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COUNTABLY EVALUATING HOMOMORPHISMS ON REAL FUNCTION ALGEBRAS

Eva Adam, Peter Biström, Andreas Kriegl

Abstract. By studying algebra homomorphisms, which act as point evaluations on each countable subset, we obtain improved results on the question when all algebra homomorphisms are point evaluations.

Introduction

The connecting thought of this article is the approach of identifying the real valued homomorphisms on algebras of functions as point evaluations through a study of their evaluation properties on countable sets in the algebra. The interplay between realcompactness and countably evaluating homomorphisms is elucidated: We show that if every countably evaluating homomorphisms is a point evaluation then the underlying space is realcompact with respect to the topology induced by the algebra. Furthermore, by modifying a result due to Corson about the weak topology on a Banach space, we prove the converse under certain denseness conditions. As direct output we obtain realcompactifications with respect to general algebras, a result that, combined with some extension properties of homomorphisms, yields a new characterization of weak realcompactness of locally convex spaces.

The approach not only enables us to reprove the main result in [GGJ] that for a locally convex space E which is injectable into $\ell^{2n}(\Gamma)$ the homomorphisms on inverse-closed algebras containing the polynomials $P(E)$ are all point evaluations but also to improve the central result in [BL$_1$] that this is equally true for the algebra of smooth functions on spaces admitting an injection into some realcompact space with C^∞-partitions of unity. We present two substantial improvements of this result.

Firstly; we prove that countably evaluating homomorphisms commute with locally finite sums provided that the cardinality of the summation index is non-measurable. Not only do we show that this property can be used for obtaining

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a series of fundamental results concerning non-measurable cardinals and realcompactness, but also that the assumption of C^∞-partition of unity can be relaxed to the existence of certain bump functions in order to guarantee point evaluating properties. In fact, we prove that the crucial spaces $c_0(\Gamma)$, and, more generally, $C(K)$ with $K^{(\omega_0)} = \emptyset$, admit bump functions that are locally finite sums of a form we can, by means of the general theorem, evaluate and produce the required results.

Secondly; the technique of pulling back evaluation results along a linear injection is now considerably improved: The injection can be replaced by an operator whose kernel has weak*-separable dual provided that the range space satisfies some mild separating property. The kernel may even be large, in fact injectability into some $c_0(\Gamma)$ is sufficient, if the operator is a quotient map. Indeed, we prove among other results the following "three space property" in this context; Given a Banach space F with a subspace H that is injectable into $C(K)$, and where F/H is realcompact and admits C^∞-smooth partitions of unity. Then, if $|K|$ is non-measurable and $K^{(\omega_0)} = \emptyset$, we have that all algebra homomorphisms on $C^\infty(F)$ are point evaluations. This conclusion even extends to spaces E with an injection (or an operator with separable kernel) into F.

In section 1 we collect some basic notations. Section 2 concerns the equivariance of homomorphisms with respect to the action of smooth functions. In section 3 the equivariance with respect to infinite sums and products is treated. The 4-th section characterizes evaluation properties of homomorphisms in terms of extension properties. In section 5 smallest reasonable algebras of smooth functions are investigated. Inheritance properties for short exact sequences are treated in section 6. And in the last section a large stable class of locally convex spaces is introduced, for which all countably evaluating homomorphisms on sufficiently large algebras are point evaluations.

1. Notation and terminology

Sets and topological spaces in general will be denoted by $X, Y, \dots$, locally convex spaces by $E, F, G, \dots$, algebras by $\mathcal{A}$ or by $\mathcal{A}_X$ to emphasize the underlying space X if $\mathcal{A} \subseteq \mathbb{R}^X$, elements in algebras by $f, g, h, \dots$, and smooth functions on finite dimensional spaces by greek letters. For a real separated locally convex topological vector space we use the abbreviation lcs.

The cardinal of a set Y is denoted by $|Y|$, the cardinal of $\mathbb{N} = \{1, 2, \dots\}$ by $\aleph_0$, and the first uncountable cardinal by $\aleph_1$. The cardinal of a set Y is *measurable* if there is a non-trivial two-valued countably additive measure on 2^Y that vanishes on point sets. It is consistent with set theory to assume that measurable cardinals do not exist.

Instead of $C(X, \mathbb{R})$ for the continuous functions on a topological space X we prefer to write $C(X)$. By $C_b(X)$ we mean the bounded functions in $C(X)$. The set of all C^∞-functions $\eta : \mathbb{R}^n \to \mathbb{R}$ is denoted by $C^\infty(\mathbb{R}^n)$. If $\mathcal{A} \subseteq \mathbb{R}^X$ is a set, we denote by $X_\mathcal{A}$ the set X endowed with the initial topology with respect to $\mathcal{A}$ and hence $\mathcal{A}$ can be considered as subset of $C(X_\mathcal{A})$. Given a function $f : X \to \mathbb{R}$, its *zeroset* $Z(f)$, *carrier set* $\operatorname{carr} f$ and *support* $\operatorname{supp} f$ are defined as $\{x \in X : f(x) = 0\}$, $X \setminus Z(f)$ and $\overline{\operatorname{carr} f}$ respectively.

The space of all linear functionals on a vector space E is denoted by E^*, whereas E' stands for the space of all continuous linear functionals on a locally convex space E.

All algebras $\mathcal{A}$ appearing in this paper are algebras of real valued functions (i.e. are subalgebras of $\mathbb{R}^X$ for some set X) with unit. Given a set $Y \subseteq \mathbb{R}^X$, we denote by $\langle Y \rangle_{alg}$ the smallest subalgebra of $\mathbb{R}^X$ that contains Y. However, if E is a lcs, we prefer to write $P_f(E)$ instead of $\langle E' \rangle_{alg}$ for the algebra of all finite type polynomials on E.

An algebra $\mathcal{A} \subseteq \mathbb{R}^X$ is a C^∞-algebra if $\eta \circ (f_1, ..., f_n) \in \mathcal{A}$ whenever $\eta \in C^\infty(\mathbb{R}^n)$ for any n and $f_1, ..., f_n \in \mathcal{A}$. For the case, where only $\eta \circ f \in \mathcal{A}$ whenever $\eta \in C^\infty(\mathbb{R})$ and $f \in \mathcal{A}$, we say that $\mathcal{A}$ is a $C^{(\infty)}$-algebra.

Definition. A C^∞-algebra (respectively $C^{(\infty)}$-algebra) is said to be a C^∞_{lfs}-algebra (respectively $C^{(\infty)}_{lfs}$-algebra), if it is closed under formation of locally finite sums over index sets of non-measurable cardinality. Similarly it is called a C^∞_{lfcs}-algebra (respectively $C^{(\infty)}_{lfcs}$-algebra), if it is closed under formation of locally finite countable sums.

The notion of C^∞_{lfs}-algebra clearly generalizes that of a C^∞-*local* algebra in [BL$_1$], where $\mathcal{A} \subseteq \mathbb{R}^X$ is called C^∞-local if it is closed under composition with functions in $C^\infty(\mathbb{R})$ and has the local property, i.e. $f \in \mathcal{A}$ if and only if f agrees locally around every point in X with a function in the algebra. Each C^∞-local algebra $\mathcal{A} \subseteq \mathbb{R}^X$ is obviously *inverse-closed*, i.e. $1/f \in \mathcal{A}$ whenever $f \in \mathcal{A}$ and $0 \notin f(X)$, a property also shared by the $C^{(\infty)}_{lfcs}$-algebras according to (2.4).

By a homomorphism on an algebra $\mathcal{A}$, we mean a multiplicative linear mapping preserving the unit and by $\operatorname{Hom} \mathcal{A}$ we mean the set of all real valued homomorphisms on $\mathcal{A}$.

Definition. Given a subset $\mathcal{B} \subseteq \mathbb{R}^X$ and a cardinal number $\boldsymbol{m}$ we say that a mapping $\phi : \mathcal{B} \to \mathbb{R}$ is $\boldsymbol{m}$-*evaluating* if for each set $B \subseteq \mathcal{B}$ with $|B| \leq \boldsymbol{m}$ there is a point $x \in X$ such that $\phi(f) = f(x) =: \delta_x(f)$ for all $f \in B$. The set of $\boldsymbol{m}$-evaluating homomorphisms on $\mathcal{A}$ is denoted by $\operatorname{Hom}_{\boldsymbol{m}} \mathcal{A}$. An algebra $\mathcal{A}$ is said to be $\boldsymbol{m}$-*evaluating* if each $\varphi \in \operatorname{Hom} \mathcal{A}$ is $\boldsymbol{m}$-*evaluating*, or equivalently, if $\operatorname{Hom}_{\boldsymbol{m}} \mathcal{A} = \operatorname{Hom} \mathcal{A}$. Instead of $\aleph_0$-evaluating we say countably evaluating.

Each inverse-closed algebra is 1-evaluating and every 1-evaluating algebra is n-evaluating for all $n \in \mathbb{N}$, by [BJL]. There are, however, 1-evaluating algebras that fail to be countably evaluating, e.g. the set $P_f(E)$ of finite type polynomials on any infinite dimensional real Banach space E, as we will show in (5.3). Hence the rational forms of the elements in $P_f(E)$ form even an inverse-closed algebra that fails to be countably evaluating, since homomorphisms on $P_f(E)$ extend to the rational forms by (4.2). On each infinite dimensional Banach space E, the algebra $P(E)$ of all continuous polynomials on E fails to be 1-evaluating [BJL].

For the case that each $\varphi \in \operatorname{Hom}\mathcal{A}$ is $|\mathcal{A}|$-evaluating, i.e. there is some $x \in X$ such that $\varphi = \delta_x$ on $\mathcal{A} \subseteq \mathbb{R}^X$, we write $X = \operatorname{Hom}\mathcal{A}$. Given a completely regular topological space X, then $X = \operatorname{Hom} C(X)$ is fulfilled precisely when X is realcompact [GJ], where a topological space is realcompact if it is a closed subspace of a power of the reals [En]. A topological space is Lindelöf if it is separated and each family of closed sets in the space with the countable intersection property has a nonempty intersection.

In the few places, where we mention smooth functions on infinite dimensional spaces, they are meant in the sense of [FK]. For Banach spaces these are exactly the Fréchet smooth maps, and for Fréchet spaces they coincide with almost all existing notions.

2. Basic evaluating properties of homomorphisms

Algebras as $C(X)$ for any topological space X, or more generally the C^∞-local algebras on such an X (see [BL$_1$]), are countably evaluating. And we start by observing that this is the best evaluating property one can hope for in general. In fact, we have the following:

2.1 Proposition. *Given two infinite cardinals $\boldsymbol{m} < \boldsymbol{n}$, there is a dense subspace E of $\mathbb{R}^{\boldsymbol{n}}$ such that for any algebra $\mathcal{A} \subset C(E)$, containing the natural projections $(\mathrm{pr}_\gamma)_{\gamma\in\boldsymbol{n}}$, there is a homomorphism φ on $\mathcal{A}$ that is $\boldsymbol{m}$-evaluating but not $\boldsymbol{n}$-evaluating.*

Proof. Set $E := \{x \in \mathbb{R}^{\boldsymbol{n}} :\ |\operatorname{supp} x| \le \boldsymbol{m}\}$ and take an algebra $\mathcal{A} \subset C(E)$ with $\mathrm{pr}_\gamma \in \mathcal{A}$ for all $\gamma \in \boldsymbol{n}$. By [En, p.118], each $f \in C(E)$ extends to $\tilde{f} \in C(\mathbb{R}^{\boldsymbol{n}})$. Since E is dense in $\mathbb{R}^{\boldsymbol{n}}$, this extension is unique and therefore each $x \in \mathbb{R}^{\boldsymbol{n}}$ gives a homomorphism ψ_x on $\mathcal{A}$ by means of the map $\psi_x(f) = \tilde{f}(x)$. Take $a \in \mathbb{R}^{\boldsymbol{n}} \setminus E$. Then ψ_a is not $\boldsymbol{n}$-evaluating, since otherwise it would evaluate the point separating family $(\mathrm{pr}_\gamma)_{\gamma\in\boldsymbol{n}}$ in a unique point $b \in E$. Since some projection pr_γ separates a from b, we have a contradiction. On the other hand, take a set $B \subseteq \mathcal{A}$ with $|B| \le \boldsymbol{m}$. According to [En, p.118], each $f \in \mathcal{A}$ depends on a countable number of coordinates $N_f \subset \boldsymbol{n}$. Set $N := \bigcup_{f\in B} N_f$. Then $|N| \le \boldsymbol{m}$. Set $\hat{a}_\gamma := a_\gamma$ if $\gamma \in N$ and zero otherwise. Then $\hat{a} := (\hat{a}_\gamma)_{\gamma\in\boldsymbol{n}} \in E$ and $f(\hat{a}) = \tilde{f}(a)$ for all $f \in B$. Thus ψ_a is $\boldsymbol{m}$-evaluating. $\square$

2.2 Proposition. *Let $\mathcal{A} \subseteq \mathbb{R}^X$ be an algebra. Take $f_1, \dots, f_n \in \mathcal{A}$ and $\varphi \in \operatorname{Hom}\mathcal{A}$. If $\eta \circ (f_1, \dots, f_n) \in \mathcal{A}$ for some $\eta \in C^\infty(\mathbb{R}^n)$, then*

$$(*) \qquad \varphi(\eta \circ (f_1, \dots, f_n)) = \eta(\varphi(f_1), \dots, \varphi(f_n))$$

provided that the map $\eta_j^a \circ (f_1, \dots, f_n) \in \mathcal{A}$ for all $j \le n$ and all $a \in \mathbb{R}^n$, where $\eta_j^a(x) := \int_0^1 \partial_j \eta(a + t(x - a))dt$.

Moreover, if $\mathcal{A}$ separates the points of X and $\varphi(f) \in \overline{f(X)}$ for all $f \in \mathcal{A}$, then $(*)$ holds for all $\eta \in C(\mathbb{R}^n)$ such that $\eta \circ (f_1, \dots, f_n) \in \mathcal{A}$.

Proof. Let $\eta \in C^\infty(\mathbb{R}^n)$ and $f_1, \dots, f_n \in \mathcal{A}$ such that $\eta \circ (f_1, \dots, f_n) \in \mathcal{A}$. Set $a := (\varphi(f_1), \dots, \varphi(f_n)) \in \mathbb{R}^n$. Then $\eta(x) - \eta(a) = \int_0^1 \sum_{j\le n} \partial_j \eta(a + t(x - a))\ dt \cdot$

$(x_j - a_j) = \sum_{j\le n} \eta_j^a(x)\cdot(x_j - a_j)$. Applying φ to this equation composed with the f_i one obtains $\varphi(\eta\circ(f_1,\dots,f_n)) - \eta(\varphi(f_1),\dots,\varphi(f_n)) = \sum_{j\le n} \varphi(\eta_j^a\circ(f_1,\dots,f_n))\cdot(\varphi(f_j) - \varphi(f_j)) = 0$.

Concerning the second statement, let $X_{\mathcal{A}}$ be the set X endowed with the initial topology with respect to $\mathcal{A}$. Then $X_{\mathcal{A}}$ is completely regular, $\mathcal{A} \subseteq C(X_{\mathcal{A}})$ and $\varphi(f) \in \overline{f(X)}$ for all $f \in \mathcal{A}$. Hence there is a point z in the Stone-Čech compactification $\beta(X_{\mathcal{A}})$ such that $\varphi(f) = \hat{f}(z)$ for all $f \in \mathcal{A}$ [BBL]. Here $\hat{f} : \beta X \to \mathbb{R}_\infty$ denotes the continuous extension of f considered as a map into the one-point compactification $\mathbb{R}_\infty$ of $\mathbb{R}$. The statement then follows by observing that

$$(\eta\circ(f_1,\dots,f_n))\widehat{\ }(z) = (\eta\circ(\widehat{f_1},\dots,\widehat{f_n}))(z). \qquad \square$$

Remark. By means of (2.2), for C^∞-algebras $\mathcal{A}$, each $\varphi \in \operatorname{Hom}\mathcal{A}$ commutes with the action of smooth functions. But also for the algebra $\mathcal{AE}(E)$ of globally convergent power series on a Banach space E it follows that $\varphi(\eta\circ f) = \eta(\varphi(f))$ for any $\varphi \in \operatorname{Hom}\mathcal{AE}(E)$, $f \in \mathcal{AE}(E)$ and $\eta \in \mathcal{AE}(\mathbb{R})$, since then $\eta_j^a \in \mathcal{AE}(\mathbb{R})$.

2.3 Proposition. *Any algebra that is stable under formation of locally finite sums is also stable under formation of locally finite products of the same cardinality, the converse holds for $C^{(\infty)}$-algebras and if the cardinality is countable also for algebras closed under bounded inversion.*

Recall that an algebra $\mathcal{A}$ is said to be closed under bounded inversion if $\frac{1}{f} \in \mathcal{A}$ for each $f \in \mathcal{A}$ with $|f| \ge \varepsilon$ for some $\varepsilon > 0$.

Proof. Since any algebra is automatically stable with respect to finite sums and products, we may assume that the cardinality of the index set is at least countable.

Let $\mathcal{A}$ be an algebra and $(f_i)_{i\in I}$ a family in $\mathcal{A}$ such that $f := \prod_{i\in I} f_i$ is locally finite, i.e. locally only finitely many factors f_i are unequal to 1. Let $g_i := f_i - 1$ and for finite subsets $J \subseteq I$ let $g_J := \prod_{j\in J} g_j \in \mathcal{A}$. Then $f = \sum_J g_J$, a locally finite sum, where the index J runs through all finite subsets of I. Since I is at least countable, the set of these indices has the same cardinality as I has.

Conversely, let $f = \sum_{i\in I} f_i$ be a locally finite sum of terms f_i in a $C^{(\infty)}$-algebra $\mathcal{A}$. Then $\exp\circ(\pm f) = \prod_{i\in I} \exp\circ(\pm f_i)$ is a locally finite product with $\exp\circ(\pm f_i) \in \mathcal{A}$. Hence, by assumption $\exp\circ(\pm f) \in \mathcal{A}$ and thus $\sinh\circ f = \frac{\exp\circ(+f) - \exp\circ(-f)}{2} \in \mathcal{A}$ and finally $f = \operatorname{Arsinh}\circ\sinh\circ f \in \mathcal{A}$.

Let $\sum_{k\in\mathbb{N}} f_k$ be a locally finite sum in an algebra closed under bounded inversion. We may assume that $1 + \sum_{k=1}^n f_k \ge \frac{7}{8}$ for all $n \in \mathbb{N}$: For this we use the identity

$$4\sum_k f_k = \sum_k\Big((3^k f_k + 3^{-k})^2 - 3^{-2k}\Big) - \sum_k\Big((3^k f_k - 3^{-k})^2 - 3^{-2k}\Big).$$

Then the partial sums of both sums on the right side are bounded from below by $-\sum_k 3^{-2k} = -\frac{1}{8}$.

Now let $g_k := \frac{f_k}{1+\sum_{j<k} f_j}$. Then $g_k \in \mathcal{A}$ and $1 + \sum_{j\le k} f_j = \prod_{j\le k}(1 + g_j)$ and the infinite product is locally finite. $\square$

2.4 Lemma. *Let $\mathcal{A} \subseteq \mathbb{R}^X$ be a subalgebra considered as a subspace of $C(X_\mathcal{A})$. If $\mathcal{A}$ is a $C^{(\infty)}_{lfcs}$-algebra, it is inverse-closed. If $\mathcal{A}$ is a C^∞_{lfcs}-algebra we have more generally, that $\eta \circ (f_1, \dots, f_n) \in \mathcal{A}$ provided $\eta \in C^\infty(U)$, where $U \subseteq \mathbb{R}^n$ is open and $f_i \in \mathcal{A}$ has values in U.*

Proof. For the second statement consider a smooth partition of unity $\{\eta_k : k \in \mathbb{N}\}$ of U, such that $\operatorname{supp} \eta_k \subseteq U$. Then $\eta_k \cdot \eta$ is a smooth function on $\mathbb{R}^n$ vanishing outside $\operatorname{supp} \eta_k$. Hence $(\eta_k \cdot \eta) \circ (f_1, \dots, f_n)$ is a well-defined element of $\mathcal{A}$. Since

$$\operatorname{carr}((\eta_k \cdot \eta) \circ (f_1, \dots, f_n)) \subseteq (f_1, \dots, f_n)^{-1}(\operatorname{carr} \eta_k),$$

the family $\{\operatorname{carr}((\eta_k \cdot \eta) \circ (f_1, \dots, f_n)) : k \in \mathbb{N}\}$ is locally finite and since $1 = \sum_{k \in \mathbb{N}} \eta_k$ on U we obtain that $\eta \circ (f_1, \dots, f_n) = \sum_k (\eta_k \cdot \eta) \circ (f_1, \dots, f_n) \in \mathcal{A}$. The first statement follows by taking $\eta : x \mapsto \frac{1}{x}$ □

2.5. Theorem. *Let $\mathcal{A} \subseteq \mathbb{R}^X$ be a $C^{(\infty)}_{lfcs}$-algebra considered as a subspace of $C(X_\mathcal{A})$. Then $\mathcal{A}$ is countably evaluating.*

Proof. Let $\varphi \in \operatorname{Hom} \mathcal{A}$ and take a sequence (f_n) in $\mathcal{A}$. Then $h_n : x \mapsto (f_n(x) - \varphi(f_n))^2$ belongs to $\mathcal{A}$ and $\varphi(h_n) = 0$. We have to show that there exists an $x \in X$ with $h_n(x) = 0$ for all n. Assume that this were not true. Take $\eta \in C^\infty(\mathbb{R}, [0,1])$ with $\operatorname{carr} \eta =]0, \infty[$ and let $g_n : x \mapsto \eta(h_n(x)) \cdot \eta(\frac{1}{n} - h_1(x)) \cdot \ldots \cdot \eta(\frac{1}{n} - h_{n-1}(x))$. Then $g_n \in \mathcal{A}$ and the sum $\sum_n \frac{1}{2^n} g_n$ is locally finite, hence defines a function $g \in \mathcal{A}$. By (2.4) and what has been said in section (1), $\mathcal{A}$ is 1-evaluating, and therefore for any n there exists an $x_n \in X$ with $h_n(x_n) = \varphi(h_n) = 0$ and $\varphi(g_n) = g_n(x_n)$. Hence $\varphi(g_n) = g_n(x_n) = \eta(h_n(x_n)) \cdot \eta(\frac{1}{n} - h_1(x_n)) \cdot \ldots \cdot \eta(\frac{1}{n} - h_{n-1}(x_n)) = 0$. By assumption on the h_n we have that $g > 0$. Hence $\varphi(g) > 0$, since $\mathcal{A}$ is 1-evaluating. Let N be so large that $2^{-N} < \varphi(g)$. Again since $\mathcal{A}$ is 1-evaluating, there is some $a \in X$ such that $\varphi(g) = g(a)$ and $\varphi(g_j) = g_j(a)$ for $j \leq N$. Then

$$\frac{1}{2^N} < \varphi(g) = g(a) = \sum_n \frac{1}{2^n} g_n(a) = \sum_{n \leq N} \frac{1}{2^n} \varphi(g_n) + \sum_{n > N} \frac{1}{2^n} g_n(a) \leq 0 + \frac{1}{2^N}$$

gives a contradiction. □

Remark. (2.5) improves the result in [BBL] that each C^∞-local algebra is countably evaluating. The improvement is not only formal since there are natural examples of C^∞_{lfcs}-algebras that are not C^∞-local. Indeed, let $\mathcal{A}$ be the algebra formed by those smooth functions on $c_0(\Gamma)$ which depend only on countably many coordinates. Obviously $\mathcal{A}$ is a C^∞_{lfcs}-algebra. The function $f := \prod_{\gamma \in \Gamma} \eta \circ \operatorname{pr}_\gamma$, where $\eta \in C^\infty(\mathbb{R}, [0,1])$ is chosen such that

$$\eta(t) = \begin{cases} 1, & \text{if } |t| < \frac{1}{2} \\ 0, & \text{iff } |t| \geq 1, \end{cases}$$

and where $(\operatorname{pr}_\gamma)_{\gamma \in \Gamma}$ is the family of coordinate projections, is a locally finite product. Hence, if $\mathcal{A}$ is C^∞-local, then f belongs to $\mathcal{A}$. This is, however, for uncountable Γ obviously not the case.

The algebra $\mathcal{A}$ of constant functions on a disconnected topological space provides an example of a C^∞_{lfs}-algebra that is not C^∞-local. However it remains to solve the following:

Open Problem. *Is there a C^∞_{lfs}-algebra separating the points that is not C^∞-local?*

3. Homomorphisms commuting with summation

For any algebra $\mathcal{A} \subseteq \mathbb{R}^X$ there are the natural embeddings

$$X_\mathcal{A} \hookrightarrow \operatorname{Hom}_{\aleph_0} \mathcal{A} \hookrightarrow \operatorname{Hom} \mathcal{A} \hookrightarrow \mathbb{R}^\mathcal{A},$$

where the sets of homomorphisms are endowed with the induced pointwise topology. Remark that the closure of $X_\mathcal{A}$ in $\mathbb{R}^\mathcal{A}$ is formed by those $\varphi \in \operatorname{Hom} \mathcal{A}$ such that $\varphi(f) \in \overline{f(X)}$ for all $f \in \mathcal{A}$. In fact, let $\varphi \in \operatorname{Hom} \mathcal{A}$ be given with $\varphi(f) \in \overline{f(X)}$ for all $f \in \mathcal{A}$. Take finitely many $f_i \in \mathcal{A}$ and an $\varepsilon > 0$.Let $f := \sum_i (f_i - \varphi(f_i))^2$. Then $0 = \varphi(f) \in \overline{f(X)}$ and hence there exists an $x \in X$ such that $|f_i(x) - \varphi(f_i)|^2 \leq f(x) \leq \varepsilon^2$ for all i. Thus δ_x is the required element in the neighborhood of φ. Trivially the above condition is satisfied by all 1-evaluating homomorphisms. Hence $X_\mathcal{A}$ is always dense in $\operatorname{Hom}_{\aleph_0} \mathcal{A}$. The set $\operatorname{Hom} \mathcal{A}$ is closed in $\mathbb{R}^\mathcal{A}$ and hence realcompact. Moreover, we have:

3.1 Theorem. *For any subalgebra $\mathcal{A} \subseteq \mathbb{R}^X$ the set $\operatorname{Hom}_{\aleph_0} \mathcal{A}$ endowed with the pointwise topology is realcompact. Hence, if $X_\mathcal{A}$ is not realcompact, then there exists a $\varphi \in \operatorname{Hom}_{\aleph_0} \mathcal{A}$ that is not a point evaluation.*

Proof. Assume that all sets of homomorphisms are endowed with the pointwise topology. Let $\mathcal{M} \subset 2^\mathcal{A}$ be the family of all countable subsets of $\mathcal{A}$ containing the unit. For $M \in \mathcal{M}$, consider the topological space $\operatorname{Hom}_{\aleph_0} \langle M \rangle_{alg}$. Obviously the family $(\delta_f)_{f \in M}$, where $\delta_f(\varphi) = \varphi(f)$, is a countable subset of $C(\operatorname{Hom}_{\aleph_0} \langle M \rangle_{alg})$ that separates the points in $\operatorname{Hom}_{\aleph_0} \langle M \rangle_{alg}$. Hence

$$\operatorname{Hom}_{\aleph_0} \langle M \rangle_{alg} = \operatorname{Hom} C(\operatorname{Hom}_{\aleph_0} \langle M \rangle_{alg})$$

by (2.5), i.e. $\operatorname{Hom}_{\aleph_0} \langle M \rangle_{alg}$ is realcompact. Now $\operatorname{Hom}_{\aleph_0} \mathcal{A}$ is an inverse limit of the spaces $\operatorname{Hom}_{\aleph_0} \langle M \rangle_{alg}$ for $M \in \mathcal{M}$. Since $\operatorname{Hom}_{\aleph_0} \langle M \rangle_{alg}$ is Hausdorff, we obtain that $\operatorname{Hom}_{\aleph_0} \mathcal{A}$ as a closed subset of a product of realcompact spaces is realcompact by definition. Since X is not realcompact in the topology $X_\mathcal{A}$, which is that induced from the embedding into $\operatorname{Hom}_{\aleph_0} \mathcal{A}$, we have that $X \neq \operatorname{Hom}_{\aleph_0} \mathcal{A}$ and the statement is proved. $\square$

Remark. This generalizes (2.3) and (2.4) in [KMS]. In fact it is obvious that the mapping $f \mapsto \check{f}$ from $\mathcal{A}$ into $C(\operatorname{Hom}_{\aleph_0} \mathcal{A})$, where $\check{f}(\varphi) = \varphi(f)$, is an extension operator. Hence the space $\operatorname{Hom}_{\aleph_0} \mathcal{A}$ is a "realcompactification" of $X_\mathcal{A}$ in the sense that $\operatorname{Hom}_{\aleph_0} \mathcal{A}$ is realcompact and contains $X_\mathcal{A}$ as a dense subspace and each $f \in \mathcal{A}$ admits a continuous extension to $\operatorname{Hom}_{\aleph_0} \mathcal{A}$. For a corresponding universal property see 7.15 in [Ad]. Since $\operatorname{Hom} \mathcal{A}$ is closed in $\mathbb{R}^\mathcal{A}$ and $\operatorname{Hom} \mathcal{A} \neq \mathcal{A}^*$, we never have that $X_{\mathcal{A}}$ is dense in $\mathcal{A}^*$ with respect to the weak topology. On the other hand, it might well be the case that $\mathcal{A} = \langle\mathcal{B}\rangle_{alg}$ and $X_{\mathcal{B}} = X_{\mathcal{A}}$ is dense in $\mathcal{B}^*$. The next result concerns such a situation. Although the main ideas in it can be found in [C], we prefer to give a full detailed proof, not only for sake of completeness, but also because of the important contribution it gives to the theory of homomorphisms.

3.2 Proposition. *Let X be a set and $\mathcal{B} \subseteq \mathbb{R}^X$ a linear subspace separating the points in X. Assume that X is realcompact and dense in $\mathcal{B}^*$ with respect to the weak topology $\sigma(\mathcal{B}^*, \mathcal{B})$. Let $\phi \in \mathcal{B}^*$ be countably evaluating. Then there exists some $a \in X$ with $\phi(g) = g(a)$ for all $g \in \mathcal{B}$. In particular, $X = \mathrm{Hom}_{\aleph_0}\langle\mathcal{B}\rangle_{alg}$.*

Proof. Let $X_{\mathcal{B}}$ be the space X endowed with the topology induced by $\mathcal{B}$. Thus we can identify $X_{\mathcal{B}}$ with its homeomorphic image in $\mathbb{R}^{\mathcal{B}}$ with respect to the map $x \mapsto (g(x))_{g\in\mathcal{B}}$. Let $\mathcal{F}$ be the filter on $X_{\mathcal{B}}$ generated by the filter base of sets $S(\mathcal{B}_0) := \{x \in X_{\mathcal{B}} : g(x) = \phi(g)\ \forall g \in \mathcal{B}_0\}$, where $\mathcal{B}_0 \subseteq \mathcal{B}$ is countable. Take a continuous function $f : X_{\mathcal{B}} \to \mathbb{R}$. For each $r \in \mathbb{Q}$, set $U_r := \{x \in X_{\mathcal{B}} : f(x) < r\}$ and $V_r := \{x \in X_{\mathcal{B}} : f(x) > r\}$. By assumption, the space $X_{\mathcal{B}}$ is a dense subspace of $\mathcal{B}^* \subseteq \mathbb{R}^{\mathcal{B}}$, and thus there are disjoint open sets $\widetilde{U}_r$ and $\widetilde{V}_r$ in $\mathcal{B}^*$ with $U_r \subseteq \widetilde{U}_r$ and $V_r \subseteq \widetilde{V}_r$. Let $(g_\iota)_{\iota\in I}$ be an algebraic basis of $\mathcal{B}$. The map $\chi : \mathcal{B}^* \to \mathbb{R}^I$, where $\chi(l) = (l(g_\iota))_{\iota\in I}$, defines a topological linear isomorphism $\mathcal{B}^* \cong \mathbb{R}^I$. As $\chi(\widetilde{U}_r)$ and $\chi(\widetilde{V}_r)$ are disjoint open sets in $\mathbb{R}^I$, there is a countable set $M \subseteq I$ such that for the canonical projections $p_M : \mathbb{R}^I \to \mathbb{R}^M$

$$p_M(\chi(\widetilde{U}_r)) \cap p_M(\chi(\widetilde{V}_r)) = \emptyset$$

(see [En, p.116]). Set $\mathcal{B}_r := \{g_\iota : \iota \in M\}$. Then $p_{\mathcal{B}_r}(\widetilde{U}_r) \cap p_{\mathcal{B}_r}(\widetilde{V}_r) = \emptyset$, and thus also $p_{\mathcal{B}_r}(U_r) \cap p_{\mathcal{B}_r}(V_r) = \emptyset$. Set $\mathcal{B}_o := \bigcup_{r\in\mathbb{Q}} \mathcal{B}_r$. For any $x, y \in X_{\mathcal{B}}$ and $r \in \mathbb{Q}$ with $f(x) < r < f(y)$, there exists by construction $g \in \mathcal{B}_o$ with $g(x) \neq g(y)$. Hence, if $x, y \in S(\mathcal{B}_o)$, then $|f(x) - f(y)| = 0$. Thus $\mathcal{F}$ is a Cauchy-filter in the uniformity generated by the continuous functions on $X_{\mathcal{B}}$. By realcompactness of $X_{\mathcal{B}}$, the filter $\mathcal{F}$ converges to a point $a \in X_{\mathcal{B}}$ [GJ, p.226]. Assume that $\phi(g_0) \neq g_0(a)$ for some $g_0 \in \mathcal{B}$. Take $\varepsilon > 0$ with $\phi(g_0) \notin g_0(a) +]{-\varepsilon}, \varepsilon[$ and set $W := \{x \in X : |g_0(x) - g_0(a)| < \varepsilon\}$. Since $\mathcal{F} \to a \in W$ there exists a countable $\mathcal{B}' \subseteq \mathcal{B}$ such that $S(\mathcal{B}') \in \mathcal{F}$ with $S(\mathcal{B}') \subseteq W$. Set $\mathcal{B}'' := \{g_0\} \cup \mathcal{B}'$. Thus $S(\mathcal{B}'') \subseteq S(\mathcal{B}') \subseteq W$. So there exists an $x \in S(\mathcal{B}'') \subseteq W$, i.e. $g_0(x) = \phi(g_0)$ and $|g_0(x) - g_0(a)| < \varepsilon$. But then $\phi(g_0) \in g_0(a) +] - \varepsilon, \varepsilon[$, which is a contradiction. Therefore $\phi(g) = g(a)$ for all $g \in \mathcal{B}$. □

Obviously (3.2) shows that the converse of (3.1) holds under certain conditions. Still there remains the following:

Open Problem. *Does there exist an algebra $\mathcal{A} \subseteq \mathbb{R}^X$ with $X \neq \mathrm{Hom}_{\aleph_0}\mathcal{A}$ and $X_{\mathcal{A}}$ realcompact?*

3.3 Theorem. *Let $\mathcal{A} \subseteq \mathbb{R}^X$ be a subalgebra and let $(f_\gamma)_{\gamma\in\Gamma}$ be a family in $\mathcal{A}$ such that $\sum_{\gamma\in\Gamma} z_\gamma f_\gamma^j$ is a pointwise convergent sum in $\mathcal{A}$ for all $z = (z_\gamma) \in \ell^\infty(\Gamma)$*

and $j \in \{1,2\}$. If $|\Gamma|$ is non-measurable, then, for each $\varphi \in \mathrm{Hom}_{\aleph_0}\mathcal{A}$, there is a point $a \in X$ with $\varphi(\sum_{\gamma\in\Gamma} f_\gamma) = \sum_{\gamma\in\Gamma} f_\gamma(a)$ and $\varphi(f_\gamma) = f_\gamma(a)$ for all $\gamma \in \Gamma$, and hence in particular $\varphi(\sum_{\gamma\in\Gamma} f_\gamma) = \sum_{\gamma\in\Gamma} \varphi(f_\gamma)$.

Proof. Take $\varphi \in \mathrm{Hom}_{\aleph_0}\mathcal{A}$. Let $x \in X$ and set $z_\gamma := sign f_\gamma(x)$ for all $\gamma \in \Gamma$. Then $z = (z_\gamma) \in \ell^\infty(\Gamma)$ and $\sum_{\gamma\in\Gamma} |f_\gamma(x)| = \sum_{\gamma\in\Gamma} z_\gamma f_\gamma(x) < \infty$, i.e. $(f_\gamma(x))_{\gamma\in\Gamma} \in \ell^1(\Gamma)$. We first observe that $(\varphi(f_\gamma))_{\gamma\in\Gamma} \in c_0(\Gamma)$ since otherwise, for some $\varepsilon > 0$, there is a countable set $\Lambda \subseteq \Gamma$ with $|\varphi(f_\gamma)| > \varepsilon$ for each $\gamma \in \Lambda$, and then, using the countably evaluating property of φ, there is a point $x \in X$ with $|f_\gamma(x)| > \varepsilon$ for each $\gamma \in \Lambda$, violating the condition $(f_\gamma(x))_{\gamma\in\Gamma} \in \ell^1(\Gamma)$. Furthermore $(\varphi(f_\gamma))_{\gamma\in\Gamma} \in \ell^1(\Gamma)$, since as a vector in $c_0(\Gamma)$ it has countable support and since φ is countably evaluating. Therefore

$$g(x) := ((f_\gamma(x) - \varphi(f_\gamma))^2)_{\gamma\in\Gamma} \in \ell^1(\Gamma)$$

for every $x \in X$. Define a map $g^* : \ell^\infty(\Gamma) = \ell^1(\Gamma)' \to \mathcal{A}$, by

$$g^*(z) : x \mapsto \langle z, g(x)\rangle = \sum_{\gamma\in\Gamma} z_\gamma \cdot (f_\gamma(x) - \varphi(f_\gamma))^2,$$

and let $\Phi : \ell^\infty(\Gamma) \to \mathbb{R}$ be the linear map $\Phi := \varphi \circ g^* : \ell^\infty(\Gamma) \to \mathcal{A} \to \mathbb{R}$. By the countably evaluating property of φ, for any sequence (z_n) in $\ell^\infty(\Gamma)$ there exists an $x \in X$ such that $\Phi(z_n) = \varphi(g^*(z_n)) = g^*(z_n)(x) = \langle z_n, g(x)\rangle$ for all n. Assume that $|\Gamma|$ is non-measurable. Then, by [Ed, p.575], the weak topology of $\ell^1(\Gamma)$ is realcompact. Since any lcs E is weak*-dense in E'^*, there is, by (3.2), a point $c \in \ell^1(\Gamma)$ such that $\Phi(z) = \langle z, c\rangle$ for all $z \in \ell^\infty(\Gamma)$. For each standard unit vector $e_\gamma \in \ell^\infty(\Gamma)$ we have $0 = \Phi(e_\gamma) = \langle e_\gamma, c\rangle = c_\gamma$. Hence $c = 0$ and Φ is the zero map on $\ell^\infty(\Gamma)$. For the constant vector $\mathbf{1}$ in $\ell^\infty(\Gamma)$, we get $0 = \Phi(\mathbf{1}) = \varphi(g^*(\mathbf{1}))$. Let $a \in X$ with $\varphi(\sum_{\gamma\in\Gamma} f_\gamma) = \big(\sum_{\gamma\in\Gamma} f_\gamma\big)(a) = \sum_{\gamma\in\Gamma} f_\gamma(a)$ and $\varphi(g^*(\mathbf{1})) = g^*(\mathbf{1})(a) = \sum_{\gamma\in\Gamma}(f_\gamma(a) - \varphi(f_\gamma))^2$. Thus $\varphi(f_\gamma) = f_\gamma(a)$ for each $\gamma \in \Gamma$ and therefore $\varphi(\sum_{\gamma\in\Gamma} f_\gamma) = \sum_{\gamma\in\Gamma} f_\gamma(a) = \sum_{\gamma\in\Gamma} \varphi(f_\gamma)$ as we should prove. □

Remark. It would be interesting to know, whether the condition for $j = 2$ can be dropped.

Using (3.3) we obtain the main result in [GGJ] in a direct way:

3.4 Corollary. *Let E be a Banach space admitting a continuous linear injection T into $\ell^{2n}(\Gamma)$ for some n and some Γ of non-measurable cardinality. Then $E = \mathrm{Hom}\,\mathcal{A}$ for any 1-evaluating algebra $\mathcal{A}$ with $P(E) \subseteq \mathcal{A} \subseteq C(E)$.*

Proof. Let $\varphi \in \mathrm{Hom}\,\mathcal{A}$ and take $f \in \mathcal{A}$. Let $\mathcal{A}_0$ be the algebra generated by f and all i-homogeneous polynomials in $P(E)$ with degree $i \leq 4n + 2$. Take a sequence (p_n) of continuous polynomials with degree $i \leq 2n + 1$. Then there is a sequence (t_n) in $\mathbb{R}_+$ such that $h_j \in \mathcal{A}$ for all $j \in \{0, 1\}$, where

$$h_j(x) := (f(x) - \varphi(f))^2 - \sum_{n=1}^{\infty} t_n (\frac{1}{n})^j (p_n(x) - \varphi(p_n))^2.$$

This technique is standard (see [BJL] or [Ar] from which it originates), and yields that φ evaluates f and the sequence (p_n) in some common point. Hence φ is countably evaluating on $\mathcal{A}_0$. Given $z = (z_\gamma) \in \ell^\infty(\Gamma)$ and $x \in E$, set

$$g_{z,j}(x) := f(x) + \sum_{\gamma\in\Gamma} z_\gamma(\mathrm{pr}_\gamma \circ T)^{(2n+1)j}(x),$$

where $j \in \{1, 2\}$. Then $g_{z,j} \in \mathcal{A}_0$ and we can apply (3.3). Thus there is a point $a \in E$ with $\varphi(f) = f(a)$ and $\varphi(\mathrm{pr}_\gamma \circ T)^{2n+1} = (\mathrm{pr}_\gamma \circ T)^{2n+1}(a)$ for all $\gamma \in \Gamma$. Hence $\varphi(\mathrm{pr}_\gamma \circ T) = (\mathrm{pr}_\gamma \circ T)(a)$ for all $\gamma \in \Gamma$, by which the point a is unique since $(\mathrm{pr}_\gamma \circ T)_{\gamma\in\Gamma}$ is point separating. Therefore a is not depending on the function $f \in \mathcal{A}_0$ and we are finished. □

Theorem (3.3) will be applied in particular to locally finite sums.

3.5 Proposition. *Let $\mathcal{A} \subseteq \mathbb{R}^X$ be an algebra that is closed under formation of locally finite sums. Take a locally finite product $f := \prod_{\gamma\in\Gamma} f_\gamma \in \mathcal{A}$ of factors f_γ in $\mathcal{A}$. If $|\Gamma|$ is non-measurable, then, for each $\varphi \in \mathrm{Hom}_{\aleph_0} \mathcal{A}$, there is some $a \in X$ with $\varphi(f) = f(a)$ and $\varphi(f_\gamma) = f_\gamma(a)$ for all $\gamma \in \Gamma$, and in particular $\varphi(\prod_{\gamma\in\Gamma} f_\gamma) = \prod_{\gamma\in\Gamma} \varphi(f_\gamma)$.*

Proof. For each $\gamma \in \Gamma$, set $g_\gamma := f_\gamma - 1$ and for finite $J \subseteq \Gamma$ set $g_J := \prod_{\gamma\in J} g_\gamma$. Then, according to (2.3), we have that $f = \sum_{J\in I} g_J$ is a locally finite sum in $\mathcal{A}$, where $I := \{J \subseteq \Gamma : |J| < \aleph_0\}$. Assume now that $|\Gamma|$ is non-measurable. Since the cardinality of the index set I is equal to that of Γ, there is by (3.3) a point $a \in X$ with $\varphi(f) = f(a)$ and $\varphi(g_J) = g_J(a)$ for all $J \in I$ and thus also $\varphi(f_\gamma) = f_\gamma(a)$ for all $\gamma \in \Gamma$. Hence $\varphi(f) = f(a) = \prod_{\gamma\in\Gamma} f_\gamma(a) = \prod_{\gamma\in\Gamma} \varphi(f_\gamma)$. □

3.6 Theorem. *Let X be a metric space where each closed discrete subset is of non-measurable cardinality. If X admits $\mathcal{A}$-partitions of unity with respect to an algebra $\mathcal{A} \subseteq C(X)$, then $X = \mathrm{Hom}_{\aleph_0} \mathcal{A}$. In particular, for $\mathcal{A} := C(X)$ we obtain that X is realcompact. Conversely, for every set Γ of measurable cardinality, there exists a metric space X, a locally finite sum $\sum_{\gamma\in\Gamma} f_\gamma \in C(X)$ of summands $f_\gamma \in C(X)$ and a $\varphi \in \mathrm{Hom}\, C(X)$, with $\varphi(\sum_{\gamma\in\Gamma} f_\gamma) \neq \sum_{\gamma\in\Gamma} \varphi(f_\gamma)$.*

Proof. Take $\varphi \in \mathrm{Hom}_{\aleph_0} \mathcal{A}$. For each n, let $(f_\gamma)_{\gamma\in\Gamma_n} \subset \mathcal{A}$ be a locally finite partition of unity subordinate to a cover of X consisting of balls of diameter $\frac{1}{n}$. Since a locally finite summation in a space where each closed discrete set is of non-measurable cardinality can only be taken over a set with a non-measurable cardinal, we have that each Γ_n has non-measurable cardinal. Now $1 = \sum_{\gamma\in\Gamma_n} f_\gamma$ and thus, by (3.3), $1 = \varphi(1) = \sum_{\gamma\in\Gamma_n} \varphi(f_\gamma)$. Since each f_γ is non-negative, there is some $\gamma_n \in \Gamma_n$ with $\varphi(f_{\gamma_n}) > 0$. Pick $a \in X$ with $\varphi(f_{\gamma_n}) = f_{\gamma_n}(a)$ for all n. By construction the point a is unique. Let $f \in \mathcal{A}$ be arbitrary. Evaluating f and all f_{γ_n} at some point shows that $\varphi(f) = f(a)$ and therefore $X = \mathrm{Hom}_{\aleph_0} \mathcal{A}$. For the special case $\mathcal{A} = C(X)$ we obtain by (2.5) that $X = \mathrm{Hom}_{\aleph_0} C(X) = \mathrm{Hom}\, C(X)$, i.e. X is realcompact.

If the second statement were not true, then by the procedure above $c_0(\Gamma)$ is realcompact although Γ is a set with a measurable cardinal. The set of unit vectors in $c_0(\Gamma)$ is a closed discrete subset of $c_0(\Gamma)$ with the same cardinality as that of Γ. But any closed discrete set in a realcompact space has a non-measurable cardinal [GJ, p.163]. $\square$

With (3.6) in hand the following version of Shirota's theorem is now easy to obtain.

3.7 Corollary. *Let X be a closed subspace of a product of metric spaces. If each closed discrete subset of X is of non-measurable cardinality, then X is realcompact.*

Proof. Let X be closed in $\prod_{i\in I} Y_i$, where each Y_i is a metric space. Set $X_i := \mathrm{pr}_i(X)$, where $\mathrm{pr}_i : \prod_{i\in I} Y_i \to Y_i$ are the natural projections. Then, for each $i \in I$, $X_i \subseteq Y_i$ is a metric space and each closed discrete set in X_i is of non-measurable cardinality. Also X is a closed subspace of $\prod_{i\in I} X_i$. Since the class of realcompact spaces is closed under formation of arbitrary products and closed subsets (see [GJ]) the statement then follows from (3.6). $\square$

Shirota's theorem states that if each closed discrete subspace of a completely regular space X has a non-measurable cardinal, then X is realcompact if and only if X admits a complete uniformity [GJ, p.229]. On the other hand, a topological space X admits a complete uniformity if and only if X is a closed subspace of a product of metric spaces [En, p.464].

4. Extensions of homomorphisms

In this section we characterize several evaluation properties of homomorphisms in terms of extension properties.

4.1 Proposition. *Let $\mathcal{A} \subseteq \mathbb{R}^X$ be a subalgebra. For a $\varphi \in \operatorname{Hom}\mathcal{A}$ the following statements are equivalent:*

(1) *$\varphi(f) \in \overline{f(X)}$ for all $f \in \mathcal{A}$;*
(2) *φ extends to a unique homomorphism on the C^∞-algebra $\mathcal{A}^\infty$ generated by $\mathcal{A}$, i.e.*

$$\mathcal{A}^\infty := \{\eta \circ (f_1, \dots, f_n) : n \in \mathbb{N}, f_i \in \mathcal{A}, \eta \in C^\infty(\mathbb{R}^n)\}.$$

Proof. ($1 \Rightarrow 2$) We define $\varphi(\eta \circ (f_1, \dots, f_n)) := \eta(\varphi(f_1), \dots, \varphi(f_n))$. By means of (2.2), this is the only candidate for an extension. This map is well defined. Indeed, let $\eta \circ (f_1, \dots, f_n) = \mu \circ (g_1, \dots, g_m)$. For each $\varepsilon > 0$ there is a point $x \in E$ such that $|\varphi(f_i) - f_i(x)| < \varepsilon$, $i = 1, ..., n$, and $|\varphi(g_j) - g_j(x)| < \varepsilon$, $j = 1, ..., m$. By continuity of η and μ we obtain that

$$\eta(\varphi(f_1), \dots, \varphi(f_n)) = \mu(\varphi(f_1), \dots, \varphi(f_m)),$$

and we therefore have a well defined extension of φ. This extension is a homomorphism, since for every polynomial ϑ on $\mathbb{R}^m$ (or even for $\vartheta \in C^\infty(\mathbb{R}^m)$) and $g_i := \eta_i \circ (f_1^i, \dots, f_{n_i}^i) \in \mathcal{A}^\infty$ we have

$$
\begin{aligned}
\varphi(\vartheta\circ(g_1,\dots,g_m)) &= \varphi(\vartheta\circ(\eta_1\times\dots\times\eta_m)\circ(f_1^1,\dots,f_{n_m}^m)) \\
&= (\vartheta\circ(\eta_1\times\dots\times\eta_m))(\varphi(f_1^1),\dots,\varphi(f_{n_m}^m)) \\
&= \vartheta(\eta_1(\varphi(f_1^1),\dots,\varphi(f_{n_1}^1)),\dots,\eta_m(\varphi(f_1^m),\dots,\varphi(f_{n_m}^m)) \\
&= \vartheta(\varphi(g_1),\dots,\varphi(g_m)).
\end{aligned}
$$

($2 \Rightarrow 1$) Suppose there is some $f \in \mathcal{A}$ with $\varphi(f) \notin \overline{f(X)}$. Then we may find an $\eta \in C^\infty(\mathbb{R})$ with $\eta(\varphi(f)) = 1$ and $\operatorname{carr}\eta \cap f(X) = \emptyset$. Since $\mathcal{A}^\infty$ is a C^∞-algebra, we conclude from (2.2) that $\varphi(\eta\circ f) = \eta(\varphi(f)) = 1$. But since $\eta \circ f = 0$ we arrive at a contradiction. □

Remark. If $\mathcal{A}$ separates the points of X, then the algebra $\mathcal{A}^\infty$ above can even be replaced by the bigger algebra $\tilde{\mathcal{A}} := \{\eta\circ(f_1,\dots,f_n) : n \in \mathbb{N}, f_i \in \mathcal{A}, \eta \in C(\mathbb{R}^n)\}$. Indeed, everything in ($1 \Rightarrow 2$) holds with $\mathcal{A}^\infty$ replaced by $\tilde{\mathcal{A}}$ with the possible exception of uniqueness. But since $\varphi(f) \in \overline{f(X)}$ for all $f \in \tilde{\mathcal{A}}$ by construction, again (2.2) ensures that the extension of φ to $\tilde{\mathcal{A}}$ is unique. Moreover, since the proof of (2.2) uses that there is a point $z \in \beta(X_{\mathcal{A}})$ such that $\varphi(f) = \hat{f}(z)$ for all $f \in \mathcal{A}$, the largest algebra φ can be extended to in this way is in fact the algebra $\overline{\mathcal{A}}^\varphi = \{g \in C(X_{\mathcal{A}}) : \hat{g}(z) \neq \infty\}$. Obviously $\tilde{\mathcal{A}} \subseteq \overline{\mathcal{A}}^\varphi$ and $C_b(X_{\mathcal{A}}) \subset \overline{\mathcal{A}}^\varphi$.

4.2 Proposition. *Let $\mathcal{A} \subseteq \mathbb{R}^X$ be a subalgebra. Then the following statements for a homomorphism $\varphi : \mathcal{A} \to \mathbb{R}$ are equivalent:*

(1) *φ is 1-evaluating.*
(2) *φ extends to a unique homomorphism on* $\mathcal{RA} := \{f/g : f, g \in \mathcal{A},\ 0 \notin g(X)\}$.
(3) *φ extends to a unique homomorphism on the following C^∞-algebra $\mathcal{A}^{\langle\infty\rangle}$ constructed from $\mathcal{A}$:*

$$
\mathcal{A}^{\langle\infty\rangle} := \{\eta\circ(f_1,\dots,f_n) : f_i \in \mathcal{A}, (f_1,\dots,f_n)(X) \subseteq U, \; U \text{ open in some } \mathbb{R}^n, \eta \in C^\infty(U)\}.
$$

Proof. ((1) $\Rightarrow$ (3)) We define $\varphi(\eta\circ(f_1,\dots,f_n)) := \eta(\varphi(f_1),\dots,\varphi(f_n))$. Since there exists an x with $\varphi(f_i) = f_i(x)$, we have $(\varphi(f_1),\dots,\varphi(f_n)) \in U$, hence the right side makes sense. The rest follows in the same way as in the proof of (4.1).

((3) $\Rightarrow$ (2)) is obvious, since $\mathcal{RA} \subseteq \mathcal{A}^{\langle\infty\rangle}$.

((2) $\Rightarrow$ (1)) Since $\mathcal{RA}$ is inverse-closed, the extension of φ to this algebra is 1-evaluating, hence the same is true for φ on $\mathcal{A}$. □

Remark. Every C^∞-algebra which is not 1-evaluating satisfies the equivalent conditions of (4.1), but not those of (4.2). An example is $C_b(X)$ for a Lindelöf non-compact X, since δ_x for every $x \in \beta X \setminus X$ defines a homomorphism that is not 1-evaluating.

Definition. Given a set $\mathcal{A} \subseteq C(X)$ and a cardinal $\boldsymbol{m}$ we denote by $\mathcal{A}^\infty_{lf\boldsymbol{m}\,s}$ the smallest C^∞-algebra containing $\mathcal{A}$ that is closed under formation of locally finite sums of cardinality $\boldsymbol{m}$. In case, where $m = \aleph_0$, we will also write $\mathcal{A}^\infty_{lfcs}$ instead of $\mathcal{A}^\infty_{lf\aleph_0 s}$

4.3 Theorem. *Let $\mathcal{A} \subseteq C(X)$ be an algebra. Then for any $\varphi \in \operatorname{Hom} \mathcal{A}$ the following statements are equivalent:*

(1) *φ is countably evaluating.*
(2) *φ extends to a unique homomorphism on $\mathcal{A}^\infty_{lfcs}$.*

Proof. ((1) $\Rightarrow$ (2)) We have to show that φ can be extended to $\mathcal{A}_{lfcs} := \{f : f = \sum_n f_n, f_n \in \mathcal{A},$ the sum is locally finite$\}$ and that this extension is also countably evaluating. By (2.4) the algebra $\mathcal{A}^\infty_{lfcs}$ is then the union of the algebras obtained by finite iterations of passing to $\mathcal{A}_{lfcs}$ and to $\mathcal{A}^\infty$. To $\mathcal{A}^\infty$ it extends by (4.1). It is countably evaluating there, since in any $f \in \mathcal{A}^\infty$ only finitely many elements of $\mathcal{A}$ are involved. For a locally finite sum $f = \sum_k f_k$ we define $\varphi(f) := \sum_k \varphi(f_k)$. This makes sense, since there exists an $x \in X$ with $\varphi(f_n) = f_n(x)$, and since $\sum_n f_n$ is point finite, we have that the sum $\sum_n \varphi(f_n) = \sum_n f_n(x)$ is in fact finite. It is well defined, since for $\sum_n f_n = \sum_n g_n$ we can choose an $x \in X$ with $\varphi(f_n) = f_n(x)$ and $\varphi(g_n) = g_n(x)$ for all n, and hence $\sum_n \varphi(f_n) = \sum_n f_n(x) = \sum_n g_n(x) = \sum_n \varphi(g_n)$. The extension is a homomorphism, since for the product for example we have

$$\varphi\Big(\big(\sum_n f_n\big)\big(\sum_k g_k\big)\Big) = \varphi\Big(\sum_{n,k} f_n\, g_k\Big) = \sum_{n,k} \varphi(f_n\, g_k)$$
$$= \sum_{n,k} \varphi(f_n)\, \varphi(g_k) = \Big(\sum_n \varphi(f_n)\Big)\Big(\sum_k \varphi(g_k)\Big).$$

Remains to show that the extension is countably evaluating. So let $f^k = \sum_n f^k_n$ be given. By assumption there exists an x such that $\varphi(f^k_n) = f^k_n(x)$ for all n and all k. Thus $\varphi(f^k) = \sum_n \varphi(f^k_n) = \sum_n f^k_n(x) = f^k(x)$ for all k.

((1) $\Leftarrow$ (2)) Since $\mathcal{A}^\infty_{lfcs}$ is a C^∞_{lfcs}-algebra we conclude from (2.5) that the extension of φ is countably evaluating. $\square$

If one tries to generalize this result to uncountable cardinalities, it is clear that the mapping $\sum_{\gamma\in\Gamma} f_\gamma \mapsto \sum_{\gamma\in\Gamma} \varphi(f_\gamma)$ is a well defined homomorphism on $\mathcal{A}_{lf|\Gamma|s}$ provided that $\varphi \in \operatorname{Hom}_{|\Gamma|} \mathcal{A}$. But in order to guarantee that this extension of φ is unique we need (3.3):

4.4 Proposition. *Let $\mathcal{A} \subseteq \mathbb{R}^X$ be an algebra and let $\boldsymbol{m} \geq \aleph_0$ be a non-measurable cardinal. Then each $\varphi \in \operatorname{Hom}_{\boldsymbol{m}} \mathcal{A}$ extends uniquely to $\mathcal{A}^\infty_{lf\boldsymbol{m}s}$.*

Proof. As in the proof of (4.3), $\mathcal{A}^\infty_{lf\boldsymbol{m}s}$ is the union of the algebras obtained by finite iterations of passing to $\mathcal{A}^\infty$ and to $\mathcal{A}_{lf\boldsymbol{m}s}$, where $\mathcal{A}_{lf\boldsymbol{m}s} := \{\sum_{\gamma\in\Gamma} f_\gamma, f_\gamma \in \mathcal{A}, |\Gamma| \leq \boldsymbol{m}$, the sum is locally finite$\}$. Then the mapping $\varphi : \mathcal{A}_{lf\boldsymbol{m}s} \ni \sum_{\gamma\in\Gamma} f_\gamma \mapsto \sum_{\gamma\in\Gamma} \varphi(f_\gamma)$ is obviously a well defined $\boldsymbol{m}$-evaluating homomorphism on $\mathcal{A}_{lf\boldsymbol{m}s}$. Hence each $\varphi \in \operatorname{Hom}_{\boldsymbol{m}} \mathcal{A}$ extends naturally to $\mathcal{A}^\infty_{lf\boldsymbol{m}s}$. However, since $\mathcal{A}^\infty_{lf\boldsymbol{m}s}$ is countably evaluating, (3.3) actually tells us that this extension is unique. $\square$

For the special case $\boldsymbol{m} = \aleph_0$, the iterated algebra $\mathcal{A}^\infty_{lf\boldsymbol{m}s}$ can be obtained in two steps by use of a telescoping process:

4.5 Proposition. *If $\mathcal{A} \subseteq \mathbb{R}^X$ is an algebra, then $\mathcal{A}^\infty_{lfcs} = (\mathcal{A}^\infty)_{lfcs}$. Moreover, the elements of $\mathcal{A}^\infty_{lfcs}$ can be written in the form $g\circ(f_1, f_2, \dots)$, where $g : \mathbb{R}^{(\mathbb{N})} \to \mathbb{R}$ is smooth and where (f_n) is a sequence in $\mathcal{A}$ such that $\sum_{n\geq 1} f_n$ is locally finite.*

Proof. Concerning the first statement we have to show that $(\mathcal{A}^\infty)_{lfcs}$ is closed under composition with smooth mappings. So take $\eta \in C^\infty(\mathbb{R}^n)$ and $\sum_{j\geq 1} f_{i,j} \in (\mathcal{A}^\infty)_{lfcs}$ for $i = 1, \dots, n$. By letting $h_0 := 0$ and $h_k := \eta \circ (\sum_{j=1}^k f_{1,j}, \dots, \sum_{j=1}^k f_{n,j}) \in \mathcal{A}^\infty$ we obtain

$$\eta \circ (\sum_{j\geq 1} f_{1,j}, \dots, \sum_{j\geq 1} f_{n,j}) = \sum_{k\geq 1} (h_k - h_{k-1}),$$

where the right summation is locally finite and hence gives an element of $(\mathcal{A}^\infty)_{lfcs}$.

The algebra $\tilde{\mathcal{A}}$ of functions $g \circ (f_1, f_2, \dots)$ in the second statement is obviously closed under composition with functions in $C^\infty(\mathbb{R}^n)$, but also with respect to locally finite sums since $\sum_{j\geq 1} : \mathbb{R}^{(\mathbb{N})} \to \mathbb{R}$ is a smooth and even a linear mapping. Conversely every element $g \circ (f_1, f_2, \dots) \in \tilde{\mathcal{A}}$ is a locally finite sum $\sum_{k\geq 1}(h_k - h_{k-1})$, where $h_0 := 0$ and $h_k := g \circ \mathrm{inj}_k \circ (f_1, \dots, f_k) \in \mathcal{A}^\infty$. Hence $\tilde{\mathcal{A}} = \mathcal{A}^\infty_{lfcs}$. $\square$

Remark. Also other factorizations belong to $\mathcal{A}^\infty_{lfcs}$, e.g. $g \circ (f_1, f_2, \dots)$, where $f_i \in \mathcal{A}$ are arbitrary and $g \in C^\infty(\mathbb{R}^\mathbb{N})$. This follows from the fact that such a g depends locally only on finitely many variables and $\mathbb{R}^\mathbb{N}$ has countable smooth partitions $\{h_j\}$ of unity. Hence $g = \sum_{j=1}^\infty h_j \cdot g$ and $h_j \cdot g$ depends globally only on finitely many variables.

5. Algebras generated by the dual

Definition. Given a lcs E, we denote the smallest C^∞_{lfcs}-algebra (respectively C^∞_{lfs}-algebra) that contains the dual E' by $C^\infty_{lfcs}(E)$ (respectively $C^\infty_{lfs}(E)$). In the terminology of (4.3) this is $C^\infty_{lfcs}(E) = (E')^\infty_{lf\aleph_0 s}$ and $C^\infty_{lfs}(E) = \bigcup_{\boldsymbol{m}} (E')^\infty_{lf\boldsymbol{m}s}$, where the union is taken over all non-measurable cardinals.

It is easy to show that $T^*(C^\infty_{lfcs}(E)) \subseteq C^\infty_{lfcs}(F)$ and $T^*(C^\infty_{lfs}(E)) \subseteq C^\infty_{lfs}(F)$ for any continuous linear map $T : F \to E$ between lcs. Therefore, if two lcs E and F are isomorphic, then $E = \mathrm{Hom}\, C^\infty_{lfcs}(E)$ (respectively $E = \mathrm{Hom}\, C^\infty_{lfs}(E)$) if and only if $F = \mathrm{Hom}\, C^\infty_{lfcs}(F)$ (respectively $F = \mathrm{Hom}\, C^\infty_{lfcs}(F)$).

Of course $P_f(E) \subset C^\infty_{lfcs}(E) \subseteq C^\infty_{lfs}(E)$. Furthermore $C^\infty_{lfcs}(E) = C^\infty_{lfs}(E) = C^\infty(E)$ if E is finite dimensional and also if $E \cong \mathbb{R}^\mathbb{N}$, since $\mathbb{R}^\mathbb{N}$ admits countable smooth partition of unity. If $|\Gamma|$ is non-measurable, then $C^\infty_{lfs}(\mathbb{R}^\Gamma) = C^\infty(\mathbb{R}^\Gamma)$. However, for any infinite dimensional Banach space E the function $f = \sum_{n=1}^\infty \frac{1}{2^n} \ell_n^2 \in P(E)$, where (ℓ_n) is a sequence of linearly independent norm-one functionals in E', is not an element of $C^\infty_{lfs}(E)$ since it locally does not depend on a finite number of elements in E'.

5.1 Theorem. *A lcs E is weakly realcompact if and only if $E = \operatorname{Hom}_{\aleph_0} P_f(E)(= \operatorname{Hom} C^\infty_{lfcs}(E))$.*

Proof. By (4.3) we have $\operatorname{Hom}_{\aleph_0} P_f(E) = \operatorname{Hom} C^\infty_{lfcs}(E)$. Suppose that E is weakly realcompact. Then, since E is $\sigma(E'^*, E')$-dense in E'^*, it follows, by applying (3.2) with $X = E$ and E' as the linear subspace $\mathcal{B}$, that any $\varphi \in \operatorname{Hom}_{\aleph_0} P_f(E) = \operatorname{Hom} C^\infty_{lfcs}(E)$ is a point evaluation. Conversely, if $E = \operatorname{Hom}_{\aleph_0} P_f(E)$, then the space E is weakly realcompact by (3.1). □

Remark. It should be mentioned that the algebra $C^\infty_{lfcs}(E)$ not only characterizes the weak realcompactness of E in terms of its homomorphisms as shown in (5.1), but also that $C^\infty_{lfcs}(E)$ is big enough to characterize sequential convergence in quasi-complete lcs E. To be more specific: a closer investigation of the main result in [BJ] reveals that a sequence (x_n) converges to x in a quasi-complete lcs E if and only if $f(x_n) \to f(x)$ for all $f \in C^\infty_{lfcs}(E)$.

Valdivia gives in [V] a characterization of the weakly realcompact lcs. Let us mention some classes of spaces that are weakly realcompact:

- all weakly Lindelöf lcs.
- all lcs E with $\sigma(E', E)$-separable E'.
- the Banach spaces E with angelic weak* dual unit ball [Ed, p.564].
- $\ell^1(\Gamma)$ for $|\Gamma|$ non-measurable [Ed, p.575].
- all closed subspaces of products of the spaces listed above.

Considering (5.1), one may ask when $E = \operatorname{Hom} P_f(E)$ or when the homomorphisms on $P_f(E)$ are countably evaluating. If $E \cong \mathbb{R}^\Gamma$ for some set Γ, then obviously $E = \operatorname{Hom} P_f(E)$. The converse is valid by the next

5.2 Proposition. *If $E = \operatorname{Hom} P_f(E)$, then $(E, \sigma(E, E')) \cong \mathbb{R}^\Gamma$ for some Γ. If $P_f(E)$ is only countably evaluating and if E is either a weakly realcompact lcs or a quasi-complete non-semi-reflexive lcs, then the same conclusion holds.*

Proof. Since each lcs E is $\sigma(E'^*, E')$-dense in E'^*, every $p \in P_f(E)$ admits a unique extension $\tilde{p} \in P_f(E'^*)$. Hence each $x \in E'^*$ defines a homomorphism on $P_f(E)$ by $p \mapsto \tilde{p}(x)$. Since $P_f(E'^*)$ separates the points in E'^* and since $E'^* \cong \mathbb{R}^\Gamma$ for some Γ (see the proof of (3.2)), it therefore follows that $E \cong \mathbb{R}^\Gamma$ for some Γ whenever $E = \operatorname{Hom} P_f(E)$. By (5.1), $P_f(E)$ is thus not countably evaluating if E is a weakly realcompact lcs with $E \not\cong \mathbb{R}^\Gamma$ for any Γ. Let E be a quasi-complete lcs. If $P_f(E)$ is countably evaluating, then E is also semi-reflexive by corollary 8 in [BL$_1$]. □

Recall that the associated Schwartz topology to a reflexive Fréchet space is complete [J, p.280]. Since each Schwartz space is isomorphic to a subspace of a product c_0^I [J, p.204], the weak topology of a reflexive Fréchet space is thus realcompact. This together with (5.2) gives

5.3 Corollary. *If E is a Fréchet space, then $P_f(E)$ is countably evaluating if and only if $E \cong \mathbb{R}^\Gamma$ for some countable Γ.*

Definition. Let c_0-inj denote the class of locally convex spaces that admit a continuous linear injection into $c_0(\Gamma)$ with $|\Gamma|$ non-measurable.

5.4 Proposition. *If $E \in c_0$-inj, then $E = \operatorname{Hom}_{\aleph_0} \mathcal{A}$ for any algebra $\mathcal{A} \supseteq C^\infty_{lfs}(E)$.*

Proof. Take $\varphi \in \operatorname{Hom}_{\aleph_0} \mathcal{A}$ and let $T : E \to c_0(\Gamma)$ be continuous, linear and one-one for some Γ of non-measurable cardinality. By the countably evaluating property of φ, it follows that $(\varphi(pr_\gamma \circ T))_{\gamma\in\Gamma} \in c_0(\Gamma)$, where $pr_\gamma : c_0(\Gamma) \to \mathbb{R}$ are the natural coordinate projections. Fix n and consider the function $f_n : c_0(\Gamma) \to \mathbb{R}$ defined by the locally finite product

$$f_n(x) := \prod_{\gamma\in\Gamma} \eta(n((pr_\gamma \circ T)(x) - \varphi(pr_\gamma \circ T))),$$

where $\eta \in C^\infty(\mathbb{R})$ is defined in the remark after (2.5). By means of (3.5), $\varphi(f_n) = \prod_{\gamma\in\Gamma} \eta(0) = 1$ for all n. Let $a \in E$ with $\varphi(f_n) = f_n(a)$ for all n. Then $a \in \bigcap_{n\geq 1} \operatorname{supp} f_n$ and hence $(pr_\gamma \circ T)(a) = \varphi(pr_\gamma \circ T)$ for all $\gamma \in \Gamma$, by construction of the functions f_n. Since $(pr_\gamma \circ T)_{\gamma\in\Gamma}$ is point separating, the point $a \in E$ is unique. Hence every $f \in \mathcal{A}$ is evaluated together with the sequence (f_n) at a, i.e. $\varphi(f) = f(a)$ for all $f \in \mathcal{A}$. □

The result in (5.4) was known before for the case of $\mathcal{A} = C^\infty(E)$, since $c_0(\Gamma)$ admits C^∞-smooth partitions of unity. However, the existence of such partitions of unity (even on non-separable Hilbert spaces) was a long standing open question, which was positively solved in [T] by a tricky construction. Thus the proposition above not only improves previous results, but also its proof requires only basic knowledge of the space $c_0(\Gamma)$.

We list some classes of the Banach spaces for which one can use projectional resolution of identity in order to show that they belong to c_0-inj (see [HWW, p.155].

- all $\ell^p(\Gamma)$ for $1 \leq p < \infty$ and Γ of non-measurable cardinality. More generally all realcompact WCG spaces and their duals.
- the $C(K)$-spaces where K is a Corson compact space of non-measurable cardinality. [N, p.1109].
- the realcompact dual spaces that have the Radon-Nikodym property. [FG].
- the realcompact spaces admitting a Markuševič basis. [JZ].

The class c_0-inj also provides some internal stability. Indeed, given a family $(E_\iota)_{\iota\in I}$ of Banach spaces in c_0-inj where $|I|$ is non-measurable, the ℓ_p-sum of this family (with $1 \leq p < \infty$) and the c_0-sum are elements of c_0-inj (they are injectable into $c_0(\Gamma)$, where Γ is the disjoint union of the sets Γ_ι for which there exists an injection of E_ι into $c_0(\Gamma_\iota)$). Recall that by the ℓ_p-sum of a family $(E_\iota)_{\iota\in I}$ of Banach spaces, one means the Banach space of elements $(x_\iota) \in \prod_{\iota\in I} E_\iota$ satisfying $\sum_{\iota\in I} \|x_\iota\|^p < \infty$ for $1 \leq p < \infty$, or, for $p = 0$, for each $\varepsilon > 0$, there is at most a finite set of ι in I with $\|x_\iota\| > \varepsilon$. These spaces are normed in the obvious ℓ^p or c_0 way. In the same manner it follows that c_0-inj is closed under formation of locally convex direct sums over index sets of non-measurable cardinality.

Definition. Given a set $\mathcal{A}$ with $E' \subseteq \mathcal{A} \subseteq C(E)$. We say that the lcs E has *small $\mathcal{A}$-zerosets* if for each $x \in E$ and each open U with $x \in U \subseteq E$ there is a sequence (f_n) in $\mathcal{A}$ such that $y \in U$ whenever $f_n(y) = f_n(x)$ for all n, or, equivalently, there is a sequence (f_n) in $\mathcal{A}$ with $x \in \bigcap_n Z(f_n) \subseteq U$.

Also we say that *each point in E is countably isolated by $\mathcal{A}$* if for each $x \in E$ there is a sequence (f_n) in $\mathcal{A}$ such that $y = x$ whenever $f_n(y) = f_n(x)$ for all n, or, equivalently, a sequence (f_n) in $\mathcal{A}$ with $\{x\} = \bigcap_n Z(f_n)$.

A lcs E is called *$\mathcal{A}$-regular* if $\mathcal{A}$ induces the topology of E, or, equivalently, for every neighborhood U of a point x, there exists an $f \in \mathcal{A}$ with $x \in E \setminus Z(f) \subseteq U$.

Remark. Since we deal in this paper frequently with at least three natural functors between the categories of locally convex spaces and function algebras, namely P_f, C^∞_{lfcs} and C^∞_{lfs}, we prefer to write that E is P_f-regular instead of $P_f(E)$-regular etc. In this terminology we therefore have that $c_0(\Gamma)$ is C^∞_{lfs}-regular if $|\Gamma|$ is non-measurable (the functions f_n in the proof of (5.4) are C^∞_{lfs}-bump functions). One can even show that $c_0(\Gamma)$ admits C^∞_{lfs}-partitions of unity for $|\Gamma|$ non-measurable.

We collect some basic relations between these concepts in the following:

5.5 Proposition.

(1) *If E is $\mathcal{A}$-regular, then E has small $\mathcal{A}$-zerosets.*
(2) *If each point in E is countably isolated by $\mathcal{A}$, then E has small $\mathcal{A}$-zerosets, the converse being true if E is metrizable.*
(3) *Let E be endowed with the initial locally convex structure with respect to linear mappings $T_i : E \to E_i$. Assume we are given algebras $\mathcal{A}$ on E and $\mathcal{A}_i$ on E_i, such that $T_i^*(\mathcal{A}_i) \subseteq \mathcal{A}$. If all E_i admit small $\mathcal{A}_i$-zerosets, then E admits small $\mathcal{A}$-zerosets.*
(4) *If E' is $\sigma(E', E)$-separable, then each point in E is countably isolated by $P_f(E)$.*
(5) *Each Lindelöf lcs admits small P_f-zerosets.*
(6) *Given $E \in c_0$-inj, each point in E is countably isolated by $C^\infty_{lfs}(E)$.*

Proof. (1), (2), (4) and (6) are trivial.

(3) Any neighborhood of $x \in E$ in the initial topology is of the form $x + \bigcap_{i \in I} T_i^{-1}(U_i)$, where I is finite and U_i are 0-neighborhoods in E_i. By assumption for any i there is a countable family $(f_{i,k})_k \subseteq \mathcal{A}_i$, such that $T_i(x) \in \bigcap_k Z(f_{i,k}) \subseteq T_i(x) + U_i$. So we obtain

$$x \in \bigcap_{i \in I, k \in \mathbb{N}} Z(f_{i,k} \circ T_i) \subseteq x + \bigcap_{i \in I} T_i^{-1}(U_i).$$

(5) Take a point x and an open set U with $x \in U \subseteq E$. For each $y \in E \setminus U$ choose a $p_y \in E' \subseteq P_f(E)$ with $p_y(x) = 0$ and $p_y(y) = 1$. Set $V_y := \{z \in E : p_y(z) > 0\}$. By the Lindelöf property, there is a sequence (y_n) in $E \setminus U$ such that $\{U\} \cup \{V_{y_n}\}_{n \in \mathbb{N}}$ is a cover of E. Hence for each $y \in E \setminus U$ there is some $n \in \mathbb{N}$ such that $y \in V_{y_n}$, i.e. $p_{y_n}(y) > 0 = p_{y_n}(x)$. □

The converse implications in (1) and (2) fail. In (1) consider an infinite dimensional separable Banach space E. Then E has small P_f-zerosets by (5), but E is not P_f-regular. In (2) consider $E = \mathbb{R}^\Gamma$ for Γ uncountable and $\mathcal{A} = P_f(\mathbb{R}^\Gamma)$.

5.6 Proposition. *Let $\mathcal{A} \supseteq P_f(E)$ be an algebra on a lcs E and take $\varphi \in \operatorname{Hom}_{\aleph_0} \mathcal{A}$. Assume that there is some point $a \in E$ with $\varphi = \delta_a$ on E' (hence in particular if $(E, \sigma(E, E'))$ is realcompact by (5.1)). Then we have*

(1) *$\varphi = \delta_a$ on $C^\infty_{lfs}(E)$ if $\mathcal{A} = C^\infty_{lfs}(E)$.*
(2) *$\varphi = \delta_a$ on $\mathcal{A}$ if E admits small C^∞_{lfs}-zerosets and $C^\infty_{lfs}(E) \subseteq \mathcal{A} \subseteq C(E)$.*
(3) *$\varphi = \delta_a$ on $\mathcal{A}$ if E admits small C^∞_{lfs}-zerosets and $C^\infty_{lfs}(E) \subseteq \mathcal{A}$ and E is metrizable.*
(4) *$\varphi = \delta_a$ on $\mathcal{A}$ if E admits small P_f-zerosets and $P_f(E) \subseteq \mathcal{A} \subseteq C(E)$.*

Proof. (1) This follows directly from (3.3) and (2.2).

(2) Let $f \in \mathcal{A}$ and take an arbitrary open set U in E with $a \in U$. By assumption there is a sequence $(f_n) \subset C^\infty_{lfs}(E)$ that separates a from $E \setminus U$. Let $a_U \in E$ with $\varphi(f) = f(a_U)$ and $\varphi(f_n) = f_n(a_U)$ for all n. Then $\varphi(f_n) = f_n(a)$ for all n by (1). Hence $a_U \in U$. In this way we obtain a net (a_U) that converges to a and that satisfies $\varphi(f) = f(a_U)$. Since f is continuous, we have that $\varphi(f) = f(a)$, i.e. $\varphi = \delta_a$.

(3) The point a is countably isolated by $C^\infty_{lfs}(E)$ according to (2) in (5.5) if E is metrizable. Hence the statement follows directly from the countably evaluating property of φ.

(4) This follows in the same way as (2). □

6. Short exact sequences

In analogy to algebraic homology a sequence of continuous linear mappings between locally convex spaces

$$\dots \to E_{k-1} \xrightarrow{f_k} E_k \xrightarrow{f_{k+1}} E_{k+1} \to \dots$$

is called *exact* at E_k if f_k induces an isomorphism from the quotient $E_{k-1}/\ker f_k$ onto the kernel of f_{k+1} as a subspace of E_k. In particular a sequence is called *short exact*, if it is of the form

$$0 \to H \xrightarrow{\iota} E \xrightarrow{\pi} F \to 0$$

and is exact at all the positions H, E and F. Up to isomorphisms this says that H is a closed subspace of E and F is the quotient E/H. A short sequence is called *left exact*, if it is of the form

$$0 \to H \xrightarrow{\iota} E \xrightarrow{\pi} F$$

and is exact at the positions H and E. Up to isomorphisms this says that H is the kernel of the map $\pi : E \to F$.

Our objective in this section is to give conditions on a short (left-)exact sequence, that allows to conclude from the assumption that homomorphisms on algebras $\mathcal{A}_H$ and $\mathcal{A}_F$ on the ends H and F are point evaluations that the same is true for an algebra $\mathcal{A}_E$ on the middle space E. We will require these algebras to satisfy $\pi^*(\mathcal{A}_F) \subseteq \mathcal{A}_E$ and $\iota^*(\mathcal{A}_E) \supseteq \mathcal{A}_H$, the latter one telling us that $\mathcal{A}_H$ functions on H can be extended to $\mathcal{A}_E$ functions on E. Although such a non-linear version of the Hahn-Banach extension theorem is even for smooth functions not true in general (see [FK]), within the context of homomorphisms a study of only those functions on H that are restrictions of $\mathcal{A}_E$ functions turns out to be fruitful. We start by investigating which separating properties the algebra $\mathcal{A}_E$ inherits from the algebras $\mathcal{A}_H$ and $\mathcal{A}_F$.

6.1 Theorem. *Let $0 \to H \xrightarrow{\iota} E \xrightarrow{\pi} F$ be a left exact sequence of locally convex spaces. Let algebras $\mathcal{A}_F$, $\mathcal{A}_E$ and $\mathcal{A}_H$ on F, E and H respectively be given, such that*

(i) $\pi^*(\mathcal{A}_F) \subseteq \mathcal{A}_E$ *and* $\iota^*(\mathcal{A}_E) \supseteq \mathcal{A}_H$.
(ii) *Every point in F is countably isolated by $\mathcal{A}_F$.*
(iii) *$\mathcal{A}_E$ is translation invariant.*

Then we have the following:

(1) *If every point in H is countably isolated by $\mathcal{A}_H$ then every point in E is countably isolated by $\mathcal{A}_E$.*
(2) *If H has small $\mathcal{A}_H$-zerosets, then also E has small $\mathcal{A}_E$-zerosets.*

Proof. (1) Let $x \in E$ be arbitrary. By (ii) there is a sequence (g_n) in $\mathcal{A}_F$ which isolates $\pi(x)$ in F. By the special assumption there exist countably many $h_n \in \mathcal{A}_H$ which isolate 0 in H. According to (i) $g_n \circ \pi \in \mathcal{A}_E$ and there exist $\tilde{h}_n \in \mathcal{A}_E$ with $\tilde{h}_n \circ \iota = h_n$. By (iii) we have that $f_n := \tilde{h}_n(\_ - x) \in \mathcal{A}_E$. Now the functions $\pi^*(g_n)$ together with the sequence (f_n) isolate x. Indeed, if $x' \in E$ is such that $(g_n \circ \pi)(x') = (g_n \circ \pi)(x)$ for all n, then $\pi(x') = \pi(x)$, i.e. $x' - x \in H$. From $f_n(x') = f_n(x)$ we conclude that $h_n(x' - x) = h_n(0)$, and hence $x' = x$.

(2) Let U be a 0-neighborhood in E. By the special assumption there are countably many $h_n \in \mathcal{A}_H$ with $0 \in \bigcap_n Z(h_n) \subseteq U \cap H$. As before consider the sequence of functions $f_n := \tilde{h}_n(\_ - x)$ and using (ii) choose a sequence (g_n) in $\mathcal{A}_F$ which isolates $\pi(x)$ and satisfies $g_n(\pi(x)) = 0$. The common kernel of the functions in the sequences (f_n) and the $(\pi^*(g_n))$ contains x and is contained in $\pi^{-1}(\pi(x)) = x + H$ and hence in $(x + U) \cap (x + H)$. □

If the algebras $\mathcal{A}_H$ and $\mathcal{A}_F$, besides separating properties, also have evaluating properties, the same applies for the algebra $\mathcal{A}_E$:

6.2 Theorem. *Let $0 \to H \xrightarrow{\iota} E \xrightarrow{\pi} F$ be a left exact sequence of locally convex spaces. Let algebras $\mathcal{A}_F$, $\mathcal{A}_E$ and $\mathcal{A}_H$ on F, E and H respectively be given, such that*

(i) $\pi^*(\mathcal{A}_F) \subseteq \mathcal{A}_E$ *and* $\iota^*(\mathcal{A}_E) \supseteq \mathcal{A}_H$.
(ii) *Every point in F is countably isolated by $\mathcal{A}_F$.*
(iii) *$\mathcal{A}_E$ is translation invariant.*
(iv) $\mathrm{Hom}_{\aleph_0} \mathcal{A}_F = F$ *and* $\mathrm{Hom}_{\aleph_0} \mathcal{A}_H = H$.

Then we have the following:

(1) *If every point in H is countably isolated by $\mathcal{A}_H$ then every countably evaluating homomorphism on $\mathcal{A}_E$ is evaluating on $\mathcal{A}_E$.*

(2) *If* $\iota^*(\mathcal{A}_0) = \mathcal{A}_H$ *for an algebra* $\mathcal{A}_0 \subseteq \mathcal{A}_E$, *then every countably evaluating homomorphism on* $\mathcal{A}_E$ *is evaluating on* $\mathcal{A}_0$.

Proof. Let $\varphi \in \mathrm{Hom}_{\aleph_0} \mathcal{A}_E$. Then $\varphi \circ \pi^* : \mathcal{A}_F \to \mathbb{R}$ is a countably evaluating homomorphism, and hence by (iv) given by the evaluation at a point $y \in F$. By (ii) there is a sequence (g_n) in $\mathcal{A}_F$ which isolates y. Since φ is countably evaluating there exists a point $x \in E$, such that $g_n(y) = \varphi(\pi^*(g_n)) = \pi^*(g_n)(x) = g_n(\pi(x))$ for all n. Hence $y = \pi(x)$. Since φ obviously evaluates each countable set in $\mathcal{A}_E$ at a point in $\pi^{-1}(y)$, it induces a countably evaluating homomorphism $\varphi_H : \mathcal{A}_H \to \mathbb{R}$ by $\varphi_H(\iota^*(f)) := \varphi(f(\_ - x))$. By (iv), φ_H is given by the evaluation at a point $z \in H$.

(1) We show that $\varphi = \delta_{z+x}$ on $\mathcal{A}_E$. Indeed, by the special assumption there is a sequence (h_n) in $\mathcal{A}_H$ which isolates z. By (i) and (iii), we may find $f_n \in \mathcal{A}_E$ such that $h_n = \iota^*(f_n(\_ + x))$. The sequences $(\pi^*(g_n))$ and (f_n) isolate $z + x$. So let $f \in \mathcal{A}_E$ be arbitrary. Then there exists a point $z' \in E$, such that $\varphi = \delta_{z'}$ for all these functions, hence $z' = z + x$.

(2) Here we have that $\mathcal{A}_H = \iota^*(\mathcal{A}_0)$, and hence

$$\varphi(f) = \varphi_H(\iota^*(f(\_ + x))) = \iota^*(f(\_ + x))(z) = f(\iota(z) + x)$$

for all $f \in \mathcal{A}_0$. So φ is evaluating on $\mathcal{A}_0$. □

Remark. The previous two results remain true if $\pi : E \to F$ is replaced by a sequence $\pi_n : E \to F_n$ and H denotes the common kernel, i.e. the intersection $\bigcap_n \pi_n^{-1}(0)$. In fact let $F := \prod_n F_n$ and $\pi = (\pi_n)_n$ and $\mathcal{A}_F$ be the algebra generated by ${\pi_n}^*(\mathcal{A}_{F_n})$. Then all assumptions involving $\mathcal{A}_F$ follow from the corresponding ones for the $\mathcal{A}_{F_n}$ and hence the conclusions for $\mathcal{A}_E$ follow from the previous result.

6.3 Corollary. *Let* $0 \to H \xrightarrow{\iota} E \xrightarrow{\pi} F$ *be a left exact sequence of locally convex spaces and let* $\mathcal{A}_F$, $\mathcal{A}_E$ *and* $\mathcal{A}_H$ *be algebras on* F, E *and* H *respectively that satisfy the assumptions (i-iii) of (6.2) and* $P_f(E) \subseteq \mathcal{A}_E$ *as well. Let furthermore* $\varphi : \mathcal{A}_E \to \mathbb{R}$ *be countably evaluating. Then there is some point* $a \in E$ *with*

(1) $\varphi = \delta_a$ *on* $\mathcal{A}_E$ *if* $(H, \sigma(H, H'))$ *is realcompact and admits small* P_f*-zerosets and* $\mathcal{A}_E \subseteq C(E)$.
(2) $\varphi = \delta_a$ *on* $\mathcal{A}_0$ *if* $(H, \sigma(H, \iota^*(\mathcal{A}_0)))$ *is Lindelöf and* $\mathcal{A}_0 \subseteq \mathcal{A}_E$ *is an algebra.*
(3) $\varphi = \delta_a$ *on* E' *if* $(H, \sigma(H, H'))$ *is realcompact.*

Proof. We are able to apply (2) of (6.2) in all three cases, because of the following arguments.

(1) Let $\mathcal{A}_0 := \mathcal{A}_E$ and $\mathcal{A}_H := \iota^*(\mathcal{A}_0)$. Then $\iota^*(\mathcal{A}_0) \subseteq C(H)$ and thus $\mathrm{Hom}_{\aleph_0} \iota^*(\mathcal{A}_0) = H$ by (5.6) using (5.1).
(2) If H is $\sigma(H, \iota^*(\mathcal{A}_0))$-Lindelöf, then $H = \mathrm{Hom}_{\aleph_0} \iota^*(\mathcal{A}_0)$.
(3) Let $\mathcal{A}_0 := P_f(E)$ and $\mathcal{A}_H := P_f(H) = \iota^*(\mathcal{A}_0)$. If H is $\sigma(H, H')$-realcompact, then $H = \mathrm{Hom}_{\aleph_0} P_f(H)$, by (5.1). □

By (5.5) all lcs that are isomorphic to closed subspaces of products of lcs E_i, where E_i is Lindelöf or E_i' is weak*-separable, belong to the class of spaces that are weakly realcompact by the remark after (5.1) and that admit small P_f-zerosets by (5.5). However $c_0(\Gamma)$ for uncountable Γ doesn't belong to this class, since finite type polynomials on it depend only on countably many coordinates.

For the case $H = c_0(\Gamma)$ we now construct a crucially big algebra $\mathcal{A}_H$ of smooth functions that extend to E. As the following generalization of proposition 1 in [DGZ] shows, the regularity conditions on the ends H and F transfer to E provided that $H \xrightarrow{\iota} E \xrightarrow{\pi} F$ is a short exact sequence.

6.4 Theorem. *Let $c_0(\Gamma) \xrightarrow{\iota} E \xrightarrow{\pi} F$ be a short exact sequence of locally convex spaces with $\mathcal{A}_E$ being translation invariant and containing $(\pi^*(\mathcal{A}_F) \cup E')^\infty_{lfs}$. If F is $\mathcal{A}_F$-regular then E is $\mathcal{A}_E$-regular. Moreover the algebra $\mathcal{A}_{c_0(\Gamma)}$ generated by all locally finite products of the form $\prod_{\lambda\in\Lambda} \eta_\lambda \circ l_\lambda$, with $\eta_\lambda \in C^\infty(\mathbb{R})$ and $l_\lambda \in c_0(\Gamma)'$, is contained in $\iota^*(\mathcal{A}_E)$ whenever $|\Lambda|$ and $|\Gamma|$ are non-measurable.*

Proof. Let us show that the function $f : x \mapsto \prod_{\lambda\in\Lambda}(\eta_\lambda \circ l_\lambda)(x)$ can be extended to a function in $\mathcal{A}_E$. Without loss of generality we may assume $l_\lambda \neq 0$, $\eta_\lambda \neq 1$, and, by composition with a homothety, $\|l_\lambda\| = 1$ for all $\lambda \in \Lambda$.
Let p be an extension of the supremum norm $\|\_\|_\infty$ on $c_0(\Gamma)$ to a continuous seminorm on E, and let $\tilde{p}$ be the corresponding quotient seminorm on F defined by $\tilde{p}(y) := \inf\{p(x) : \pi(x) = y\}$. Let furthermore $\tilde{l}_\lambda$ be continuous linear extensions of $l_\lambda : c_0(\Gamma) \to \mathbb{R}$ which satisfy $|\tilde{l}_\lambda(x)| \le p(x)$ for all $x \in E$.

We first claim that there exists a finite subset $F \subseteq \Lambda$ and a $\tau > 0$ such that $\eta_\lambda = 1$ on $[-\tau, \tau]$ for all $\lambda \notin F$:
Since the product is assumed to be locally finite, there exists a finite subset $F \subseteq \Lambda$ and a $\tau > 0$ such that for all $\|x\| \le \tau$ and all $\lambda \notin F$ we have $\eta_\lambda(l_\lambda(x)) = 1$. Since $\|l_\lambda\| = 1$ we have $l_\lambda(\{x : \|x\| \le \tau\}) \supseteq]-\tau, \tau[$. Hence $\eta_\lambda(t) = 1$ for all $|t| < \tau$.

Without loss of generality we may assume that $F = \emptyset$ and we define $\varepsilon_\lambda := \inf\{|t| : \eta_\lambda(t) \neq 1\} > 0$ and choose a positive $\varepsilon < \inf\{\varepsilon_\lambda : \lambda\}$.

We show first, that for $x \in E$ with $\tilde{p}(\pi(x)) < \varepsilon$ the product $\prod_{\lambda\in\Lambda} \eta_\lambda(\tilde{l}_\lambda(x))$ is locally finite as well.

We consider first the case that $\varepsilon_\lambda \to \infty$ (i.e. $\{\lambda : \varepsilon_\lambda < N\}$ is finite for all N). Thus $|\tilde{l_\lambda}(x)| \le N$ for all x with $p(x) \le N$ and hence $\tilde{f} := \prod_{\lambda\in\Lambda} \eta_\lambda(\tilde{l_\lambda})$ yields a locally finite extension of f.

Next consider the case where ε_λ is bounded from above by some N.
We claim that $l_\lambda \to 0$ pointwise. Suppose this is not the case. Then there exist $x \in c_0(\Gamma)$ and $\tau > 0$ such that $|l_\lambda(x)| \ge \tau$ for infinitely many λ. So we may find infinitely many $s_\lambda \in [N, N+1]$ such that$|l_\lambda(\frac{s_\lambda}{\tau}x)| \ge N$. Since the s_λ are bounded we find an accumulation point s. So the product is not locally finite at $\frac{s}{\tau}x$.
Next we show point finiteness. For this it suffices to show that $\Lambda_x := \{\lambda : |\tilde{l}_\lambda(x)| \ge \varepsilon\}$ is finite. In fact, assume this set where infinite, say, containing a sequence (λ_n). Since the family $\{\tilde{l}_{\lambda_n} : n \in \mathbb{N}\}$ is equicontinuous it is by the Alaoglu-Bourbaki-theorem relatively compact for the topology of pointwise convergence. So the sequence $(\tilde{l}_{\lambda_n})$ has some cluster point $\tilde{l} \in E'$. From $|\tilde{l}_\lambda| \le p$ we conclude $|\tilde{l}| \le p$ and furthermore for $z \in c_0(\Gamma)$ we have $\tilde{l}(z) = \lim_n \tilde{l}_{\lambda_n}(z) = 0$ since $l_\lambda \to 0$ pointwise. So

$$\varepsilon > \tilde{p}(\pi(x)) = \inf\{p(x') : \pi(x') = \pi(x)\} \geq \inf\{|\tilde{l}(x')| : \pi(x') = \pi(x)\} = |\tilde{l}(x)| \geq \varepsilon$$

gives a contradiction.
Now the local finiteness. For this take $y \in E$ with $p(y-x) \leq \inf\{\varepsilon_\lambda : \lambda\} - \varepsilon$. Then for $\lambda \notin \Lambda_x$ we have

$$|\tilde{l}_\lambda(y)| \leq |\tilde{l}_\lambda(x)| + |\tilde{l}_\lambda(y-x)| < \varepsilon + p(y-x) \leq \inf\{\varepsilon_\lambda : \lambda\},$$

and hence the product is finite on $\{y : \tilde{p}(y-x) < \inf\{\varepsilon_\lambda : \lambda\} - \varepsilon\}$.

For the general case let $\Lambda_n := \{\lambda \in \Lambda : n < \varepsilon_\lambda \leq n+1\}$. By the previous case there exists a finite subset $F_n \subseteq \Lambda_n$ such that for $\lambda \in \Lambda_n \setminus F_n$ the corresponding factor is identical to 1 on the set $\{y : \tilde{p}(y-x) < \inf\{\varepsilon_\lambda : \lambda\} - \varepsilon\}$. The remaining product $\prod_n \prod_{\lambda \in F_n} \eta_\lambda \circ \tilde{l}_\lambda$ is locally finite. So the full product $\prod_{\lambda\in\Lambda} \eta_\lambda \circ \tilde{l}_\lambda$ locally coincides with it and hence is a locally finite extension of f on $\{x : \tilde{p}(\pi(x)) < \varepsilon\}$.

In order to obtain the required extension to all of E, we choose a function $g \in \mathcal{A}_F$ with carrier contained inside $\{z : \tilde{p}(z) \leq \varepsilon\}$ and with $g(0) = 1$. Then $f : E \to \mathbb{R}$, defined by

$$f(x) := g(\pi(x)) \prod_{\lambda\in\Lambda} \eta_\lambda(\tilde{l}_\lambda(x)),$$

is an extension belonging to $(\pi^*(\mathcal{A}_F) \cup E')^\infty_{lfs}$.

Let us now show that we can find such an extension with arbitrary small carrier, for $\Lambda := \Gamma$, $l_\gamma := \mathrm{pr}_\gamma$ and $\eta_\gamma = \eta \in C^\infty(\mathbb{R})$, where $\eta = 1$ locally around 0 and has compact support. In this case the product is automatically locally finite.
So let an arbitrary seminorm p on E be given. Then there exists an $M > 0$ such that $p|_{c_0(\Gamma)} \leq M\, \|\text{-}\|_\infty$. Let q be an extension of $\|\text{-}\|_\infty$ to a continuous seminorm on E. By replacing p by $\max\{p, M\,q\}$ we may assume that $p|_{c_0(\Gamma)} = M\,\|\text{-}\|_\infty$. Without loss of generality we may assume that $M = 1$. Let again $\tilde{p}$ be the corresponding quotient norm on F. Then the construction above gives a function $f \in \mathcal{A}_E$, where the coordinate projections have to be extended to functionals $\tilde{\mathrm{pr}}_\gamma$ satisfying $|\tilde{\mathrm{pr}}_\gamma| \leq p$. It remains to show that the support of f is bounded with respect to the p-seminorm (using a homothety we can then achieve that its support is small with respect to p): Let $x \in E$ be such that $f(x) \neq 0$. Then on one hand $g(\pi(x)) \neq 0$ and hence $\tilde{p}(\pi(x)) \leq \varepsilon$ and on the other hand $\eta(\tilde{\mathrm{pr}}_\gamma(x)) \neq 0$ for all $\gamma \in \Gamma$ and hence $|\tilde{\mathrm{pr}}_\gamma(x)| \leq K$, where $K := \sup\{|t| : t \in \operatorname{supp}\eta\} < \infty$. We consider the space $E/\ker p$ with the norm induced by p. The map

$$\tilde{T} : \ell_1(\Gamma) \cong c_0(\Gamma)' \xrightarrow{T} (E/\ker p)' \xrightarrow{\pi_1^*} E'$$

is continuous and linear, where T denotes the unique norm 1 extension map, which maps pr_γ to $\tilde{l}_\gamma$ defined by $\tilde{l}_\gamma \circ \pi_1 = \tilde{\mathrm{pr}}_\gamma$, and π_1^* denotes the map induced by the continuous projection $\pi_1 : E \to E/\ker p$. Then $\iota^* \circ \tilde{T} = 1$ and $\tilde{T}(\mathrm{pr}_\gamma) = \tilde{l}_\gamma \circ \pi_1 = \tilde{\mathrm{pr}}_\gamma$ for all $\gamma \in \Gamma$. So $|\tilde{T}\,\mathrm{pr}_\gamma(x)| = |\tilde{\mathrm{pr}}_\gamma(x)| \leq K$ and since the unit ball of $\ell_1(\Gamma)$

is the closed absolutely convex hull of $\{\mathrm{pr}_\gamma : \gamma \in \Gamma\}$, we have for all h in the unit ball of $\ell_1(\Gamma)$ that $|\tilde{T}h(x)| \le K$.

Now let $l \in E'$ be such that $|l| \le p$, hence $l|_{c_0(\Gamma)}$ is in the unit ball of $\ell_1(\Gamma)$ and $|\tilde{T}(\iota^*(l))(x)| \le K$. Then $l_0 := (\tilde{T} \circ \iota^* - 1)(l) = \tilde{T}(l|_{c_0(\Gamma)}) - l \in E'$ vanishes on $c_0(\Gamma)$ and $|l_0| \le 2|p|$ and hence $|l_0| \le 2\tilde{p} \circ \pi$ as above. So $|l(x)| \le |\tilde{T}(l|_{c_0(\Gamma)})(x)| + |l_0(x)| \le K + 2\tilde{p}(\pi(x)) \le K + 2\varepsilon$ and thus $p(x) = \sup\{|l(x)| : |l| \le p\} \le K + 2\varepsilon$. □

Open Problem. *Let $H \xrightarrow{\iota} E \xrightarrow{\pi} F$ be a short exact sequence of locally convex spaces with algebras $\mathcal{A}_F$, $\mathcal{A}_E$ and $\mathcal{A}_H$ on F, E and H respectively. If H is $\mathcal{A}_H$-regular, F is $\mathcal{A}_F$-regular and $\iota^*(\mathcal{A}_E) \supseteq \mathcal{A}_H$, is then E always $\mathcal{A}_E$-regular?*

Remark. The standard example of a short exact sequence of Banach spaces, where the ends H and F are Hilbert spaces and hence C^∞-regular, but E is not, does not give a counter example here because it fails the extension property in (i) of (6.1). Indeed, let $\ell^2 \to E \to F$ be a short exact sequence of Banach spaces, which does not split. We claim that the square of the norm on ℓ^2 does not extend to a smooth function on E. Assume that such a smooth extension $f : E \to \mathbb{R}$ would exist. Then the second derivative at 0 would be a continuous bilinear symmetric mapping $b : E \times E \to \mathbb{R}$, which extends the inner product on ℓ^2. However, since the sequence does not split, the isomorphism $\ell^2 \to (\ell^2)'$ given by $x \mapsto \langle x, \_ \rangle$ has no extension to a continuous linear map $E \to (\ell^2)'$, but the map $x \mapsto b(x, \_)$ would be such an extension.

6.5 Corollary. *Let $H = c_0(\Gamma) \xrightarrow{\iota} E \xrightarrow{\pi} F$ be a short exact sequence of locally convex spaces and let $\mathcal{A}_E$ translation invariant and containing $(\pi^*(\mathcal{A}_F) \cup E')^\infty_{lfs}$. Assume furthermore that F be $\mathcal{A}_F$-regular and each point in F is isolated by a countable set of functions in $\mathcal{A}_F$. Then the points in E are countably isolated with respect to $\mathcal{A}_E$ and E is $\mathcal{A}_E$-regular. Moreover, $F = \mathrm{Hom}_{\aleph_0} \mathcal{A}_F$ yields that $E = \mathrm{Hom}_{\aleph_0} \mathcal{A}_E$.*

Proof. Because of the extension property $\mathcal{A}_{c_0(\Gamma)} \subseteq \iota^*(\mathcal{A}_E)$ in (6.4) we can apply (1) of (6.1) to obtain the first statement. The second then follows from (1) in (6.2). □

That in (6.5) we can in fact replace $c_0(\Gamma)$ by spaces from a huge class is made possible by the following result:

6.6 Lemma. *Given a short exact sequence $H \xrightarrow{\iota} E \xrightarrow{\pi} F$ and a continuous linear map $T : H \to H_1$. Then there is a short exact sequence $H_1 \xrightarrow{\iota_1} E_1 \xrightarrow{\pi_1} F$ and an extension $\tilde{T} : E \to E_1$ of T, with $\ker T = \ker \tilde{T}$, such that the following diagram commutes*

$$\begin{array}{ccccc} H & \xrightarrow{\iota} & E & \xrightarrow{\pi} & F \\ {\scriptstyle T}\downarrow & & {\scriptstyle \tilde{T}}\downarrow & & \| \\ H_1 & \xrightarrow{\iota_1} & E_1 & \xrightarrow{\pi_1} & F \end{array}$$

Moreover, if T is a quotient map then so is $\tilde{T}$, i.e. $\ker T \to E \xrightarrow{\tilde{T}} E_1$ is short exact.

Proof. Without loss of generality we may assume that $\iota : H \to E$ is the embedding of a closed subspace and $\pi : E \to F$ the quotient map to $F = E/H$. Set $E_1 := (H_1 \times E)/\{(T(z), z) : z \in H\}$. Since H is closed in E, also E_1 is a Hausdorff lcs. Define $\iota_1 : H_1 \to E_1$ by $\iota_1(u) = [(u, 0)]$ and $\pi_1 : E_1 \to F$ by $\pi_1([(u, z)]) = -\pi(z)$. We have that ι_1 is injective, since $(u, 0) \in \{(T(z), z) : z \in H\}$ implies $0 = z$ and $u = T(z) = T(0) = 0$. The mapping π_1 is obviously well-defined and $0 = \pi_1[(u, z)] = -\pi(z)$ iff $z \in H$, i.e. $[(u, z)] = [(u - Tz, 0)]$ lies in the image of ι_1. In order to show that the sequence

$$0 \to H_1 \xrightarrow{\iota_1} E_1 \xrightarrow{\pi_1} F,$$

is short exact, it remains to prove that ι_1 is a topological embedding. So let U be an absolutely convex 0-neighborhood in H_1. Since ι is a topological embedding there is a 0-neighborhood W in E with $W \cap H = T^{-1}(U)$. Now consider the image of $U \times W \subseteq H_1 \times E$ under the quotient map $H_1 \times E \to E_1$. This is a 0-neighborhood in E_1 and its inverse image under ι_1 is contained in $2U$. Indeed, if $[(u, 0)] = [(x, z)]$ with $u \in H_1$, $x \in U$ and $z \in W$, then $x - u = T(z)$ and $z \in H \cap W$, by which $u = x - T(z) \in U - U = 2U$. Hence ι_1 embeds H_1 topologically into E_1.

The map $\tilde{T} : E \to E_1$, given by $\tilde{T}(x) = [(0, -x)]$, is obviously a well defined continuous linear operator. Also, since $T(x) = 0$ if and only if $[(0, -x)] = [(0, 0)]$, we have that $\ker T = \ker \tilde{T}$.

Assume that T, in addition, is a quotient map. Given any $[(y, x)] \in E_1$, there is then some $z \in H$ with $T(z) = y$. Thus $\tilde{T}(z - x) = [(0, x - z)] = [(T(z), x)] = [(y, x)]$ and $\tilde{T}$ is onto. Remains to prove that $\tilde{T}$ is open. Take a 0-neighborhood $U \subseteq E$. Then $U \cap H$ is a 0-neighborhood in H and hence $T(U \cap H)$ is a 0-neighborhood in H_1. Let V be the 0-neighborhood given by image of $T(U \cap H) \times U$ under the quotient map $H_1 \times E \to E_1$. For any $v \in V$ there is some representant $(T(h), u) \in T(U \cap H) \times U$ with $v = [(T(h), u)] = [(0, u - h)] = \tilde{T}(h - u)$ and $h - u \in U - U = 2U$. Therefore $V \subseteq 2\tilde{T}(U)$ and $\tilde{T}$ is open. □

Definition. Let c_0-ext be the class of spaces H, for which there are finitely many short exact sequences $c_0(\Gamma_j) \to H_j \to H_{j+1}$, $j = 1, ..., n$, with $|\Gamma_j|$ non-measurable, $H_{n+1} = \{0\}$ and $T : H \to H_1$ an operator whose kernel has a weak*-separable dual.

Remark. Of course c_0-inj is a subclass of c_0-ext. Moreover, there are natural spaces in c_0-ext besides those in c_0-inj. For example let K be a compact space with $|K|$ non-measurable and $K^{(\omega_0)} = \emptyset$, where ω_0 is the first infinite ordinal and $K^{(\omega_0)}$ the corresponding ω_0-th derived set. Then the Banach space $C(K)$ belongs to c_0-ext, but is in general not even injectable into some $c_0(\Gamma)$, see [GPWZ]. In fact, from $K^{(\omega_0)} = \emptyset$ and the compactness of K, we conclude that $K^{(n)} = \emptyset$ for some integer n. We have the short exact sequence

$$c_0(K \setminus K^{(1)}) \cong E \xrightarrow{\iota} C(K) \xrightarrow{\pi} C(K)/E \cong C(K^{(1)}),$$

where $E := \{f \in C(K) : f|_{K^{(1)}} = 0\}$. By using (6.4) inductively the space $C(K)$ is C^∞_{lfs}-regular.

6.7 Theorem. *Let $H \xrightarrow{\iota} E \xrightarrow{\pi} F$ be a short exact sequence of locally convex spaces and let $\mathcal{A}_F \supseteq C^\infty_{lfs}(F)$ where F is C^∞_{lfs}-regular and each point in F is isolated by a countable set of functions in $C^\infty_{lfs}(F)$. Let $\mathcal{A}_E$ be translation invariant and containing $(\pi^*(\mathcal{A}_F) \cup E')^\infty_{lfs}$, $\mathcal{A}_H \supseteq C^\infty_{lfs}(H)$ and let H be of class c_0-ext.*

Then the points in E are countably isolated with respect to $\mathcal{A}_E$ and E is C^∞_{lfs}-regular. Under the above assumptions, $F = \operatorname{Hom}_{\aleph_0} \mathcal{A}_F$ implies that $E = \operatorname{Hom}_{\aleph_0} \mathcal{A}_E$.

Proof. According to (6.6) the diagram

$$\begin{tikzcd} H \arrow[r, "\iota"] \arrow[d, "T"'] & E \arrow[r, "\pi"] \arrow[d, "\tilde{T}"'] & F \arrow[d, equal] \\ H_1 \arrow[r, "\iota_1"] & E_1 \arrow[r, "\pi_1"] & F \end{tikzcd}$$

commutes, where $(\ker \tilde{T})'$ is weak*-separable. Hence the statement follows by (1) in (6.2) provided that the points in E_1 are separated by the algebra $\mathcal{A}_{E_1} := (\pi_1^*(\mathcal{A}_F) \cup E_1')^\infty_{lfs}$ and that $E_1 = \operatorname{Hom}_{\aleph_0} \mathcal{A}_{E_1}$. On the other hand, this latter condition, together with the additional property that E_1 is $\mathcal{A}_{E_1}$-regular, can be proved by induction on the length n of the resolution $H_1, H_2, \ldots H_{n+1}$ using (6.6) and (6.5). We can assume that this result holds already for sequences starting with H_2. According to (6.6) we have the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & F \arrow[r, equal] & F \\ c_0(\Gamma_1) \arrow[u] \arrow[r] & E_1 \arrow[u] \arrow[r] & (E_1)_1 \arrow[u] \\ c_0(\Gamma_1) \arrow[u, equal] \arrow[r] & H_1 \arrow[u] \arrow[r] & H_2 \arrow[u] \\ 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] \end{tikzcd}$$

where $c_0(\Gamma_1) \to E_1 \to (E_1)_1$ is short exact. By (6.5) the properties for the space $(E_1)_1$ transform to the space E_1, which proves the theorem. □

7. A stable class

Definition. Let RZ denote the class of all lcs E admitting small C^∞_{lfs}-zerosets and such that $E = \operatorname{Hom}_{\aleph_0} \mathcal{A}$ for any algebra $\mathcal{A}$ with $C^\infty_{lfs}(E) \subseteq \mathcal{A} \subseteq C(E)$.

A combination of (6.1), (6.2) and (6.7) yield:

7.1 Theorem. *Let E be a lcs and assume that there is a continuous linear map $T : E \to F$, where $\ker T$ admits small P_f-zerosets and is weakly realcompact and where F is a lcs admitting a closed subspace $H \in c_0$-ext such that F/H is*

C^∞_{lfs}-regular, each point in F/H is countably isolated by $C^\infty_{lfs}(F/H)$ and $F/H = \operatorname{Hom} C^\infty_{lfs}(F/H)$. Then $E \in RZ$.

The class RZ is big. Of course we have that c_0-ext is a subclass of RZ. Members of RZ are also the following spaces:

- the spaces $C(K)$ where K is the one-point compactification of the topological disjoint union of a sequence of compact spaces K_n with $K_n^{(\omega_0)} = \emptyset$. The restriction maps $C(K) \to C(K_n)$ give an embedding $T : C(K) \to F := C(\{\infty\}) \times \prod_n C(K_n)$. Applying (7.1) with $H = \{0\}$ we see that $C(K)$ is in RZ (the factors $C(K_n)$ belong to c_0-ext according to the remark before (6.7) and the properties needed are stable under formation of countable products, see (7.7)). Remark that in such a situation we might have $K^{(\omega_0)} = \{\infty\} \neq \emptyset$.

- The space $D[0,1]$ of all functions $f : [0,1] \to \mathbb{R}$ which are right continuous and have left limits and endowed with the sup norm is in RZ. Indeed it contains $C[0,1]$ as a subspace and $D[0,1]/C[0,1] \cong c_0[0,1]$ according to [C].

For $H = \{0\}$ the condition on the quotient space $F = F/H$ in (7.1) to be C^∞_{lfs}-regular can be replaced by its metrizability still yielding the same result by (6.3). In particular, we then have:

7.2 Corollary. *Let H be a closed subspace of E, where H is $\sigma(H, H')$-realcompact and admits small P_f-zerosets and where $E/H \in RZ$ is a metrizable lcs. Then $E \in RZ$.*

Taking the remark preceding (6.3) in consideration we get:

7.3 Corollary. *Let (E_n) be a sequence of metrizable spaces in RZ and set $E := \bigoplus_{n\in\mathbb{N}} E_n$, the countable locally convex direct sum of (E_n). For an algebra $\mathcal{A}$ with $C^\infty_{lfs}(E) \subseteq \mathcal{A}$ we have that $E = \operatorname{Hom}_{\aleph_0} \mathcal{A}$.*

By a similar argumentation as that in (7.3) we get:

7.4 Corollary. *The class of Banach spaces in RZ is closed under formation of countable c_0-sums and ℓ_p-sums with $1 \le p \le \infty$.*

Only cosmetic changes in theorem 1 in [BL$_2$] need to be done in order to get:

7.5 Proposition. *The class RZ is closed under formation of arbitrary products and closed subspaces.*

Results in [CO] yield that each reflexive Fréchet space is a closed subspace of a product of Banach spaces the dual unit balls being Talagrand compact in the weak* topology. Since Talagrand compact spaces are Corson compact, (7.5) yields:

7.6 Corollary. *The class RZ contains all reflexive Fréchet spaces of non-measurable cardinality.* □

In [KM] it was proved that the property $E = \operatorname{Hom} \mathcal{A}$ is closed under formation of arbitrary products and closed subspaces provided that the spaces E admit large carriers of class $\mathcal{A}$ and that the algebras are 1-evaluating. Now the weak topology is C^∞_{lfs}-regular and hence, by means of (5.6) we arrive at:

7.7 Proposition. *The class of lcs E with $E = \operatorname{Hom} C^\infty_{lfs}(E)$ is closed under formation of arbitrary products and closed subspaces.*

Each Banach space is a closed subspace of $\ell^\infty(\Gamma)$, where Γ is the closed dual unit-ball. Hence, given a set Γ with $|\Gamma|$ non-measurable, an affirmative answer to the question $\ell^\infty(\Gamma) \in RZ$ would translate to $E \in RZ$ for any complete lcs E of non-measurable cardinality. Although $\ell^\infty(\Gamma) = \operatorname{Hom} C^\infty_{lfs}(\ell^\infty(\Gamma))$ remains unsolved for $|\Gamma|$ non-measurable, it should be of interest to solve the following:

Open Problem. *Are the points in $\ell^\infty(\Gamma)$ countably isolated by $C^\infty_{lfs}(\ell^\infty(\Gamma))$ for $|\Gamma|$ non-measurable?*

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