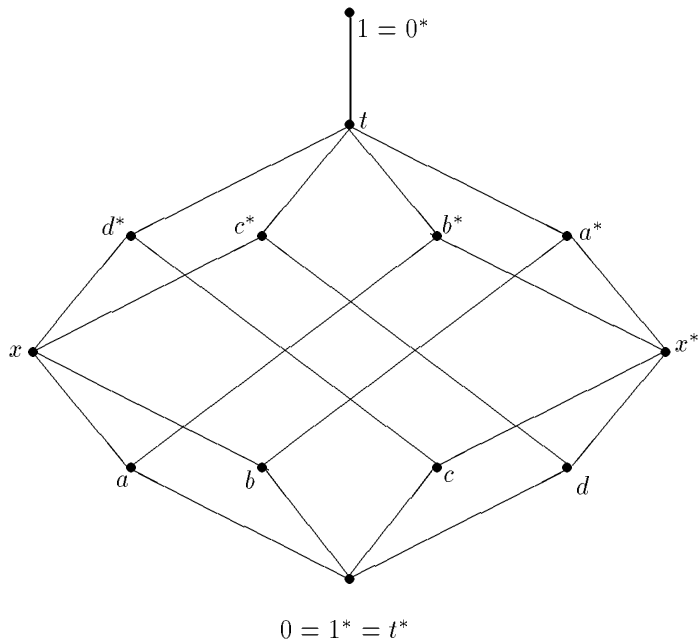


Fig. 5

Therefore, the lower cone of an arbitrary element $a$ of a Stone ordered set is possible to express as a lower cone of the set $B$, which is the union of some element of $\mathcal{B}(A)$ and some subset of $\mathcal{D}(A)$.

**Definition 4.** An ordered set $A$ is called a *Brouwer* ordered set if for all elements $a, b \in A$ there exists the greatest element $b : a \in A$ satisfying the condition $L(a, x) \subseteq L(b)$, $x \in A$.

**Proposition 6.** *Every Brouwer ordered set $A$ is distributive.*

**Proof.** Let $D = U(L(a,b), L(a,c)), a, b, c \in A$, $d \in D$. Then $x \leq d$, $y \leq d$ for all elements $x \in L(a,b)$, $y \in L(a,c)$, hence $L(d) \supseteq L(a,b)$, $L(d) \supseteq L(a,c)$ and $b \leq d : a$, $c \leq d : a$, hence $U(b,c) \supseteq U(d : a)$. Then $L(a, U(b,c)) \subseteq L(a, U(d : a)) = L(a, (d : a)) \subseteq L(d)$. But $d$ is an arbitrary element from $D$, hence it is valid: $L(a, U(b,c)) \subseteq \bigcap_i L(d) = L(D) = LU(L(a,b), L(a,c))$. But the opposite inclusion is always valid in $A$ and $A$ is distributive. □

**Remark.** An ordered set depicted in Fig. 3 is a Brouwer set which is not lattice.

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