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# COVARIANTS UNDER THE FULL LINEAR GROUP

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The matrix of transformations of homogeneous products of degree $r$ in the coefficients of a ground form of type $\{n\}$ corresponds to the expression $\{n\} \otimes \{r\}$ [1]. Also the matrix of transformation of homogeneous products of degree $s$ in the coefficients of a ground form of type $\{m\}$ corresponds to the expression $\{m\} \otimes \{s\}$. The matrix of transformation of homogeneous products which are of degree $r$ in the coefficients of the first ground form & of degree $s$ in the coefficients of the second ground form corresponds to the direct product of these two matrices & since the spur of a direct product is the product of the spurs, therefore it corresponds to the expression $(\{n\} \otimes \{r\})(\{m\} \otimes \{s\})$. Hence if $(\{n\} \otimes \{r\})(\{m\} \otimes \{s\}) = \sum \Gamma_\lambda \{\lambda\}$ there will be a simultaneous concomitant of the two ground forms of degree $r$ in the first & of degree $s$ in the second for each $\{\lambda\}$ in the summation. But it is known that the principal parts of the products of terms in the expression of $(\{n\} \otimes \{r\})(\{m\} \otimes \{r\})$ appear as terms in $\{m + n\} \otimes \{r\}$.[2] Hence it is clear that simultaneous concomitants of degree $r$ in the coefficients of 2 ground forms of types $\{n\}$ & $\{m\}$ are related to concomitants that are of degree $r$ in the coefficients of a ground form of type $\{m + n\}$.

**Theorem.** *If $f = gh$ and $G$ is an irreducible covariant of degree $r$ in the coefficients of $g$ & $H$ is an irreducible covariant of degree $r$ in the coefficients of $h$, then there exists an irreducible covariant of degree $r$ in the coefficients of $f$ which can be expressed as a function of $G$, $H$ & the simultaneous covariant of $g$ & $h$ that is of degree $r$ in the coefficients.*

Proof. Given $f = gh$, let $g$ be a ground form of type $\{n\}$, $h$ be a ground form of type $\{m\}$ and that $\Phi$, $\Psi$ be symbolic expressions for two irreducible covariants of degree $r$ in the coefficients of $g$ & $h$, that appear as terms in $\{n\} \otimes \{r\}$, $\{m\} \otimes \{r\}$ respectively.

If we express $\Phi$ in terms of the symbols $\alpha_i \alpha'_i \alpha''_i \dots$ & $\Psi$ in terms of $\beta_i \beta'_i \beta''_i \cdots$

[1]) Littlewood [2].
[2]) Ibrahim [1].

then $\Phi\Psi$ either gives the symbolic expression for the direct product of the two covariants of $g$ & $h$ that are of degree $r$ in the coefficients or the symbolic expression for a simultaneous covariant of degree $r$ in the coefficients of the 2 quantics $g$ & $h$. Hence an irreducible form of type $\{\lambda\}$ that appears in $\{n + m\} \otimes \{r\} = \sum \Gamma_\lambda \{\lambda\}$ could be a function of the corresponding irreducible covariants, of degree $r$ in the coefficients of $g$ & $h$; and of their simultaneous covariant that is of degree $r$ in the coefficients of $g$ & $h$. The irreducible covariants appear as terms in $\{n\} \otimes \{r\}$ & & $\{m\} \otimes \{r\}$ and the simultaneous covariant appears as a term in $(\{n\} \otimes \{r\})$ . . $(\{m\} \otimes \{r\})$. The existence of this relation proves the theorem.

For the binary cubic $f(xy) = \{3\} = ax^3 + 3bx^2y + 3cxy^2 + dy^3$ there is an irreducible covariant $\{42\}$ which appears as a term in $\{3\} \otimes \{2\}$. It is of the second degree in the coefficients of $f$. In symbols it is denoted by $(\alpha\beta)^2 \alpha_x \beta_x$ where $f(xy) = = \alpha_x^3 = \beta_x^3$ and in full it is $(ac - b^2) x^2 + (ad - bc) xy + (bd - c^2) y^2$ .

For a binary form $g = \{1\} = a_1 x + b_1 y$, the form itself is a covariant of the first degree in the coefficients.

For a binary quadratic $h = \{2\} = a_2 x^2 + 2h_2 xy + b_2 y^2$ there is the invariant $\{2^2\}$ which appears as a term in $\{2\} \otimes \{2\}$. It is of the second degree in the coefficients of $h$. In symbols it is $(\alpha\beta)^2$ and in full it is $a_2 b_2 - h_2^2$.

A simultaneous covariant of a binary linear form & a binary quadratic of the second degree in the coefficients could be given by [3])

$$\left(\frac{\partial}{\partial x_1}\frac{\partial}{\partial y_2} - \frac{\partial}{\partial x_2}\frac{\partial}{\partial y_1}\right)^2 (a_1 x_1 + b_1 y_1)^2 (a_2 x_2^2 + 2h_2 x_2 y_2 + b_2 y_2^2)^2$$

after putting $x_1 = x_2 = x$ & $y_1 = y_2 = y$. This gives

$$(a_2 b_2 - h_2^2)(a_1 x + b_1 y)^2 + 3\{(a_2 b_1 - a_1 h_2) x + (h_2 b_1 - b_2 a_1) y\}^2 .$$

The second part can be given symbolically by $[(\alpha\beta)\alpha_x]^2$ taking into consideration that we are dealing with the quadratic $=\alpha_x^2$ & the linear form $\beta_x$. In fact

$$\begin{aligned}(ac - b^2) x^2 + (ad - bc) xy + (bd - c^2) y^2 = \\ = (1/3)(a_1 x + b_1 y)^2 (a_2 b_2 - h_2^2) - (1/9)\{x(a_2 b_1 - a_1 h_2) + \\ + y(b_1 h_2 - a_1 b_2)\}^2 .\end{aligned}$$

The quartic $f = \{4\} = \alpha_x^4 = \beta_x^4 = ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4$ has the irreducible invariant $\{4^2\}$ which appears as a term in $\{4\} \otimes \{2\}$. It is of the second degree in the coefficients of $f$. In symbols it is $(\alpha\beta)^4$ & in full it is $ae - 4bd + 3c^2$.

The binary quadratic $g = \{2\} = a_1 x^2 + 2h_1 xy + b_1 y^2$ has the invariant $\{2^2\}$ which appears as a term in $\{2\} \otimes \{2\}$. It is of the second degree in the coefficients of $g$.

[3]) Turnbull [3].

In symbols it is denoted by $(\alpha\beta)^2$ where $g = \alpha_x^2 = \beta_x^2$. In full it is $a_1 b_1 - h_1^2$. The same for the binary quadratic $h = a_2 x^2 + 2h_2 xy + b_2 y^2$; it has the invariant $a_2 b_2 - h_2^2$.

A simultaneous invariant of the two quadratics is given by

$$\left(a_2 \frac{\partial}{\partial a_1} + b_2 \frac{\partial}{\partial b_1} + h_2 \frac{\partial}{\partial h_1}\right)(a_1 b_1 - h_1^2) = a_2 b_1 + a_1 b_2 - 2h_1 h_2 .$$

In symbols it is $(\alpha\beta)^2$ where $g = \alpha_x^2$, $h = \beta_x^2$.

In fact

$$ae - 4bd + 3c^2 = (a_1 b_1 - h_1^2)(a_2 b_2 - h_2^2) + (1/12)(a_1 b_2 + a_2 b_1 - 2h_1 h_2)^2 .$$

The quintic $f = \{5\} = \alpha_x^5 = \beta_x^5 = ax^5 + 5bx^4 y + 1acx^3 y^2 + 10dx^2 y^3 + 5exy^4 + fy^5$ has the irreducible covariant $\{64\}$ which appears as terms in $\{5\} \otimes \{2\}$. It is of the second degree in the coefficients of $f$. In symbols it is given by $(\alpha\beta)^4 \alpha_x \beta_x$ & in full it is $(2ae + 6c^2 - 8bd)x^2 + (2af - 6be + 4cd)xy + (2bf - 8ce + 6d^2)y^2$.

The binary quadratic $g = \{2\} = a_1 x^2 + 2h_1 xy + b_1 y^2$ has the invariant $\{2^2\} = a_1 b_1 - h_1^2$ as we have mentioned before.

The binary cubic $h = \{3\} = a_2 x^3 + 3b_2 x^2 y + 3c_2 xy^2 + d_2 y^3$ has the irreducible covariant $\{42\}$ which is

$$(a_2 c_2 - b_2^2)x^2 + (a_2 d_2 - b_2 c_2)xy + (b_2 d_2 - c_2^2)y^2$$

as we have shown before.

A simultaneous covariant of the second degree in the coefficients of $g$ & $h$ is given by

$$\left(\frac{\partial}{\partial x_1}\frac{\partial}{\partial y_2} = \frac{\partial}{\partial x_2}\frac{\partial}{\partial y_1}\right)^3 (a_1 x_1^2 + 2h_1 x_1 y_1 + b_1 y_1^2)(a_2 x_2^3 + 3b_2 x_2^2 y_2 + 3c_2 x_2 y_2^2 + d_2 y_2^3)^2$$

or in symbols by $[(\alpha\beta)^2 \alpha_x]^2$ where $\{3\} = \alpha_x^3$, $\{2\} = \beta_x^2$.

In full it is $[(a_2 b_1 + c_2 a_1 - 2b_2 h_1)x + (b_2 b_1 + d_2 a_1 - 2c_2 h_1)y]^2$. In fact

$$\begin{gathered}(2ac + 6c^2 - 8bd)x^2 + (2af - 6be + 4cd)xy + (2bf - 8ce + 6d^2)y^2 = \\ = \lambda\{(a_1 b_1 - h_1^2)[(a_2 c_2 - b_2^2)x^2 + (a_2 d_2 - b_2 c_2)xy + (b_2 d_2 - c_2^2)y^2]\} + \\ + \mu\{a_2 b_1 + c_2 a_1 - 2b_2 h_1)x + (b_2 b_1 + d_2 a_1 - 2c_2 h_1)y\}^2 .\end{gathered}$$

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