Paulo R. C. Ruffino; Luiz A. B. San Martin

Lyapunov exponents for stochastic differential equations on semi-simple Lie groups

Archivum Mathematicum, Vol. 37 (2001), No. 3, 207--231

Persistent URL: `http://dml.cz/dmlcz/107799`

ARCHIVUM MATHEMATICUM (BRNO)
Tomus 37 (2001), 207 – 231

LYAPUNOV EXPONENTS FOR STOCHASTIC DIFFERENTIAL EQUATIONS ON SEMI-SIMPLE LIE GROUPS

PAULO R. C. RUFFINO* AND LUIZ A. B. SAN MARTIN**

Abstract. With an intrinsic approach on semi-simple Lie groups we find a Furstenberg–Khasminskii type formula for the limit of the diagonal component in the Iwasawa decomposition. It is an integral formula with respect to the invariant measure in the maximal flag manifold of the group (i.e. the Furstenberg boundary $B = G/MAN$). Its integrand involves the Borel type Riemannian metric in the flag manifolds. When applied to linear stochastic systems which generate a semi-simple group the formula provides a diagonal matrix whose entries are the Lyapunov spectrum. Some Brownian motions on homogeneous spaces are discussed.

1. Introduction

In this article we consider right invariant stochastic differential equations in a semi-simple Lie group G with the purpose of studying the asymptotic time average of the logarithm of the A-part of the Iwasawa decomposition of the trajectories. After constructing a convenient radial-spherical decomposition, we get an integral formula by applying a Furstenberg-Khasminskii type argument. Interesting algebraic and geometrical interpretations come out of this formula when we consider the Borel type metric on the flag manifolds.

The motivation for having such a formula is that, for many well known interesting systems, that limit describes the stability of the system since it contains the Lyapunov spectrum. Among those systems, the linear ones have been quite well studied by several authors who developed formulae in different contexts. We mention, for instance, Khasminskii [16], Arnold, Kliemann and Oeljeklaus [1], Arnold, Oeljeklaus and Pardoux [2] for linear systems, and Carverhill [7], [8], Arnold and San Martin [24] for extensions to nonlinear systems.

2000 *Mathematics Subject Classification*: 60H10 (58G32, 22E46).
Key words and phrases: Lyapunov exponents, stochastic differential equations, semi-simple Lie groups, flag manifolds.
*Research partially supported by CNPq grant no. 300670/95-8 and FAPESP 96/3134-0.
**Research partially supported by CNPq grant no. 301060/94-0 and FAPESP 96/3134-0.
Received February 11, 2000.

Most of those formulae detect only the top Lyapunov exponent. A creative method to calculate the whole Lyapunov spectrum was established by Baxendale [5]. He used the same kind of argument for the calculation of the top exponent but applied to the induced system on the Grassmannian $\mathrm{Gr}_k(n)$. Another approach to find the whole spectrum is due to Arnold and Imkeller [3], who got formulae to calculate these numbers via anticipative calculus, where each exponent is given by a Khasminskii type formula plus a correction term which are expressed in terms of a Malliavin derivative of the orthogonal projectors on the Osseledets spaces.

Dealing with the Iwasawa decomposition of systems in $\mathrm{Sl}(n,\mathbb{R})$, Liao [19], in a geometrical context analogous to [5], obtained the whole spectrum as an integral formula with respect to an invariant measure for the induced system on the special orthogonal group. The intrinsic geometric approach of this paper allows to extend the results in [19] to systems evolving in arbitrary semi-simple Lie groups. Here however we work intrinsically in a general semi-simple Lie group. The link to linear systems is established either by taking linear representations of the group or by starting with a linear system and assuming that the Lie algebra generated by its coefficients is semi-simple. The advantage of this intrinsic set up is that the assumptions regarding the non-degeneracy of the systems are less demanding, in the sense that it requires only that the Lie algebra generated by the system is semi-simple.

The intrinsic approach also allows applications to other systems, like the geodesic flow in symmetric spaces (see Malliavin and Malliavin [21] and Carverhill and Elworthy [10]) or geodesic systems and other kinds of Brownian motions.

This article is organized as follows: in section 2 we present some algebraic preliminaries on semi-simple Lie algebra, flag manifolds and the Borel type metric. Section 3 shows that the homogeneous space G/MN is a trivial principal fibre bundle over the maximal flag manifold with A as the structural group such that there exists a spherical-radial decomposition of this space. In section 4 we study the asymptotic behavior of solutions of stochastic differential equations in these radial fibres. Then, we close the argument in section 5 where we show that for many interesting systems (*cf.* Guivarc'h and Raugi [13]) this asymptotic behavior, as limiting elements in $\mathfrak{a}$, provides the Lyapunov spectrum of the system. Finally, in section 6 we calculate some geometrically interesting examples.

We mention that although we work in the semi-simple context, the results are easily extended to a reductive Lie algebra, that is, which decomposes as a sum of a semi-simple Lie algebra plus the center. At this regard recall that a Lie algebra of matrices which is irreducible in the sense that it does not have invariant proper subspaces, is reductive. This implies that this method applies also to linear systems which generate an irreducible Lie algebra of matrices.

After the conclusion of this paper we became aware of similar results of Liao [20], which also work in the general setting of semi-simple Lie groups. Contrary to [20], here we write a formula for the Lyapunov exponent as an integral on the flag manifolds, factoring further the formula of [20].

2. Algebraic Preliminaries

The purpose of this section is to present some known algebraic and geometrical facts about semi-simple Lie groups, their algebras and associated flag manifolds. We refer to Helgason [14] or Warner [27] for unexplained concepts.

Before starting we set the following notation: if G is a Lie group, a homogeneous space of G is a coset space G/H with H a closed subgroup. By left translation, G acts transitively on G/H. Let $\mathfrak{g}$ be the Lie algebra of G and take $X \in \mathfrak{g}$. Then X induces the vector field $\tilde{X}$ on G/H given by

$$\tilde{X}(x) = \frac{d}{dt}(\exp tX)(x)_{|t=0}$$

whose flow is the action of $\exp(tX)$, $t \in \mathbb{R}$, on G/H. When it is necessary to emphasize the specific homogeneous space G/H the induced vector field will be denoted by $\tilde{X}\Big|_{G/H}$. Since the action is transitive the tangent space at x is $T_x(G/H) = \{\tilde{X}(x) : X \in \mathfrak{g}\}$.

2.1. **Semi-simple Lie algebras.** Let $\mathfrak{g}$ be a semi-simple Lie algebra. Given a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$, let θ stand for the corresponding Cartan involution ($\theta = \mathrm{id}$ in $\mathfrak{k}$ and $\theta = -\mathrm{id}$ in $\mathfrak{s}$). Let $\Pi \subset \mathfrak{a}^*$ stand for the set of roots of the pair $(\mathfrak{g}, \mathfrak{a})$; the eigenvalues of $\mathrm{ad}_{\mathfrak{g}}(H)$, $H \in \mathfrak{a}$ are 0 and $\alpha(H)$, $\alpha \in \Pi$. The root space

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : [H, X] = \alpha(H) X \text{ for all } H \in \mathfrak{a}\}$$

is the common eigenspace for $\mathrm{ad}_{\mathfrak{g}}(H)$ associated with the eigenvalue $\alpha(H)$, $\alpha \in \Pi$. By fixing a lexicographic order in the dual $\mathfrak{a}^*$ of $\mathfrak{a}$ we have $\Pi = \Pi^+ \cup \Pi^-$ where Π^+ is the set of positive roots with respect to this order, and $\Pi^- = -\Pi^+$. The direct sum

$$\mathfrak{n}^+ = \sum_{\alpha \in \Pi^+} \mathfrak{g}_\alpha$$

is a nilpotent subalgebra of $\mathfrak{g}$. We denote by $\mathfrak{a}^+$ the Weyl chamber associated with Π^+:

$$\mathfrak{a}^+ = \{H \in \mathfrak{a} : \alpha(H) > 0, \alpha \in \Pi^+\}.$$

The choice of one among Π^+, $\mathfrak{n}^+$ or $\mathfrak{a}^+$ determines the others. From the decomposition of $\mathfrak{g}$ into $\mathrm{ad}(\mathfrak{a})$-eigenspaces we have

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}^+$$

where

$$\mathfrak{n}^- = \theta(\mathfrak{n}^+) = \sum_{\alpha \in \Pi^-} \mathfrak{g}_\alpha$$

is the subalgebra opposed to $\mathfrak{n}^+$ and $\mathfrak{m} = \{X \in \mathfrak{k} : [X, \mathfrak{a}] = 0\}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. A Weyl chamber $\mathfrak{a}^+$ determines the Iwasawa decomposition:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}^+.$$

We shall denote by $\mathrm{pr}_{\mathfrak{i}}$ the projection of $\mathfrak{g}$ onto the Iwasawa component $\mathfrak{i} = \mathfrak{k}$, $\mathfrak{a}$ or $\mathfrak{n}$. In the particular case of $\mathfrak{sl}(n, \mathbb{R})$ with the canonical Iwasawa decomposition, $\mathfrak{k}$ is the algebra of skew-symmetric matrices, $\mathfrak{a}$ is the abelian algebra of diagonal matrices and $\mathfrak{n}^+$ the upper triangular matrices with zeros on the main diagonal.

Denote by $\langle \cdot, \cdot \rangle$ the Cartan-Killing form of $\mathfrak{g}$. We recall the following facts (see *e.g.* [14]):

- $\theta(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$.
- $\langle \mathfrak{g}_\alpha, \mathfrak{g}_\beta \rangle = 0$ unless $\beta = -\alpha$.
- The bilinear form in $\mathfrak{g}$ defined by $B_\theta(X, Y) = -\langle X, \theta Y \rangle$ is an inner product. In particular, the restriction of $\langle \cdot, \cdot \rangle$ to $\mathfrak{s}$ is an inner product and for every $0 \neq X \in \mathfrak{g}_\alpha$, $\langle X, \theta X \rangle \neq 0$.

2.2. **Flag Manifolds.** Let G be a connected and noncompact semi-simple Lie group with Lie algebra $\mathfrak{g}$. An Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}^+$ extends to an Iwasawa decomposition $G = KAN$ where the groups K, A and N are the exponentials of $\mathfrak{k}$, $\mathfrak{a}$ and $\mathfrak{n}^+$ respectively.

Let M be the centralizer of A in K. The Lie algebra of M is the centralizer $\mathfrak{m}$ of $\mathfrak{a}$ in $\mathfrak{k}$. The product $P = MAN$ is a closed subgroup of G with Lie algebra $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}^+$. The subgroup P is the normalizer of $\mathfrak{p}$ in G . It is a minimal parabolic subgroup and the quotient $\mathbf{B} = G/P$ is a compact homogeneous space of G known as the maximal flag manifold or the Furstenberg boundary of G. The subgroup K also acts transitively on $\mathbf{B}$. Through the transitive action of K we have $\mathbf{B} = K/M$. We remark that $\mathbf{B}$ is the same, regardless the specific G having Lie algebra $\mathfrak{g}$. This is due to the fact that M contains the center of G so that the action of G on $\mathbf{B}$ factors through the group of inner automorphisms of $\mathfrak{g}$ which is centerless.

For the construction of the non maximal flag manifolds we need the simple system of roots Σ associated to Π^+. This is a basis of $\mathfrak{a}^*$ such that every $\alpha \in \Pi^+$ is a linear combination of Σ with nonnegative integers as coefficients.

Given $\Theta \subset \Sigma$ let $\langle \Theta \rangle$ be the subset of positive roots generated by Θ. Denote by $\mathfrak{n}^-(\Theta)$ the subalgebra spanned by the root spaces $\mathfrak{g}_{-\alpha}$, $\alpha \in \langle \Theta \rangle$ and let $\mathfrak{p}_\Theta$ be the parabolic subalgebra defined by

$$\mathfrak{p}_\Theta = \mathfrak{n}^-(\Theta) \oplus \mathfrak{p} \,.$$

Its normalizer P_Θ in G is a parabolic subgroup whose Lie algebra is $\mathfrak{p}_\Theta$. We put $\mathbf{B}_\Theta = G/P_\Theta$ for the corresponding flag manifold. If $M_\Theta = P_\Theta \cap K$ then $\mathbf{B}_\Theta = K/M_\Theta$, that is, K is transitive in $\mathbf{B}_\Theta$. It turns out that M_Θ is the centralizer in K of any H in the "subchamber"

$$\{H \in \mathrm{cl}\left(\mathfrak{a}^+\right) : \alpha(H) = 0 \text{ if } \alpha \in \langle \Theta \rangle \text{ and } \alpha(H) > 0 \text{ if } \alpha \in \Pi^+ - \langle \Theta \rangle\} \,.$$

By this transitivity of K, $\mathbf{B}_\Theta$ identifies with the $\mathrm{Ad}(K)$-orbit of H in $\mathfrak{s}$. In this case the notation $\mathbf{B}_H$ and M_H are also used instead of $\mathbf{B}_\Theta$ and M_Θ respectively. The Lie algebra of M_H is the centralizer $\mathfrak{m}_H$ of H in $\mathfrak{k}$:

$$\mathfrak{m}_H = \{X \in \mathfrak{k} : [X, H] = 0\}.$$

A relation between the Iwasawa and Cartan components is provided by the following simple algebraic lemma, which generalizes the symmetrization and skew-symmetrization of matrices.

Lemma 2.1. *Let $\mathfrak{a}^+$, and hence $\mathfrak{n}^+$ be given.*

(1) *Define $\mathfrak{k}_{\mathfrak{a}^+} = \{Y + \theta(Y) : Y \in \mathfrak{n}^+\}$. Then $\mathfrak{k}_{\mathfrak{a}^+} \subset \mathfrak{k}$ and $\mathfrak{k} = \mathfrak{k}_{\mathfrak{a}^+} \oplus \mathfrak{m}$. Moreover, the skew-symmetrization map*

$$\chi : Y \in \mathfrak{n}^+ \longmapsto Y + \theta(Y) \in \mathfrak{k}_{\mathfrak{a}^+}$$

is an isomorphism of vector spaces.

(2) *Define $\mathfrak{s}_{\mathfrak{a}^+} = \{Y - \theta(Y) : Y \in \mathfrak{n}^+\}$. Then $\mathfrak{s}_{\mathfrak{a}^+} \subset \mathfrak{s}$ and $\mathfrak{s} = \mathfrak{s}_{\mathfrak{a}^+} \oplus \mathfrak{a}$. Moreover, the symmetrization map*

$$\sigma : Y \in \mathfrak{n}^+ \longmapsto Y - \theta(Y) \in \mathfrak{s}_{\mathfrak{a}^+}$$

is an isomorphism of vector spaces.

Proof. To see item (1) note that $\mathfrak{k}$ is the subspace of points fixed by θ so that $\mathfrak{k}_{\mathfrak{a}^+} \subset \mathfrak{k}$. We have $\mathfrak{k} \subset \mathfrak{n}^- \oplus \mathfrak{m} \oplus \mathfrak{n}^+$. Hence $X \in \mathfrak{k}$ is written uniquely as $X = Z + A + Y$, with $Z \in \mathfrak{n}^-$, $A \in \mathfrak{m}$ and $Z \in \mathfrak{n}^+$. Then

$$X = \theta(X) = \theta(Z) + A + \theta(Y)$$

with $\theta(Z) \in \mathfrak{n}^+$and $\theta(Y) \in \mathfrak{n}^-$. Therefore $\theta(Y) = Z$ so that $X = (Y + \theta(Y)) + A \in \mathfrak{k}_{\mathfrak{a}^+} \oplus \mathfrak{m}$. The isomorphism is a consequence of the fact that for $Y \in \mathfrak{n}^+$, $Y + \theta(Y) = 0$ if and only if $Y = 0$.

Item (2) follows in the same way: $\mathfrak{s} \subset \mathfrak{n}^- \oplus \mathfrak{a} \oplus \mathfrak{n}^+$ and consider $Y - \theta(Y)$ instead of $Y + \theta(Y)$. □

With the isomorphisms χ and σ of this lemma we construct the isomorphism

$$\zeta = \sigma \circ \chi^{-1} : \mathfrak{k}_{\mathfrak{a}^+} \to \mathfrak{s}_{\mathfrak{a}^+},$$

which extends to $\mathfrak{k}$ by declaring it to be zero at $\mathfrak{m}$.

For $H \in \mathfrak{a}$ the flag manifold $\mathbf{B}_H$ is the $\mathrm{Ad}(K)$-orbit of H so that its tangent space at H is

$$T_H\mathbf{B}_H = \{\tilde{X}(H) = [X, H] : X \in \mathfrak{k}\} \subset \mathfrak{s}.$$

Clearly $[\mathfrak{m}_H, H] = 0$. Hence $T_H\mathbf{B}_H$ is the subspace of tangent vectors $\tilde{X}(H)$ with X running through the subspace $\chi(\mathfrak{n}_H)$ where

$$\mathfrak{n}_H = \sum \{\mathfrak{g}_\alpha : \alpha \in \Pi^+, \alpha(H) \neq 0\}. \tag{1}$$

An easy computation shows that $T_H\mathbf{B}_H$, as a subspace of $\mathfrak{s}$, coincides with $\sigma(\mathfrak{n}_H)$ so that it is the orthogonal complement of $\zeta(\mathfrak{m}_H)$.

Later on we will use the following facts relating the isomorphism $\sigma : \mathfrak{n}^+ \to \mathfrak{s}_{\mathfrak{a}^+}$ with the Cartan-Killing form: If $Y \in \mathfrak{g}_\alpha$, $\alpha > 0$ then $\sigma Y = Y - \theta Y$, and since $\langle Y, Y \rangle = 0$,

$$\langle \sigma Y, \sigma Y \rangle = \langle Y - \theta Y, Y - \theta Y \rangle = 2B_\theta(Y, Y).$$

Also, if $Z \in \mathfrak{g}_\beta$, with $\alpha \neq \beta > 0$ then $\langle Y, Z\rangle = \langle Y, \theta Z\rangle = \langle \theta Y, \theta Z\rangle = 0$ so that $\langle \sigma Y, \sigma Z\rangle = 0$. From this fact we can construct orthonormal bases of $\mathfrak{s}$ as follows: take a basis $\{Y_1, \ldots, Y_m\}$ of $\mathfrak{n}^+$ which is the union of bases of the root spaces $\mathfrak{g}_\alpha$, $\alpha > 0$. Then $\{\sigma Y_1, \ldots, \sigma Y_m\}$ is a basis of $\mathfrak{s}_{\mathfrak{a}^+}$ which can be complemented with a basis of $\mathfrak{a}$ to get a basis of $\mathfrak{s}$. Any such basis will be called *adapted to* $\mathfrak{a}^+$. In particular, if $\{Y_1, \ldots, Y_m\}$ is orthonormal with respect to the inner product B_θ then

$$\{\frac{\sqrt{2}}{2}\sigma Y_1, \ldots, \frac{\sqrt{2}}{2}\sigma Y_m\} \tag{2}$$

is orthonormal in $\mathfrak{s}_{\mathfrak{a}^+}$ which can be complemented to an orthonormal basis of $\mathfrak{s}$.

2.3. **Borel metric.** It is possible to endow a flag manifold with a special Riemannian metric which depends on its realization as an $\mathrm{Ad}\,(K)$-orbit, namely the *Borel* (B) metric (see Borel [6] and Duistermmat, Kolk and Varadarajan [11]). For the definition of the B metric take $H \in \mathrm{cl}\,(\mathfrak{a}^+)$. Then at the tangent space $T_H\mathbf{B}_H$ the B metric is given by

$$\left(\tilde{X}\,(H)\,, \tilde{Y}\,(H)\right)_H = \langle H, [X, \zeta\,(Y)]\rangle \tag{3}$$

for $X, Y \in \mathfrak{k}$. This expression actually defines an inner product in $T_H\mathbf{B}_H$ which is invariant under M_H so that it extends to a K-invariant Riemannian metric in $\mathbf{B}_H$. This metric will play an essential role in the sequel for the computation of the Lyapunov exponents.

A crucial fact about the B metric is that the vector fields induced by $\mathfrak{s}$ are gradient. More precisely, for $X \in \mathfrak{s}$ let $\tilde{X}$ be the vector field it induces in $\mathbf{B}_H$ through the G-action in this flag manifold. Since $\mathbf{B}_H$ is embedded in $\mathfrak{s}$, it makes sense to define the function $f_X : \mathbf{B}_H \to \mathbb{R}$ by $f_X\,(Y) = \langle X, Y\rangle$.

Lemma 2.2. *For any $X \in \mathfrak{s}$, $\tilde{X} = \mathrm{grad} f_X$ where the gradient is taken with respect to B, that is, $d\,(f_X) = \left(\tilde{X}, \cdot\right)$.*

Proof. See Proposition 3.3 in [11]. □

The right hand side of equation (3) is linear in H showing that the B metric changes linearly with H. The exact meaning of this linear dependence is as follows: fix $t > 0$ and put $H_1 = tH$. The centralizers of H and H_1 in K coincide so that $\mathbf{B}_{H_1} = \mathbf{B}_H$, that is, both orbits $\mathrm{Ad}\,(K)\,H_1$ and $\mathrm{Ad}\,(K)\,H$ identify with the same homogeneous space of K. Under these identifications H_1 and H give the same base point. The vector field $\tilde{X}$ is defined by means of the K-action and it is independent of the specific realization. From (3) we see that the B metric defined by H_1 is t times the metric defined by H.

Another aspect about the B metric which needs to be discussed concerns its values on the vectors of $\mathfrak{s}$ which are tangent to $\mathrm{Ad}\,(K)\,H$ at H. Any such vector is of the form $\sigma\,(A)$ with $A \in \mathfrak{n}_H$, defined in (1). We have,

Lemma 2.3. *Let $\alpha > 0$ be a root such that $\alpha(H) \neq 0$. Let $A \in \mathfrak{g}_\alpha$ and view $\sigma(A) \in \mathfrak{s}$ as a tangent vector to* $\mathrm{Ad}(K) H$ *at H. Then*

$$(\sigma(A), \sigma(A))_H = \frac{1}{\alpha(H)} \langle \sigma(A), \sigma(A) \rangle .$$

Moreover, if $\beta \neq \alpha$ is another positive root and $B \in \mathfrak{g}_\beta$ then $(\sigma(B), \sigma(A))_H = 0$.

Proof. Let χ be the isomorphism of Lemma 2.1 and put $X = -\frac{1}{\alpha(H)} \chi(A)$. Direct computations show that

$$\begin{aligned} (\sigma(A), \sigma(A))_H &= \left(\tilde{X}, \tilde{X}\right)_H = \langle H, [X, \zeta X] \rangle \\ &= \langle [H, X], \zeta X \rangle = \langle -\sigma(A), \zeta X \rangle \\ &= \tfrac{1}{\alpha(H)} \langle \sigma(A), \sigma(A) \rangle . \end{aligned}$$

The orthogonality between $\sigma(A)$ and $\sigma(B)$ follows if we perform the computations with ζY instead of ζX where $Y = -\frac{1}{\beta(H)} \chi(B)$. □

This lemma has the following interesting consequence: suppose that H is such that $\alpha(H) = 1$ for every positive root α such that $\alpha(H) \neq 0$. Then the B metric in $\mathbf{B}_H$ is just the metric induced by its immersion in $\mathfrak{s}$. When this happens we say that $\mathbf{B}_H$ is an *immersed flag manifold.*

For later reference we include here the computation of the B metric in the vector fields induced by the elements in $\mathfrak{s}$.

Lemma 2.4. *Take $H \in \mathrm{cl}(\mathfrak{a}^+)$ and denote by x_0 the origin of $\mathbf{B}_H$. Let α be a positive root. For $X \in \mathfrak{g}_\alpha$ put $S = \sigma(X)$. Then*

$$\left| \tilde{S}(x_0) \right|^2 = \alpha(H) \langle \sigma(X), \sigma(X) \rangle .$$

Moreover, if $\beta \neq \alpha$ is another positive root and $Y \in \mathfrak{g}_\beta$ then $\left(\widetilde{\sigma(X)}, \widetilde{\sigma(Y)}\right)_H = 0$.

Proof. At H the vector field $\widetilde{\sigma(X)}$ is equal to $(\mathrm{pr}_{\mathfrak{k}} \sigma(X))^{\sim}$. We have

$$\sigma(X) = X - \theta(X) = (-X - \theta(X)) + 2X .$$

The right hand side of this equality is the Iwasawa decomposition of $\sigma(X)$ because $-X - \theta(X) \in \mathfrak{k}$ and $2X \in \mathfrak{n}^+$. Hence $\mathrm{pr}_{\mathfrak{k}} \sigma(X) = -(X + \theta(X))$. Since $\zeta(\sigma(X)) = X - \theta(X)$, a similar formula for Y yields

$$\left(\widetilde{\sigma(X)}, \widetilde{\sigma(Y)}\right)_H = \langle H, [X + \theta(X), Y - \theta(Y)] \rangle .$$

Now the Cartan-Killing form is invariant under the adjoint representation so that

$$\left(\widetilde{\sigma(Y)}, \widetilde{\sigma(Y)}\right)_H = \langle [H, X + \theta(X)], Y - \theta(Y) \rangle .$$

But $[H, X] = \alpha(H) X$ and $[H, \theta(X)] = -\alpha(H) X$ because $X \in \mathfrak{g}_\alpha$ and $\theta(X) \in \mathfrak{g}_{-\alpha}$. Since $\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}$ is orthogonal to $\mathfrak{g}_\beta + \mathfrak{g}_{-\beta}$ if $\beta \neq \alpha$, this implies the second statement. On the other hand,

$$\left| \widetilde{\sigma(X)}(x_0) \right|^2 = \alpha(H) \langle X - \theta(X), X - \theta(X) \rangle$$

as claimed. ☐

3. Decomposition of G/MN

In this section we show that the homogeneous space G/MN is a trivial principal fiber bundle over the maximal flag manifold $\mathbf{B}$ such that the structural group is the A part of the Iwasawa decomposition $G = KAN$, i.e., there exists a kind of spherical-radial decomposition of this space. In the next subsections we study the asymptotic behavior of the trajectories in the fibers, so we find the Lyapunov spectrum in the entries of limiting elements in A.

The product $L = MN \subset P$ is a closed subgroup which is normal in P because A normalizes N and M. Consider the canonical fibration

$$\pi : gL \in G/L \longmapsto gP \in G/P$$

whose fiber is $A = P/L$. Since L is normal in P it turns out that G/L is a principal bundle over G/P with A as structural group. The right action of A on G/L is given by $R_h(gL) = gLh = ghL$, $h \in A$. It is clear that for any $g \in G$, $gR_h = R_h g$. Also, π is equivariant with respect to the actions of G on G/L and G/P in the sense that $\pi g = g\pi$. The next proposition shows that the above principal bundle is trivial:

Proposition 3.1. *The map*

$$\phi : (uM, h) \in (K/M) \times A \longmapsto uhL \in G/L\,. \tag{4}$$

is a diffeomorphism between G/L and $\mathbf{B} \times A$. Its inverse ϕ^{-1} maps $gL \in G/L$ in $(uM, h) \in K/M \times A$ where $g = uhn$ is the Iwasawa decomposition of g. Moreover, ϕ is a bundle map in the sense that $\phi(b, R_{a_1}(a)) = R_{a_1}\phi(b, a)$ for all $(b, a) \in B \times A$ and $a_1 \in A$.

Proof. Note that, by (4), ϕ does not depend on the representative $u \in K$ because if $m \in M$ and $h \in A$ then $umhL = uhmL = uhL$. Let ψ be the map $uhL \mapsto (uM, h)$; we claim that ψ is the inverse of ϕ. Firstly we check that ψ is also well defined: let $g = uhn$ and suppose that $g_1 = u_1 h_1 n_1$ is in the coset gL, i.e.

$$n^{-1}h^{-1}u^{-1}u_1 h_1 n_1 \in L\,.$$

Since $N \subset L$ it follows that $h^{-1}u^{-1}u_1 h_1 \in L \subset P$. Hence $u^{-1}u_1 \in P \cap K = M$. So that $u^{-1}u_1$ commutes with h^{-1} and hence $u^{-1}u_1 h^{-1}h_1 \in L$ which implies that $h^{-1}h_1 \in L \cap A = \{1\}$. Therefore $h_1 = h$ and $uM = u_1 M$ showing that ψ is well defined. The composition $\phi\psi$ is the identity because if $g = uhn$ then $gL = uhL$. On the other hand, $\psi\phi(uM, h) = (uM, h)$ because uh is already written in Iwasawa decomposition. Since ϕ and ψ are differentiable we conclude that ϕ is a diffeomorphism between G/L and $B \times A$. It is a bundle map because A normalizes L. ☐

This decomposition has an evident meaning as a polar decomposition with $\mathbf{B}$ playing the role of the spherical component while A is the radial component. Also, by identifying G/L with $\mathbf{B} \times A$ through ϕ, the A-component of gL becomes the A-component of the Iwasawa decomposition of g.

3.1. **Vector Fields.** We look now at the behavior of vector fields in G/L under the above decomposition. Recall that the g-translation of $\tilde{X}$ is given by the adjoint in $\mathfrak{g}$:

$$g_*\tilde{X} = (\mathrm{Ad}\,(g)\,X)^{\sim} \ . \tag{5}$$

The vector field induced by X on G/L is right invariant under $h \in A$, i.e., $R_{h*}X = X$ because the action of G on G/L commutes with the right action of A. Also, taking the decomposition $G/L = \mathbf{B} \times A$ and considering the trivial connection on this bundle, X decomposes as

$$X\,(b,h) = X_H\,(b,h) + X_V\,(b,h)$$

where X_H is the *horizontal component* (in the direction of $\mathbf{B}$) while X_V stands for the *vertical component* (in the direction of A). We shall find explicit expressions for these components.

The horizontal component is just the vector field induced by X on $\mathbf{B}$. In fact, the projection

$$\pi : G/L \longrightarrow G/P$$

is equivariant which implies that $\pi_* \left.\tilde{X}\right|_{G/L} = \left.\tilde{X}\right|_{G/P}$. Since $\pi_* X_V = 0$ it follows that $\pi_* \left.\tilde{X}\right|_{G/L} = X_H$. Therefore $X_H\,(b,h) = X_H\,(b)$ is independent of $h \in A$ and coincides with the vector field induced by X on $\mathbf{B}$.

In order to get the vertical component we denote by H^* the vertical vector field induced in $\mathbf{B} \times A$ by $H \in \mathfrak{a}$ as an element of the Lie algebra of the structural group, hence, now the action is on the right. For every vertical vector v at (b,h) there exists $H \in \mathfrak{a}$ such that $v = H^*\,(b,h)$. Hence with a given $X \in \mathfrak{g}$ we have defined a map $b \in \mathbf{B} \mapsto H_{X,b} \in \mathfrak{a}$ such that

$$X\,(b,1) = X_H\,(b) + H^*_{X,b}\,(b,1) \ .$$

From this equality we can obtain the vertical component. In fact, X is right invariant, i.e., $X\,(b,h) = R_{h*}X\,(b,1)$. Now

$$\begin{aligned} R_{h*}\left(X_H\,(b) + H^*_{X,b}\,(b,1)\right) &= X_H\,(b) + R_{h*}\left(H^*_{X,b}\,(b,1)\right) \\ &= X_H\,(b) + \left(\mathrm{Ad}\left(h^{-1}\right) H_{X,b}\right)^*(b,h) \ . \end{aligned}$$

Since A is abelian $\mathrm{Ad}\left(h^{-1}\right) H = H$, hence

$$X\,(b,h) = X_H + H^*_{X,b}\,(b,h)$$

i.e. the vertical component is determined by $H_{X,b} \in \mathfrak{a}$ which depends only on X and on b. We will find an explicit expression for this map. Consider the Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ and let $b_0 = P$ be the origin of $\mathbf{B}$. Since N is contained in the isotropy subgroup at $(b_0, 1)$, the $\mathfrak{n}$-component of X becomes zero at this point. Under the diffeomorphism ϕ the horizontal component $\left.\tilde{X}\right|_{G/P}(b_0)$ is given by $\left.(\mathrm{pr}_{\mathfrak{k}})^{\tilde{}}\right|_{G/P}$ and the vertical component is given by $\left.(\mathrm{pr}_{\mathfrak{a}})^{\tilde{}}\right|_{A}$. Hence, since $\mathfrak{a}$ is abelian, $H_{X,b_0} = \mathrm{pr}_{\mathfrak{a}} X$.

For the values of $H_{X,b}$ at other points of $\mathbf{B}$, take $u \in K$ and put $b = ub_0$. Then, by equation (5) and the fact that $u_* H^* = H^*$ we have:

$$\begin{aligned} \tilde{X}(b,1) &= u_* \left(\left(\mathrm{Ad}\left(u^{-1}\right) X \right)^{\sim} (b_0, 1) \right) \\ &= u_* \left(\left(\mathrm{Ad}\left(u^{-1}\right) X \right)^{\sim} \Big|_{G/P} (b_0) \right) + u_* H^*_{\mathrm{Ad}(u^{-1})X, b_0} (b_0, 1) \\ &= X_H (b_0) + H^*_{\mathrm{Ad}(u^{-1})X, b_0} (b, 1) \, . \end{aligned}$$

So, $X_V (b,1) = H^*_{\mathrm{Ad}(u^{-1})X,b_0} (b,1)$, and we get for $b = ub_0$, $u \in K$ the desired expression of $H_{X,b}$:

$$H_{X,b} = H_{\mathrm{Ad}(u^{-1})X,b_0} = \mathrm{pr}_{\mathfrak{a}} \left(\mathrm{Ad}\left(u^{-1}\right) X \right) \, .$$

The group A is diffeomorphic to its Lie algebra $\mathfrak{a}$ through the exponential map. Hence G/L is also diffeomorphic to $\mathbf{B} \times \mathfrak{a}$ and there is a decomposition of the vector fields at this level too. We have then:

Proposition 3.2. *The differential equation induced in $G/L = \mathbf{B} \times \mathfrak{a}$ by $X \in \mathfrak{g}$ decomposes into the equations*

$$\begin{cases} \frac{db}{dt} &= \tilde{X}\Big|_{G/P} (b) \, , \text{ with } b \in \mathbf{B} \text{ and} \\ \\ \frac{dH}{dt} &= \mathrm{pr}_{\mathfrak{a}} \left(\mathrm{Ad}\left(u^{-1}\right) X \right) \text{ where } b = ub_0 \quad \text{and } H \in \mathfrak{a} \end{cases}$$

Proof. We only remark that the second equation means that if $a_t = \exp H(t) \in A$ then $\dot{a}_t = \mathrm{pr}_{\mathfrak{a}} \left(\mathrm{Ad}\left(u^{-1}\right) X \right) a_t$. □

In the following sections it will be convenient to use the notation

$$q_X(b) = \mathrm{pr}_{\mathfrak{a}} \left(\mathrm{Ad}\left(u^{-1}\right) X \right) \in \mathfrak{a} \tag{6}$$

with $X \in \mathfrak{g}$, $b = ub_0$ where $u \in K$. Note that (6) does not depend on the representative $u \in K$ which satisfies $ub_0 = b$. In fact, if $u_1 b_0 = b$ then $u_1 = um$ for some $m \in M$, and $\mathrm{pr}_{\mathfrak{a}} \circ \mathrm{Ad}(m) = \mathrm{pr}_{\mathfrak{a}}$ because m centralizes $\mathfrak{a}$.

4. Stochastic Differential Equations

Consider the stochastic differential equation on the semi-simple Lie group G:

$$dg = X(g)\, dt + \sum_{i=1}^{m} Y^i (g) \circ dW_i \, . \tag{7}$$

We shall assume the *accessibility property* of this system which means that X and $Y^1, \ldots, Y^m$ generate the Lie algebra $\mathfrak{g}$ of G. As in the case of vector fields this equation induces stochastic equations in the homogeneous spaces of G, in particular in G/L. By the preceding section, there is a decomposition of the process in G/L into radial and spherical components. In fact, using Itô's formula the induced equation in $\mathbf{B} \times \mathfrak{a}$ has the components

$$db = X_H (b)\, dt + \sum_{i=1}^{m} Y^i_H (b) \circ dW_i \tag{8}$$

in the direction of $\mathbf{B}$ and

$$dH = q_X(b)\,dt + \sum_{i=1}^{m} q_{Y^i}(b) \circ dW_i \tag{9}$$

in the direction of $\mathfrak{a}$. Let g_t be the solution of (7) starting at the identity $1 \in G$. Write $g_t = u_t h_t n_t$ for its Iwasawa decomposition and put $H_t = \log h_t$. Then H_t is driven by equation (9). In order to describe the asymptotic behavior of H_t it will be convenient to convert the Stratonovich equation (9) in Itô form:

$$dH = q_X(b)\,dt + \frac{1}{2}\sum_{i=1}^{m} r_{Y^i}(b)\,dt + \sum_{i=1}^{m} q_{Y^i}(b)\,dW_i \tag{10}$$

where r_Z for $Z \in \mathfrak{g}$ stands for the directional derivative:

$$r_Z(b) = (Z_H \cdot q_Z)(b) \; .$$

We will find an expression for r_Z by reducing the computation of the derivative at the origin $b_0 \in B$. Given $b \in B$ let $u \in K$ be such that $b = ub_0$. Then

$$\begin{aligned} r_Z(b) &= d(q_Z)_b(Z_H(b)) \\ &= d(q_Z)_b \circ du_{b_0} \circ du_b^{-1}(Z_H(b)) \end{aligned}$$

with u viewed as a diffeomorphism $u : \mathbf{B} \to \mathbf{B}$. Hence:

$$r_Z(b) = d(q_Z \circ u)_{b_0}\left(du_b^{-1}(Z_H(b))\right) \; .$$

Now, from (5), $du_b^{-1}(Z_H(b))$ is the vector field induced on $\mathbf{B}$ by $\mathrm{Ad}\left(u^{-1}\right) Z$ at b_0, i.e.,

$$r_Z(b) = d(q_Z \circ u)_{b_0}\left(\left(\mathrm{Ad}\left(u^{-1}\right) Z\right)^{\sim}(b_0)\right) \; . \tag{11}$$

We recall from the previous section that at the origin b_0 of B, given $X \in \mathfrak{g}$, $\widetilde{X}(b_0) = (\mathrm{pr}_{\mathfrak{k}} X)^{\sim}(b_0)$ (because $\mathfrak{a} + \mathfrak{n}$ is contained in the isotropy subalgebra at b_0). Hence, if we denote

$$W(u) = \mathrm{pr}_{\mathfrak{k}}\,\mathrm{Ad}\left(u^{-1}\right) Z \, ,$$

then

$$r_Z(b) = \frac{d}{dt}(q_Z \circ u)\left(e^{tW(u)} b_0\right)_{|t=0} \; ,$$

which by the definition of q_Z becomes:

$$r_Z(b) = \frac{d}{dt}\,\mathrm{pr}_{\mathfrak{a}}\left(\mathrm{Ad}\left(e^{-tW(u)} u^{-1}\right) Z\right)_{|t=0} \; .$$

A direct calculation shows that

$$r_Z(b) = \mathrm{pr}_{\mathfrak{a}}[\mathrm{Ad}\left(u^{-1}\right) Z, W(u)] \, .$$

Summarizing, we have the following formula:

Proposition 4.1. *If $Z \in \mathfrak{g}$ and $b = ub_0 \in B$ with $u \in K$ then*

$$r_Z(b) = \mathrm{pr}_{\mathfrak{a}}[\mathrm{Ad}\left(u^{-1}\right) Z \, , \mathrm{pr}_{\mathfrak{k}}\,\mathrm{Ad}\left(u^{-1}\right) Z]$$

4.1. **The Integral Formula.** The assumption that $\{X, Y^1, \ldots, Y^m\}$ generates $\mathfrak{g}$ guarantees that in each compact homogeneous space of G there exists a unique (ergodic) invariant probability measure for the diffusion process which is the solution of the induced stochastic differential equation. In particular there exists a unique invariant probability measure ν on the maximal flag manifold for the process in this space. Applying the ergodic theorem to the skew-symmetric flow (see *e.g.* Arnold, Kliemann and Oeljeklaus [1] or Carverhill [7]) we have the following well known special case of the Law of Large Numbers:

$$\lim_{t\to\infty} \frac{1}{t} \log a_t = \lim_{t\to\infty} \frac{1}{t} \int_0^t Q(b_s)\, ds = \int_{\mathbf{B}} Q(b)\nu(db) \text{ for } \nu \otimes \mathbb{P}\text{-almost every } (b, \omega)$$

where the function $Q : \mathbf{B} \to \mathfrak{a}$ is given by:

$$Q(b) = q_X + \sum_{i=1}^{m} r_{Y^i}(b) \tag{12}$$

with $q_X(b) = \mathrm{pr}_{\mathfrak{a}}\left(\mathrm{Ad}\left(u^{-1}\right) X\right)$ and $r_{Y^i}(b) = \mathrm{pr}_{\mathfrak{a}}[\mathrm{Ad}\left(u^{-1}\right) Y^i, \mathrm{pr}_{\mathfrak{k}}\mathrm{Ad}\left(u^{-1}\right) Y^i]$ where $b = ub_0$.

4.2. **The Integrand.** We shall find an expression for the quadratic part $r_Z(b)$, $Z \in \mathfrak{g}$, of the integrand in terms of the B metric.

The restriction of the Cartan-Killing form $\langle\cdot,\cdot\rangle$ to $\mathfrak{a}$ is an inner product so that we determine $r_Z(b)$ if we compute $\langle H, r_Z(b)\rangle$ for every H in a basis of $\mathfrak{a}$. In other words, we must calculate

$$\langle H, r_Z(b)\rangle = \langle H, \mathrm{pr}_{\mathfrak{a}}[\mathrm{Ad}\left(u^{-1}\right) Z, \mathrm{pr}_{\mathfrak{k}}\mathrm{Ad}\left(u^{-1}\right) Z]\rangle$$

for generic $H \in \mathfrak{a}$. Under the Cartan-Killing form $\mathfrak{k}$ and $\mathfrak{n}^+$ are orthogonal to $\mathfrak{a}$ so that

$$\langle H, r_Z(b)\rangle = \langle H, [\mathrm{Ad}\left(u^{-1}\right) Z, \mathrm{pr}_{\mathfrak{k}}\mathrm{Ad}\left(u^{-1}\right) Z]\rangle\,. \tag{13}$$

Note first that if $Z \in \mathfrak{k}$ then $\mathrm{Ad}\left(u^{-1}\right) Z \in \mathfrak{k}$. Hence $\mathrm{pr}_{\mathfrak{k}}\mathrm{Ad}\left(u^{-1}\right) Z$ coincides with $\mathrm{Ad}\left(u^{-1}\right) Z$ so that (13) vanishes trivially.

On the other hand for $Z \in \mathfrak{s}$ we can relate $\langle H, r_Z(b)\rangle$ with the B metric $(\cdot,\cdot)$ in $\mathbf{B}_H$.

Lemma 4.2. *If $Z \in \mathfrak{s}$ then*

$$\langle H, [Z, \mathrm{pr}_{\mathfrak{k}} Z]\rangle = ((\mathrm{pr}_{\mathfrak{k}} Z)^{\sim}, (\mathrm{pr}_{\mathfrak{k}} Z)^{\sim})_H \tag{14}$$

where $(\mathrm{pr}_{\mathfrak{k}} Z)^{\sim}$ means the vector field on $\mathbf{B}_H$ induced by $\mathrm{pr}_{\mathfrak{k}} Z$.

Proof. By Lemma 2.1 there are $Y \in \mathfrak{n}^+$ and $H' \in \mathfrak{a}$ such that

$$Z = Y - \theta(Y) + H' = (-Y - \theta(Y)) + H' + 2Y\,.$$

The right hand side of this equality is the Iwasawa decomposition of Z because $-Y - \theta(Y) \in \mathfrak{k}$, $H' \in \mathfrak{a}$ and $Y \in \mathfrak{n}^+$. Hence

$$\mathrm{pr}_{\mathfrak{k}} Z = -Y - \theta(Y)\,,$$

and

$$[Z, \mathrm{pr}_{\mathfrak{k}} Z] = [Y - \theta(Y), \mathrm{pr}_{\mathfrak{k}} Z] + [H', \mathrm{pr}_{\mathfrak{k}} Z]. \tag{15}$$

The term $[H', \mathrm{pr}_{\mathfrak{k}} Z]$ is orthogonal to H so that it does not contribute to (14). In fact, $\mathrm{pr}_{\mathfrak{k}} Z = -Y - \theta(Y)$ belongs to $\mathfrak{n}^- + \mathfrak{n}^+$ and this subspace is orthogonal to $\mathfrak{a}$ and invariant under $\mathrm{ad}(\mathfrak{a})$. On the other hand, the first term in the right hand side of (15) is $-[\zeta(\mathrm{pr}_{\mathfrak{k}} Z), \mathrm{pr}_{\mathfrak{k}} Z]$. The Cartan-Killing product of this term with H is by definition $((\mathrm{pr}_{\mathfrak{k}} Z)^{\sim}, (\mathrm{pr}_{\mathfrak{k}} Z)^{\sim})_H$, proving the lemma. □

The right hand side of (14) can be given an interpretation in terms of the G-action on $\mathbf{B}_H$: For $X \in \mathfrak{g}$ let as before $\tilde{X}$ stand for the vector field induced by X on $\mathbf{B}_H$. If $x_0 \in \mathbf{B}_H$ corresponds to H then $\mathfrak{a} + \mathfrak{n}^+$ is contained in the isotropy subalgebra at x_0. Hence for $X \in \mathfrak{g}$,

$$\tilde{X}(x_0) = (\mathrm{pr}_{\mathfrak{k}} X)^{\sim}(x_0).$$

Therefore we have

Corollary 4.3. *For $Z \in \mathfrak{s}$ it holds*

$$\langle H, [Z, \mathrm{pr}_{\mathfrak{k}} Z] \rangle = (\tilde{Z}, \tilde{Z})_{x_0} = \left| \tilde{Z}(x_0) \right|^2.$$

Using K-invariance we can transport this formula to every point of $\mathbf{B}_H$.

Corollary 4.4. *If $Z \in \mathfrak{s}$ then*

$$\langle H, r_Z(b) \rangle = (\tilde{Z}, \tilde{Z})_{u x_0}$$

where $(\cdot, \cdot)$ is the B metric in $\mathbf{B}_H$. Here u and b are related by $b = u x_0$ and x_0 is the origin of $\mathbf{B}_H$.

Proof. Put $U = \mathrm{Ad}(u^{-1}) Z$. By definition of r_Z and the above lemma,

$$\langle H, r_Z(b) \rangle = \left| \tilde{U}(x_0) \right|^2$$

with the norm given by the B metric in $\mathbf{B}_H$. However,

$$\tilde{U}(x_0) = \left(\mathrm{Ad}(u^{-1}) Z \right)^{\sim}(x_0) = u_*^{-1}(Z(u x_0)).$$

Since the metric is K-invariant it follows that

$$\left| \tilde{U}(x_0) \right|^2 = \left| u_*^{-1}\left(\tilde{Z}(u x_0) \right) \right|^2 = \left| \tilde{Z}(u x_0) \right|^2$$

as claimed. □

In general, let $Z = A + S$ with $A \in \mathfrak{k}$ and $S \in \mathfrak{s}$ and for $u \in K$ put $Z_u = \mathrm{Ad}(u^{-1}) Z$, $A_u = \mathrm{Ad}(u^{-1}) A$ and $S_u = \mathrm{Ad}(u^{-1}) S$. Then $Z_u = A_u + S_u$. Plugging this into formula (13) and taking into account that $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$ is orthogonal to H we get

$$\langle H, r_Z(b) \rangle = \langle H, [S_u, A_u] \rangle + \left| \tilde{S}(u x_0) \right|^2$$

where $b = ub_0$. Now,

$$S_u = \frac{Z_u - \theta Z_u}{2}, \qquad A_u = \frac{Z_u + \theta Z_u}{2}$$

and $\mathrm{Ad}\,(u)$ commutes with θ. Hence there is the following expression for $\langle H, r_Z\,(b)\rangle$, which holds for arbitrary Z.

Proposition 4.5. *If $Z \in \mathfrak{g}$ and $H \in \mathfrak{a}$ then*

$$\langle H, r_Z\,(b)\rangle = \frac{1}{2}\langle H, [\mathrm{Ad}\,(u)\,Z, \mathrm{Ad}\,(u)\,(\theta Z)]\rangle + \left|\tilde{S}\,(ux_0)\right|^2$$

where $b = ub_0$ and b_0 and x_0 are the origins in the flag manifolds $\mathbf{B}$ and $\mathbf{B}_H$ respectively.

5. Lyapunov Exponents

The right invariant stochastic differential equation (7) on G induces a stochastic differential

$$dx = \tilde{X}\,(x) + \sum_{i=1}^{m} \tilde{Y}^i\,(x) \circ dW_i$$

on each space endowed with a G action. For many of the induced systems their Lyapunov exponents are described by the asymptotic of the A-part in the Iwasawa decomposition. We present below two classical situations covered by this construction, namely the linear systems induced by representations of the group and, secondly, Brownian motions in flag manifolds and symmetric spaces.

5.1. **Linear Systems.** Let $\rho : G \to \mathrm{Gl}\,(d, \mathbb{R})$ be a representation of G in $\mathbb{R}^{\,d}$. It induces a representation of $\mathfrak{g}$ (also denoted by ρ) so that the right invariant vector fields in G are mapped into linear vector fields in $\mathbb{R}^d$. Therefore, under the representation, the system given by equation (7) is mapped into the linear differential equation:

$$dx = \rho\,(X)\,x dt + \sum_{i=1}^{m} \rho\left(Y^i\right) x \circ dW_i \qquad x \in \mathbb{R}^{\,d}. \tag{16}$$

The relation between the systems (7) and (16) is that if x_t is the solution of (16) starting at x_0 then $x_t = \rho\,(g_t)\,x_0$ where g_t is the solution of (7) starting at the identity. Clearly, $\rho\,(g_t)$ is the solution of a right invariant differential equation in the Lie group $\rho\,(G)$, image of (7) under ρ. Every data about (16) is contained in this image system and not in G itself. Since we are primarily interested in (16) we assume from now on that ρ is a faithful representation. This amounts to assume that G is a linear group and ρ is just the inclusion of G into the general linear group. Alternatively we may start with a linear system and make the assumption that the Lie algebra generated by the coefficients is a semi-simple subalgebra of matrices.

Our purpose here is to sketch a proof of the easily suspected fact that the A-part in the Iwasawa decomposition gives, through the representation, the Lyapunov exponents of (16)

$$\lim_{t\to+\infty}\frac{1}{t}\log\|x_t\| \,. \tag{17}$$

There are certainly different ways to prove this fact. All of them require some regularity property of the system. For our systems the regularity comes from the accessibility property, i.e., the assumption that the coefficients of the system generate $\mathfrak{g}$.

To find the Lyapunov exponents of equation (16) we use the theory of Guivarc'h and Raugi [13]. The first thing to do is to change our continuous-time system into a discrete-time one. This is easily achieved by taking the solution g_t of (7) starting at the identity at time 1. Let μ be the law of g_1. Then the law of g_k is the k-th convolution power $\mu^{\star k}$. Also, the flow property of g_t implies that for a random element

$$g_k(\omega)=g_1(\theta_{k-1}(\omega))\cdots g_1(\theta(\omega))\cdot g_1(\omega) \tag{18}$$

where θ is the shift in probability space. Therefore, in a convenient probability space, g_k can be regarded as a product of an i.i.d. sequence of random elements in G. On the other hand the limit in (17) can be discretized with the same results, that is, the Lyapunov exponents of the sequence $\rho(g_n)$ of random matrices coincides with the Lyapunov exponents of the system (16) (see Carverhill [7] for further discussions about the continuous vs. discrete-time stochastic systems).

With this in mind we observe that the support suppμ of μ has nonempty interior in G. In fact, by the support theorem supp$\mu = \mathrm{cl}(\mathcal{A}(1))$, where $\mathcal{A}(1)$ is the attainable set from the identity in G, at time 1, of the right invariant control system in G obtained from (7):

$$\dot{g}=X(g)+\sum_{i=1}^{m}u_i(t)\,Y^i(g)$$

with $u_i(t)$ piecewise constant controls (see *e.g.* Ikeda and Watanabe [15]). A general result of Sussmann and Jurdjevic [25] says that the attainable set of an analytic control system at a fixed time has nonvoid interior inside the leaf of a certain integrable distribution of codimension zero or one in the state space. An application of this result to a right invariant control system on a Lie group proves that $\mathcal{A}(1)$ has nonvoid interior in a coset of a connected normal subgroup $H\subset G$ with codimension zero or one. Since we are working with a semi-simple Lie group, there are no normal subgroup of codimension one. Hence $H=G$ and suppμ has nonvoid interior in G.

This fact ensures that the probability measure μ and the corresponding random product are under the basic assumptions of [13], namely that the semigroup T_μ generated by suppμ is contracting and strongly irreducible.

Consider now the Iwasawa decomposition of the product (18)

$$g_k(\omega)=u_n(\omega)\,h_k(\omega)\,n_k(\omega)\in KAN\,.$$

Write also the polar decomposition

$$g_k(\omega) = x_k(\omega)\, a_k(\omega)\, y_k(\omega) \in K\bar{A}^+K\,, \tag{19}$$

where $\bar{A}^+$ stands for the exponential of the closure of a Weyl chamber in $\mathfrak{a}$. By [13, Cor. 2.8] $h_k(\omega)\, a_k^{-1}(\omega)$ converges almost surely so that

$$\lim \frac{1}{k} \log h_k = \lim \frac{1}{k} \log a_k\,.$$

At this point we need the following well known fact about Cartan decompositions of Lie algebras and subalgebras (see *e.g.* [14], [27]):

Lemma 5.1. *Let $\mathfrak{g} \subset \widetilde{\mathfrak{g}}$ be noncompact semi-simple Lie algebras and consider a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ of $\mathfrak{g}$. Then there exists a Cartan decomposition $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{k}} \oplus \widetilde{\mathfrak{s}}$ such that $\mathfrak{k} \subset \widetilde{\mathfrak{k}}$ and $\mathfrak{s} \subset \widetilde{\mathfrak{s}}$. Also, if $\mathfrak{a} \subset \mathfrak{s}$ is maximal abelian then there exists a maximal abelian $\widetilde{\mathfrak{a}} \subset \widetilde{\mathfrak{s}}$ such that $\mathfrak{a} \subset \widetilde{\mathfrak{a}}$.*

The compatible Cartan decompositions extend to the group level: Let $G \subset \tilde{G}$ be semi-simple Lie groups with Lie algebras $\mathfrak{g} \subset \widetilde{\mathfrak{g}}$. A Cartan decomposition $G = KS$ comes from a Cartan decomposition of $\mathfrak{g}$, through the exponential mapping. Hence there exists a Cartan decomposition $\tilde{G} = \tilde{K}\tilde{S}$ such that $K \subset \tilde{K}$ and $S \subset \tilde{S}$. We can apply this fact to our linear group G. Since G is semi-simple it is contained in $\mathrm{Sl}(d, \mathbb{R})$ so that a Cartan decomposition $G = KS$ extends to a Cartan decomposition of $\mathrm{Sl}(d, \mathbb{R})$. This means that there is an inner product of $\mathbb{R}^d$ such that with respect to it the elements of K are orthogonal matrices and those of S are symmetric and positive definite. The same way for a polar decomposition $G = K\bar{A}^+K$ there is a group $\tilde{A}$ of diagonal matrices in $\mathrm{Sl}(d, \mathbb{R})$ containing A. We remark that it is not true in general that $\bar{A}^+$ is contained in a unique Weyl chamber of $\tilde{A}$. Given the decomposition (19) of $g_k(\omega)$, the eigenvalues of

$$\lambda = \lim \frac{1}{2k} \log \left(g_k(\omega)\, g_k(\omega)^* \right)$$

are exactly the eigenvalues of $\lim \frac{1}{k} \log a_k(\omega)$, which coincides with our previously defined Lyapunov exponent matrix. By the approach in Ruelle [23], the eigenvalues of λ are the Lyapunov exponents of our system (16).

We state now these facts using the language of representation theory. Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of the semi-simple Lie algebra in the real vector space V. If $\mathfrak{a} \subset \mathfrak{g}$ has the same meaning as before, a linear functional $\lambda : \mathfrak{a} \to \mathbb{R}$ is said to be a weight of the representation if the weight space

$$V_\lambda = \{ v \in V : \rho(H)\, v = \lambda(H)\, v \text{ for all } H \in \mathfrak{a} \}$$

is not zero. If $H \in \mathfrak{a}$ then $\rho(H)$ is diagonalizable and its eigenvalues are $\lambda(H)$ with λ running through the set of weights. With this terminology we have the formula:

Theorem 5.2. *Assume that the right invariant system (7) in G satisfies the accessibility property. Consider the linear differential equation (16) induced by the* *representation ρ. Then the Lyapunov exponents of (16) are the entries of*

$$\rho\left(\int Q\left(b\right)\nu\left(db\right)\right),$$

which are $\lambda\left(\int Q\left(b\right)\nu\left(db\right)\right)$ with λ running through the weights of the representation. Here Q is given by (12) and ν is the unique invariant probability measure in the maximal flag manifold of G.

5.2. **Systems with Symmetric Vector Fields.** Consider a system

$$dg=\sum_{i=1}^{m}Y^{i}\left(g\right)\circ dW_{i} \tag{20}$$

without drift such that $\{Y^1,\ldots,Y^m\}$ is an orthonormal basis of $\mathfrak{s}$. Under the inverse mapping of G the right invariant vector field Y^i is mapped into the left invariant vector field whose value at the identity is $-Y^i$. It was proved by Malliavin and Malliavin [21] that the left invariant system thus obtained is the horizontal diffusion in the symmetric space G/K (see also Liao [18] and Taylor [26]). In our computations below we shall recover a result of [21] on the limit behavior of the A-part of the Iwasawa decomposition of the horizontal diffusion.

For the system (20) the integrand in the formula (12) becomes

$$Q\left(b\right)=\frac{1}{2}\sum_{i=1}^{m}r_{Y^{i}}\left(b\right).$$

By Corollary 4.4 if $H\in\mathfrak{a}$ then

$$\left\langle H,Q\left(b\right)\right\rangle=\sum_{i=1}^{m}\left|\tilde{Y}^{i}\left(ux_{0}\right)\right|^{2} \tag{21}$$

with $b=ub_0$ and the norm is with respect to B metric in $\mathbf{B}_H$. We shall compute this expression explicitly. Firstly note that the right hand side of (21) is independent of the orthonormal basis, in fact:

Lemma 5.3. *Let $\left(Y^i\right)_{i=1,\ldots,m}$ and $\left(Z^i\right)_{i=1,\ldots,m}$ be orthonormal bases of $\mathfrak{s}$. Then*

$$\sum_{i=1}^{m}\left|\tilde{Y}^{i}\left(ux_{0}\right)\right|^{2}=\sum_{i=1}^{m}\left|\tilde{Z}^{i}\left(ux_{0}\right)\right|^{2}$$

for all $u\in K$.

Proof. Let $a^i_j\in\mathbb{R}$, $i,j=1,\ldots,m$, be such that

$$Z^{i}=\sum a_{j}^{i}Y^{j}.$$

Since the restriction to $\mathfrak{s}$ of the Cartan-Killing form is an inner product, the $m\times m$ matrix $\left(a^i_j\right)_{i,j}$ is orthogonal. Hence

$$\sum_{i=1}^{m}\left|\tilde{Z}^{i}\left(ux_{0}\right)\right|^{2}=\sum_{i,j=1}^{m}\left(a_{j}^{i}\right)^{2}\left|\tilde{Y}^{j}\left(ux_{0}\right)\right|^{2}+\sum_{i}^{m}\sum_{j\neq k}^{m}a_{j}^{i}a_{k}^{i}\left(\tilde{Y}^{j}\left(ux_{0}\right),\tilde{Y}^{k}\left(ux_{0}\right)\right)$$

shows the lemma. □

Corollary 5.4. *If* $\left(Y^i\right)_{i=1,\dots,m}$ *is an orthonormal basis of* $\mathfrak{s}$ *then* $\sum_{i=1}^m \left|\tilde{Y}^i\left(ux_0\right)\right|^2$ *is independent of* $u \in K$.

Proof. The translation formula for vector fields in a homogeneous spaces yields

$$\tilde{Y}\left(ux_0\right) = u_* \left(\mathrm{Ad}\left(u^{-1}\right) Y^i\right)^{\sim}\left(x_0\right) .$$

Hence

$$\sum_{i=1}^m \left|\tilde{Y}^i\left(ux_0\right)\right|^2 = \sum_{i=1}^m \left|\left(\mathrm{Ad}\left(u^{-1}\right) Y^i\right)^{\sim}\left(x_0\right)\right|^2 .$$

By the previous lemma the right hand side is equal to $\sum_{i=1}^m \left|\tilde{Y}^i\left(x_0\right)\right|^2$ because $\mathrm{Ad}\left(u^{-1}\right) Y^i$, $i = 1, \dots, m$ form an orthonormal basis. □

In the light of this corollary we write

$$c_H = \sum_{i=1}^m \left|\tilde{Y}^i\left(ux_0\right)\right|^2 .$$

This is a constant independent of $u \in K$ and the orthonormal basis $\left(Y^i\right)_{i=1,\dots,m}$ of $\mathfrak{s}$. In order to compute this constant we put $u = 1$ and choose an orthonormal adapted basis, which complements

$$\{S_1, \dots, S_m\} = \{\frac{\sqrt{2}}{2}\sigma Y_1, \dots, \frac{\sqrt{2}}{2}\sigma Y_m\}$$

where $\{Y_1, \dots, Y_m\}$ is a basis of $\mathfrak{n}^+$, as the construction in (2). By Lemma 2.4 $\{\tilde{S}_1\left(x_0\right), \dots, \tilde{S}_m\left(x_0\right)\}$ is an orthogonal basis of $T_{x_0}\mathbf{B}_H$ with respect to the B metric. Moreover, for $j = 1, \dots, m$,

$$\langle S_j, S_j\rangle = \frac{1}{2}\langle Y_j - \theta Y_j, Y_j - \theta Y_j\rangle = B_\theta\left(Y_j, Y_j\right) = 1$$

so that

$$\left|\tilde{S}_j\left(x_0\right)\right|^2 = \alpha\left(H\right)$$

if $Y_j \in \mathfrak{g}_\alpha$. Therefore, summing up over $\alpha > 0$ gives the value of c_H:

Proposition 5.5. $c_H = \sum_{\alpha>0} d_\alpha\, \alpha\left(H\right)$ *where* $d_\alpha = \dim \mathfrak{g}_\alpha$.

From this proposition we get easily an expression for the Lyapunov exponent matrix for the symmetric system. For a root α define H_α by $\alpha\left(\cdot\right) = \langle H_\alpha, \cdot\rangle$. Let $Q \in \mathfrak{a}$ be the constant function $Q\left(b\right)$. Then Q is the only element of $\mathfrak{a}$ satisfying

$$\langle Q, H\rangle = \frac{c_H}{2} = \frac{1}{2}\sum_{\alpha>0} d_\alpha \alpha\left(H\right) .$$

for every $H \in \mathfrak{a}$. Hence

$$Q = \frac{1}{2} \sum_{\alpha > 0} d_\alpha H_\alpha \,.$$

After integrating this constant with respect to the invariant probability measure on $\mathbf{B}$ we get

Theorem 5.6. *For the symmetric system (20) the Lyapunov exponent matrix is*

$$\Lambda = \frac{1}{2} \sum_{\alpha > 0} d_\alpha H_\alpha \tag{22}$$

This result agrees with Theorem 8.2 in Malliavin and Malliavin [21] (see also Taylor [26, Cor. 5.2]). The factor 1/2 appearing in (22) is due to our normalization of (20) which is given by an orthonormal basis of $\mathfrak{s}$. Also, in [21] appears a minus sign which is due to the fact that the horizontal diffusion is given by g_t^{-1} where g_t is the solution of (20) and the Iwasawa decomposition considered in [21] is NAK so that the two A -parts are inverse to each other.

The symmetric stochastic differential equation (20) also induces Brownian motions in the immersed flag manifolds. Indeed, by Lemma 2.3 and the remarks afterwards, if $H \in \mathfrak{a}^+$ is such that $\alpha(H) = 1$ for every positive root α such that $\alpha(H) \neq 0$ then the B metric in $\mathbf{B}_H$ is induced by its immersion in $\mathfrak{s}$. On the other hand Lemma 2.2 ensures that if $Z \in \mathfrak{s}$ then the vector field $\tilde{Z}$ induced on $\mathbf{B}_H$ is the gradient of the height function $X \mapsto \langle Z, X \rangle$. Since (20) is made up from an orthonormal basis of $\mathfrak{s}$, it follows that the diffusion generated by the induced differential equation on $\mathbf{B}_H$ is a gradient Brownian system with respect to the B metric. More generally, Liao [18, Thm. 1] shows that in each flag manifold there is a canonical K-invariant Riemannian metric for which our symmetric system induces a Brownian motion.

Like in the case of linear systems, here, the A-part of the Iwasawa decomposition gives the Lyapunov spectrum of the induced system. Hence, if Λ stands for the Lyapunov exponent matrix then the eigenvalues of $\mathrm{ad}(\Lambda)$ in the tangent space at the origin of the flag manifold are the Lyapunov exponents, given by:

$$\{-\alpha(\Lambda) : \alpha \in \Pi^+, \alpha(H) \neq 0\}$$

(see [18, Section 5] for a detailed discussion of these Lyapunov exponents).

6. Examples

We will describe here some of the semi-simple Lie groups together with their flag manifolds, emphasizing the B metric which by Proposition 4.5 enters in the formula for the Lyapunov exponent matrix.

6.1. **Real rank 1 groups.** The real rank of $\mathfrak{g}$ (or G) is the dimension of the subalgebra $\mathfrak{a}$. For a rank one Lie group there is just one flag manifold, which is diffeomorphic to a sphere S^n with dimension $n = \dim \mathfrak{s} - 1$. There is just one simple root α and the positive roots are α and possibly 2α. Hence for a symmetric system the Lyapunov exponent matrix is

$$\Lambda = d_\alpha H_\alpha + d_{2\alpha} H_{2\alpha} = (d_\alpha + 2d_{2\alpha}) H_\alpha$$

with $d_{2\alpha} = 0$ if 2α is not a root. Since $\mathrm{Ad}(K)$ is transitive on the spheres of $\mathfrak{s}$, modulo constant multiples there is just one B metric. It is given by the isometric immersion of S^n into $\mathfrak{s}$ when $d_{2\alpha} = 0$, hence in this case it is the canonical Riemannian metric in S^n.

The simple rank one Lie algebras are composed of three series of algebras and an exceptional one (see [14, Ch. X, Table V]). Below we list them with the corresponding dimensions of the root spaces and of $\mathfrak{s}$.

- $\mathfrak{so}(1,n)$; $d_\alpha = n-1$, $d_{2\alpha} = 0$; $\dim \mathfrak{s} = n$. (This class includes $\mathfrak{sl}(2,\mathbb{R}) \approx \mathfrak{sp}(1,\mathbb{R}) \approx \mathfrak{so}(1,2)$ and $\mathfrak{su}^*(4) \approx \mathfrak{so}(1,5)$);
- $\mathfrak{su}(1,n)$; $d_\alpha = 2(n-1)$, $d_{2\alpha} = 1$; $\dim \mathfrak{s} = 2n$. (This class includes $\mathfrak{so}^*(6) \approx \mathfrak{su}(1,3)$);
- $\mathfrak{sp}(1,n)$; $d_\alpha = 4(n-1)$, $d_{2\alpha} = 3$; $\dim \mathfrak{s} = 4n$;
- A real form of the exceptional Lie algebra F_4; $d_\alpha = 8$, $d_{2\alpha} = 7$; $\dim \mathfrak{s} = 16$.

Symmetric systems in $\mathfrak{so}(1,n)$ were studied by Baxendale [4]. This is the Lie algebra of real matrices of the form

$$\begin{pmatrix} 0 & \gamma \\ \gamma^t & B \end{pmatrix} \tag{23}$$

with B skew-symmetric and γ a $1 \times n$-matrix. For a Cartan decomposition we can take $\mathfrak{s}$ to be the subspace of matrices in (23) with $B = 0$ and

$$\mathfrak{a} = \{H(\gamma) = \begin{pmatrix} 0 & \gamma \\ \gamma^t & 0 \end{pmatrix} : \gamma = (x, 0, \ldots, 0), x \in \mathbb{R}\}. \tag{24}$$

The computation of adjoints and the Cartan-Killing form gives, for the simple root α,

$$H_\alpha = \frac{1}{2(n-1)} H(\gamma)$$

where $\gamma = (1, 0, \ldots, 0)$. Hence the matrix Lyapunov exponent is $\Lambda = \frac{1}{4} H(\gamma)$. The eigenvalues of this matrix are the Lyapunov exponents computed in [4, Thm. 2.6], with $\lambda = \sqrt{1/(2(n-1))}$ (the notation is as in [4]). This normalization of the system of [4] comes from the fact that we consider systems made up of an orthonormal basis in $\mathfrak{s}$. In fact, with $H(\gamma)$ and γ as in (23), $\langle H(\gamma), H(\gamma)\rangle = 2(n-1)|\gamma|^2$ with $|\gamma|$ given by the canonical inner product in $\mathbb{R}^n$.

The Lie algebra $\mathfrak{su}(1,n)$ is the algebra of the skew-Hermitian matrices with respect to a Hermitian form of signature $(1,n)$. It is realized as the algebra of complex matrices of the form

$$\begin{pmatrix} it & z \\ z^* & B \end{pmatrix} \tag{25}$$

where $t \in \mathbb{R}$, $z \in \mathbb{C}^n$ is a $1 \times n$ complex matrix and $B + B^* = 0$. We denote the matrix in (25) by $(t, z, B) \in \mathbb{R} \times \mathbb{C}^n \times \mathfrak{u}(n)$. There are the following data:

$$\mathfrak{k} = \{(t, 0, B) : \mathrm{tr} B = -it\}, \qquad \mathfrak{s} = \{(0, z, 0) : z \in \mathbb{C}^n\} \approx \mathbb{C}^n$$

and $\mathfrak{a} = \mathrm{span}_{\mathbb{R}} H_0$ with $H_0 = (0; 1, 0, \ldots, 0; 0)$ is a maximal abelian subalgebra of $\mathfrak{s}$. The eigenvalues of $\mathrm{ad}(H_0)$ are $0, \pm 1, \pm 2$, so that the roots are $\pm\alpha, \pm 2\alpha$ with $\alpha(H_0) = 1$. Since $\dim \mathfrak{g}_{\pm\alpha} = 2(n-1)$ and $\dim \mathfrak{g}_{\pm 2\alpha} = 1$,

$$\langle H_0, H_0 \rangle = 4(n-1) + 8 = 4(n+1) .$$

Hence the Cartan-Killing form is $\langle C, D \rangle = 2(n+1) \mathrm{tr}(CD)$ for $C, D \in \mathfrak{su}(1, n)$.

If $z \in \mathbb{C}^n$ is such that its first component is purely imaginary we denote by $[z]$ the $n \times n$ skew-Hermitian matrix $B = (b_{jk})$ whose first row is z and $b_{jk} = 0$ if $j, k \geq 2$. By computing the eigenspaces of $\mathrm{ad}(H_0)$, it follows that $\mathfrak{n}^+ = \mathfrak{g}_\alpha + \mathfrak{g}_{2\alpha}$ is the subalgebra whose elements are of the form $(t; z; [z])$ with $z = (-it, z_2, \ldots, z_n)$. If $Y = (t; z; [z]) \in \mathfrak{n}^+$ then $Y + \theta(Y) = 2(t; 0; [z])$ because θ fixes $\mathfrak{k}$ and changes the sign in $\mathfrak{s}$. This implies that

$$\zeta : \begin{pmatrix} it & 0 \\ 0 & [z] \end{pmatrix} \longmapsto \begin{pmatrix} 0 & z \\ z^* & 0 \end{pmatrix} .$$

Now, let $X = (t; 0; [z])$ and $Y = (s; 0; [w])$ with $z_1 = it$ and $w_1 = is$. Then

$$[H_0, X] = (0; 2it, z_2, \ldots, z_n; 0) .$$

So that

$$\left(\tilde{X}, \tilde{Y}\right)_{H_0} = 4(n+1)\left(2ts + \mathrm{Re}\left(z_2 \bar{w}_2 + \cdots + z_n \bar{w}_n\right)\right) .$$

Through the identification of $\mathfrak{s}$ with $\mathbb{C}^n$, $\tilde{X}(H_0)$ becomes the tangent vector

$$-(2it, z_2, \ldots, z_n) .$$

Since a similar expression holds for $\tilde{Y}$, the B metric in the tangent space to S^{2n-1} at $(1, 0, \ldots, 0)$ is

$$((it, z_2, \ldots, z_n), (is, w_2, \ldots, w_n)) = (n+1)\left(2ts + \mathrm{Re}\left(z_2 \bar{w}_2 + \cdots + z_n \bar{w}_n\right)\right) .$$

Note that $(it, 0, \ldots, 0)$ is in the direction of the complex line spanned by $(1, 0, \ldots, 0)$. This inner product extends to the whole sphere through the action of $\mathrm{SU}(n)$ which is contained in K. Since $\mathrm{SU}(n)$ maps complex subspaces into complex subspaces, we have the following description of the B metric at $z \in S^{2n-1}$: it is $2(n+1)$ times the canonical inner product in the direction of the complex line spanned by z and $(n+1)$ times the canonical inner product in the subspace orthogonal to this complex line.

We refrain to write down the details for the algebra $\mathfrak{sp}(1, n)$. The description is similar to $\mathfrak{su}(1, n)$ with a quaternionic space playing the role of $\mathbb{C}^n$.

6.2. **Real Forms.** Up to isomorphism each complex semi-simple Lie algebra $\mathfrak{g}_{\mathbb{C}}$ has just one normal real form $\mathfrak{g}$ (see *e.g.* [14]). For such a real form a maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{s}$ is a Cartan subalgebra of $\mathfrak{g}$, the root spaces $\mathfrak{g}_\alpha$ are one dimensional and the subalgebra $\mathfrak{m} \subset \mathfrak{k}$ reduces to zero. We describe below the normal real forms of the simple complex classical Lie algebras, together with their flag manifolds. We use the notation E_{ij} for the matrix whose only nonzero entry is 1 in position i, j. Also,

$$\lambda_i : \mathrm{diag}\{a_1, \dots, a_n\} \longmapsto a_i$$

is a linear functional in the space of diagonal matrices.

Example A_l**.** The Lie algebra $\mathfrak{sl}\,(n, \mathbb{R})$ is the normal real form of $\mathfrak{sl}\,(n, \mathbb{C})$. We have $\mathfrak{k} = \mathfrak{so}\,(n, \mathbb{R})$, $\mathfrak{s}$ the space of trace zero symmetric $n \times n$ matrices and $\mathfrak{a}$ the subalgebra of diagonal matrices in $\mathfrak{sl}\,(n, \mathbb{R})$. The Cartan-Killing form is

$$\langle X, Y \rangle = 2n\,\mathrm{tr}\,(XY)\ .$$

The set of $(\mathfrak{g}, \mathfrak{a})$-roots is $\Pi = \{\alpha_{ij} = \lambda_i - \lambda_j : i \neq j\}$. The root space $g_{\alpha_{ij}}$ is spanned by E_{ij}, $i \neq j$. The co-root α_{ij} with respect to the Cartan-Killing is the matrix

$$H_{\alpha_{ij}} = \frac{E_{ii} - E_{jj}}{2n}\,.$$

We can declare $\alpha_{ij} > 0$ if $i < j$. Hence for a symmetric system normalized by the Cartan-Killing form its Lyapunov exponent matrix is

$$\Lambda = \frac{1}{2n}\mathrm{diag}\{n-1, n-3 \dots, -n+1\}\,.$$

The flag manifolds of a group whose Lie algebra is $\mathfrak{sl}\,(n, \mathbb{R})$ are the standard flag manifolds: Given a sequence of positive integers $\mathbf{r} = (r_1, \dots r_k)$ with $r_1 + \cdots + r_k = n$ let $\mathbb{F}\,(\mathbf{r})$ stand for the manifold of all flags $(V_1 \subset \cdots \subset V_k)$ where V_i is a subspace of $\mathbb{R}^n$ with $\dim V_i = r_1 + \cdots + r_i$. Let

$$H = \mathrm{diag}\{a_1, \dots, a_n\} \qquad a_1 + \cdots + a_n = 0$$

be such that $a_1 = \cdots = a_{r_1} > a_{r_1+1} = \cdots = a_{r_1+r_2} > \cdots$. Then $\mathbb{F}\,(\mathbf{r})$ identifies with the orbit of H under conjugations by orthogonal matrices. By varying H we get different embeddings of $\mathbb{F}\,(\mathbf{r})$ into $\mathfrak{s}$ with corresponding B metric. In case H has only two eigenvalues with multiplicities d and $n-d$, $\mathbb{F}\,(\mathbf{r})$ is the Grassmannian $\mathrm{Gr}_d\,(n)$ of d-dimensional subspaces of $\mathbb{R}^n$. In this case the B metric is just the metric induced by the embedding in $\mathfrak{s}$. It is a multiple of the canonical metric in the $\mathrm{Gr}_d\,(n)$ which turns it into a locally symmetric space (see Kobayashi and Nomizu [17]). The Grassmannians are the only immersed flag manifolds.

Example C_l**.** The normal real form of the complex simple Lie algebra $\mathfrak{sp}\,(n, \mathbb{C})$ is the Lie algebra $\mathfrak{sp}\,(n, \mathbb{R})$ of real symplectic matrices. It is the Lie algebra of matrices which are skew with respect to the canonical symplectic form

$$\omega\,(u, v) = u^t \begin{pmatrix} 0 & -1_{n\times n} \\ 1_{n\times n} & 0 \end{pmatrix} v \qquad u, v \in \mathbb{R}^{2n}\,.$$

Hence it is given by the $2n \times 2n$ real matrices of the form

$$(A, B, C) = \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}$$

with B and C symmetric $n \times n$ matrices. A Cartan decomposition is given by the symmetric and skew symmetric matrices in $\mathfrak{sp}(n, \mathbb{R})$, that is,

$$\mathfrak{k} = \{(A, B, -B^t) : A + A^t = 0\} \approx \mathfrak{u}(n) \qquad \mathfrak{s} = \{(A, B, B^t) : A = A^t\}.$$

We can choose

$$\mathfrak{a} = \{(A, 0, 0) : A = \operatorname{diag}(a_1, \ldots, a_n)\}.$$

Then the roots are $\alpha_{ij} = \lambda_i - \lambda_j$, $i \neq j$ and $\pm\beta_{ij} = \pm(\lambda_i + \lambda_j)$ with $\Pi^+ = \{\alpha_{ij}, i < j; \beta_{ij}\}$. Since the Cartan-Killing form is $\langle C, D\rangle = 2n\operatorname{tr}(CD)$, the co-roots are

$$H_{\alpha_{ij}} = \frac{1}{2n}(E_{ii} - E_{jj}, 0, 0) \qquad H_{\beta_{ij}} = \frac{1}{2n}(E_{ii} + E_{jj}, 0, 0).$$

Adding up the positive co-roots we get the Lyapunov exponent matrix for a symmetric system

$$\Lambda = \frac{1}{2(n+1)}(\operatorname{diag}\{2n-1, 2n-3, \ldots, 1\}, 0, 0).$$

A subspace $V \subset \mathbb{R}^{2n}$ is Lagrangian if the restriction of ω to V is identically zero. By force $\dim V \leq n$. Denote by $\mathrm{L}_d(n)$, $d = 1, \ldots, n$ the set of all d -dimensional Lagrangian subspaces of $\mathbb{R}^{2n}$. Similarly, denote by $\mathbb{F}\mathrm{L}(\mathbf{r})$ the subset of $\mathbb{F}(\mathbf{r})$ consisting of flags $(V_1 \subset \cdots \subset V_k)$ which are made up of Lagrangian subspaces. Any flag manifold of $\mathfrak{sp}(n, \mathbb{R})$ is some $\mathbb{F}\mathrm{L}(\mathbf{r})$. In particular, the minimal flag manifolds are $\mathrm{L}_d(n)$, $d = 1, \ldots, n$. For a sequence $\mathbf{r}$ let

$$A_{\mathbf{r}} = \operatorname{diag}\{a_1, \ldots, a_n\}$$

be such that $a_1 = \cdots = a_{r_1} > a_{r_1+1} = \cdots = a_{r_1+r_2} > \cdots$, and put $H_{\mathbf{r}} = (A_{\mathbf{r}}, 0, 0)$. Then the $\operatorname{Ad}(K)$-orbit of H identifies with $\mathbb{F}\mathrm{L}(\mathbf{r})$. Since $H_{\mathbf{r}}$ has just one positive eigenvalue if and only if $\mathbf{r} = (n)$, $\mathrm{L}_n(n)$ is the only flag manifold of $\mathfrak{sp}(n, \mathbb{R})$ which is immersed.

Example B-D_l**.** The normal real form of $\mathfrak{so}(n, \mathbb{C})$ is $\mathfrak{so}(l, n-l)$ the Lie algebra of the matrices which are skew-symmetric with respect to a quadratic form of signature $(l, n-l)$. Here $n = 2l$ or $2l+1$ according if it is even or odd. The description here parallels that of the symplectic Lie algebra, with the quadratic form instead of the symplectic one. We shall avoid it here, but record that, in the canonical realization of the algebras, the Lyapunov exponent matrix for a symmetric system is given by

$$\Lambda = \frac{1}{2(l-1)}\begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix} \qquad A = \operatorname{diag}\{2l-2, 2l-4, \ldots, 0\}$$

for $n = 2l$ even and

$$\Lambda = \frac{1}{2l-1}\begin{pmatrix} 0 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & -A \end{pmatrix} \qquad A = \operatorname{diag}\{2l-1, 2l-3, \ldots, 1\}$$

for $n = 2l + 1$ odd.

References

[1] Arnold, L., Kliemann, W. and Oeljeklaus, E., *Lyapunov exponents of linear stochastic systems.* In Lyapunov Exponents (Eds. L. Arnold and W. Wihstutz), Lecture Notes Math. - Springer **1186** (1986), 85–128.

[2] Arnold, L., Oeljeklaus, E. and Pardoux, E., *Almost sure and moment stability for linear Itô equations.* In Lyapunov Exponents (Eds. L. Arnold and W. Wihstutz), Lecture Notes Math. - Springer **1186** (1986), 129–159.

[3] Arnold, L. and Imkeller, P., *Furstenberg-Khasminskii formulas for Lyapunov exponents via antecipative calculus*, Stochastics and Stochastics Reports, **54** (1+2) (1995), 127–168.

[4] Baxendale, P. H., *Asymptotic behavior of stochastic flows of diffeomorphisms: Two case studies*, Probab. Theory Related Fields, **73** (1986), 51–85.

[5] Baxendale, P. H., *The Lyapunov spectrum of a stochastic flow of diffeomorphisms.* In Lyapunov Exponents (Eds. L. Arnold and W. Wihstutz), Lecture Notes Math. - Springer **1186** (1986), 322–337.

[6] Borel, A., *Kählerian coset spaces of semi-simple Lie groups.* Proc. Nat. Acad. Sci. **40** (1954), 1147–1151.

[7] Carverhill, A. P., *Flows of stochastic dynamical systems: Ergodic Theory*, Stochastics **14** (1985), 273–317.

[8] Carverhill, A. P., *A Formula for the Lyapunov numbers of a stochastic flow. Application to a perturbation theorem*, Stochastics **14** (1985), 209–226.

[9] Carverhill, A. P., *A non-random Lyapunov spectrum for non-linear stochastic systems*, Stochastics **17** (1986), 253–287.

[10] Carverhill, A. P. and Elworthy, K. D., *Lyapunov exponents for a stochastic analogue of the geodesic flow*, Trans. Amer. Math. Soc. **295** (1986), 85–105.

[11] Duistermaat, J. J., Kolk, J. A. C. and Varadarajan, V., *Functions, flows and oscillatory integrals on flag manifolds and conjugacy classes in real semisimple Lie groups*, Compositio Math. **49** (1983), 309–398.

[12] Furstenberg, H. and Kesten, H., *Products of random matrices*, Ann. Math. Stat. **31** (1960), 457–469.

[13] Guivarc'h, Y. and Raugi, A., *Frontière de Furstenberg, propriétés de contraction et théorèmes de convergence*, Z. Wahrscheinlinchkeitstheor. Verw. Geb. **69** (1985), 187–242.

[14] Helgason, S., Differential geometry, Lie groups and symmetric spaces, Academic Press (1978).

[15] Ikeda, N. and Watanabe, S., Stochastic differential equations and diffusion processes, North-Holland (1981).

[16] Khashminskii, R. Z., Stochastic stability of differential equations, Sijthoff and Noordhoff, Alphen (1980).

[17] Kobayashi, S. and Nomizu, K., Foundations of differential geometry, Interscience Publishers (1963 and 1969).

[18] Liao, M., *Stochastic flows on the boundaries of Lie groups*, Stochastics Stochastics Rep. **39** (1992), 213–237.

[19] Liao, M., *Liapunov Exponents of Stochastic Flows*, Ann. Probab. **25** (1997), 1241–1256.

[20] Liao, M., *Invariant diffusion processes in Lie groups and stochastic flows*, Proc. of Symposia in Pure Math. **57** (1995), 575–591.

[21] Malliavin, M. P. and Malliavin, P., *Factorisations et lois limites de la diffusion horizontale au-dessus d'un espace Riemannien symmetrique*, Lecture Notes Math. **404** (1974), 164–217.

[22] Oseledec, V. I., *A multiplicative ergodic theorem. Lyapunov characteristic numbers for dynamical systems*, Trans. Moscow Math. Soc. **19** (1968), 197–231.

[23] Ruelle, D., *Ergodic theory of differentiable dynamical systems*, I.H.E.S. – Publ. Math. **50**, (1979), 275–306.

[24] San Martin, L. A. B. and Arnold, L., *A Control problem related to the Lyapunov spectrum of stochastic flows*, Mat. Apl. Comput. **5** (1986), 31–64.

[25] Sussmann, H. and Jurdjevic, V., *Controllability of nonlinear systems*, J. Differential Equations **12** (1972), 95–116.

[26] Taylor, J. C., *The Iwasawa decomposition and the limiting behavior of Brownian motion on a symmetric space of non-compact type*, Contemp. Math. AMS **73** (1988), 303–302.

[27] Warner, G., Harmonic Analysis on Semi-simple Lie Groups, Springer-Verlag (1972).

Instituto de Matemática, Universidade Estadual de Campinas
Cx. Postal 6065, 13.081-970 Campinas-SP, BRASIL
E-mail: `smartin@ime.unicamp.br`, `ruffino@ime.unicamp.br`