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Časopis pro pěstování matematiky, Vol. 109 (1984), No. 2, 168--193

Persistent URL: `http://dml.cz/dmlcz/108502`

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A MAXIMUM PROBLEM FOR OPERATORS

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(Received July 5, 1983)

The investigation of the first maximum problem seems to have reached the stage where the basic problems are solved, the proofs have been simplified and the connections with other branches of mathematics cleared up so that a report about the present state of the theory might be in order.

The present paper, which is predominantly expository, is intended as a summary of the results obtained thus far.

The first maximum problem was introduced in the author's paper [5]. It consists in identifying, among all contractions T on the n-dimensional Hilbert space H_n which are annihilated by a given polynomial p, those operators T which maximize the norm of T^n.

The solution given by the author in [6] was based on a technical device which consists in considering sequences of the type

$$x, Tx, T^2x, \dots$$

and which makes it possible to linearize the problem and to describe an operator for which the required maximum is attained. While the main idea of this method remains unchanged a number of technical simplifications make it possible to present a more transparent and considerably shorter proof. This is done in the second chapter of the present paper.

If $\mathscr{A}(p)$ stands for the set of all contractions which are annihilated by a given polynomial p one of the two main results of [5, 6] was the construction of an operator $T_0 \in \mathscr{A}(p)$ such that the norm of T^n assumes its maximum for $T \in \mathscr{A}(p)$ at T_0.

In spite of the fact that the problem is a finite dimensional one, infinite dimensional methods will be applied. The interest of the method of linearization used in [6] lies in the direct and natural manner in which the passage into infinite dimensional spaces is motivated.

The author's paper [6] was soon followed by a paper of B. Sz-Nagy [22] which describes another way of identifying the extremal operators — his method is based on the fact that, for any contraction T on a Hilbert space H, the mapping

$$x \mapsto ((1 - T^*T)^{1/2}\, x,\ (1 - T^*T)^{1/2}\, Tx,\ (1 - T^*T)^{1/2}\, T^2x, \dots)$$

is an isometry taking H into a subspace of $l^2(\mathscr{D})$ which intertwines T and the backward shift operator on $l^2(\mathscr{D})$. Here $\mathscr{D}$ stands for the closure of the range of the operator $1 - T^*T$. In the same paper Sz-Nagy observed that the extremal operator T_0 described in [6] maximizes, in fact, the norm of any analytic function $f(T)$, not only the norm of T^n. It is not difficult to see that only a slight technical modification of the author's original proof suffices to obtain this interesting fact by the original method. Accordingly, the proof presented in the second chapter is already formulated to include this stronger result.

It is no coincidence that the sequences

$$x,\ Tx,\ T^2x, \ldots$$

play a decisive role in both proofs: in the finite-dimensional case, the first n terms of this sequence already determine the rest. The role of the Gram matrix of these vectors in our method corresponds, in a manner of speaking, to that of the factor $(1 - T^*T)^{1/2}$ in the mapping used by Sz-Nagy: both are used to restore the isometry of certain mappings.

The second section of this paper is devoted to the description of an operator $T_0 \in \in \mathscr{A}(p)$ which maximizes, for any polynomial f, the norm of $f(T)$ if T ranges over $\mathscr{A}(p)$.

We prove this result using the original method — it is the harder way but it is also the more direct one; besides, we shall get some more information about the structure of the extremal operators as well.

It is an interesting coincidence that, about the same time, D. Sarason published his famous paper on generalized interpolation in H^∞. In spite of the fact that the motivation, the method as well as the aim of his investigation are entirely different, it turned out later that our maximum problem is related to questions treated by Sarason. The connection is not at all obvious: to the casual observer it would probably never occur that those two papers might have anything in common at all. Indeed, it takes some effort to see that there is a connection between our maximum problem, formulated as it is in terms of linear algebra with a motivation in numerical mathematics and a study of interpolation problems in spaces of analytic functions.

The first indication of this connection came in a letter of P. J. Williams to the present author: the problem admits a reformulation in the language of complex functions to which the results of Sarason and von Neumann may be applied. The author did not pursue this line of investigation further because it requires the use of very powerful theorems. It was not until 1978 when the work in this direction was resumed. A part of the work of the functional analysis seminar in Prague was devoted to several topics in the theory of complex functions centered about the Carathéodory and Nevanlinna-Pick problems and the work of I. Schur on bounded analytic functions.

Several reports in the seminar of functional analysis presented by J. Fuka, N. J. Young and the author have contributed to the clarification of the connection between the first maximum problem and classical function theory. These connections are summarized in chapter three. In the second part of that chapter we give a proof of the unitary equivalence of our extremal operator $S \mid \operatorname{Ker} \varphi(S)$ and the model operator $S(\varphi)$ used in the theory of Sz-Nagy and Foias. The author is indebted to D. Voiculescu and B. Sz-Nagy for highly stimulating discussions on this matter. The result and its proof are based on conversations with them.

The last section is devoted to a recent improvement due to V. V. Peller of an interesting conjecture formulated originally by N. J. Young. The proof of this conjecture given in [12] is based on the fact that a linear mapping in a finite-dimensional linear space is injective if and only if it is onto. It seems that the real meaning of the essential assumption in the conjecture may only be properly understood if the conjecture is formulated in its full generality where the assumption appears in the form of the condition that the winding number of a certain curve be zero. It was V. V. Peller who observed that a result stronger than the original conjecture may be deduced from a general identity for Hankel and Toeplitz operators.

The survey is preceded by a section which collects some preliminary information about Gram matrices and related topics. Since the notation is not stabilized in the literature we prefer to explain it in detail in particular as it differs, in some points, from the notation used in quite a few standard textbooks.

As an example, we believe that some of the formulae to be used later become neater if we define the ik-th element of the Gram matrix as (g_k, g_i) in disctintion to the notation used by many authors.

The author wishes to acknowledge a debt of gratitude to P. Vrbová whose comments have contributed to a significant improvement of the presentation.

1. NOTATION AND PRELIMINARIES

In the whole paper n will be a fixed natural number, H_n will be an abstract n-dimensional Hilbert space, $\mathscr{L}(H_n)$ the algebra of all linear operators on H_n. We shall also occasionally consider the concrete n-dimensional Hilbert space C^n whose elements are column vectors indexed by $0, 1, \ldots, n-1$. Thus $x \in C^n$ means

$$x = (x_0, x_1, \ldots, x_{n-1})^{\mathsf{T}}.$$

We shall denote by $e_0, e_1, \ldots, e_{n-1}$ the standard unit vectors

$$\begin{aligned} e_0 &= (1, 0, 0, \ldots, 0)^{\mathsf{T}} \\ e_1 &= (0, 1, 0, \ldots, 0)^{\mathsf{T}} \\ &\ldots \end{aligned}$$

An operator A on C^n will be identified with its matrix

$$a_{ik} = (Ae_k, e_i).$$

An n-tuple of vectors $b_0, \ldots, b_{n-1}$ in H_n will be interpreted in two ways. We shall view it either as a row vector $B = (b_0, \ldots, b_{n-1})$ or as a linear operator B from C^n into H_n defined by the relations

$$Be_k = b_k \quad k = 0, 1, \ldots, n-1 .$$

In the particular case where $H_n = C^n$, in other words, if the b_j are column vectors, the row vector $(b_0, \ldots, b_{n-1})$ will become a matrix which happens to be the matrix of the operator B just defined.

Similarly, if $T \in \mathscr{L}(H_n)$ we shall interpret the product TB either as the row vector $(Tb_0, \ldots, Tb_{n-1})$ or as the operator obtained as the superposition of B and T. If $A \in \mathscr{L}(C^n)$ then BA will be understood either as the row vector

$$\left(\sum_i b_i a_{i0}, \ldots, \sum_i b_i a_{i,n-1}\right)$$

– in other words, the formal product of the 1 by n matrix B and the n by n matrix – or as the product of an operator in $\mathscr{L}(C^n)$ and an operator from C^n into H.

If $B = (b_0, b_1, \ldots, b_{n-1})$, $b_j \in H_n$ is a basis of H_n and if $T \in \mathscr{L}(H_n)$ we define $\mathscr{M}(T, B)$, the matrix of T with respect to the basis B, by the requirement that

$$Tb_i = \sum_s m_{si} b_s$$

for every i.

It is easy to see that the equation

$$TB = B\mathscr{M}(T, B)$$

is valid in both interpretations of B, either as the equality of two row vectors or of two operators from C^n into H_n. In the particular case where $H_n = C^n$ the operator T as well as B become matrices for which $TB = B\mathscr{M}(T, B)$.

Summing up: if B is a basis, T an operator and $M = \mathscr{M}(T, B)$, then

$$TB = BM ;$$

this equality characterizes the matrix of T with respect to the basis B. Indeed, if $M' \in \mathscr{L}(C^n)$ satisfies $TB = BM'$ then $M' = \mathscr{M}(T, B)$.

To illustrate the advantages of the formal multiplication introduced above we intend to describe the relation between the matrices of an operator in two different bases.

Suppose X is a basis of H_n and that a new basis Y is introduced by means of the invertible matrix W

$$y_j = \sum_r w_{rj} x_r .$$

If we interpret bases as row vectors $B = (x_0, \ldots, x_{n-1})$ and $B' = (y_0, \ldots, y_{n-1})$ then $B' = BW$ and

$$TB = B\,\mathscr{M}(T, X)$$

whence

$$TB' = TBW = B\,\mathcal{M}(T, X)\,W = BWW^{-1}\,\mathcal{M}(T, X)\,W = B'W^{-1}\,\mathcal{M}(T, X)\,W$$

so that

$$\mathcal{M}(T, Y) = W^{-1}\,\mathcal{M}(T, X)\,W.$$

We shall frequently work with cyclic bases. A vector x is said to be cyclic for the operator T if the vectors $x, Tx, \ldots, T^{n-1}x$ are linearly independent; a basis constructed in this manner from a vector x will be called a cyclic basis for T. It will be convenient to introduce a special notation for the operator B in this particular case. We shall write

$$M(x, T) = (x, Tx, \ldots, T^{n-1}x)$$

so that $M(x, T)$ is the operator from C^n into H_n which takes the column vector $u = (u_0, \ldots, u_{n-1})^{\mathsf{T}}$ into the sum $\sum u_j T^j x$.

If a and b are two elements of some Hilbert space H we denote by b^*a the scalar product (a, b) and by ab^* the operator

$$x \mapsto (x, b)\,a\,.$$

This notation has the advantage that the operator ab^* coincides with the matrix ab^* in the case of the space C^n. Also, it behaves nicely with respect to multiplication; indeed

$$A(ab^*)\,B^* = Aa(Bb)^*$$

for any $A, B \in \mathscr{L}(H)$.

If we denote ab^* by T we have the following formulae

$$T^* = ba^*$$
$$T^*T = |a|^2\,bb^*$$
$$TT^* = |b|^2\,aa^*$$

so that the norm of T equals $|a|\,|b|$.

Since $Ta = ab^*a = (b^*a)\,a$ the vector a is either zero or an eigenvector of T with eigenvalue $b^*a = (a, b)$. The spectrum of T consists thus of at most two numbers; first (a, b) and then zero with multiplicity $n - 1$ since the $(n - 1)$ dimensional subspace $b^{\perp}$ is annihilated by T.

Consider a Hilbert space H. If a vector $u \in H$ is represented as a linear combination of n given vectors $f_0, \ldots, f_{n-1}$

$$u = x_0 f_0 + \ldots + x_{n-1} f_{n-1}$$

then its scalar product with the vector

$$v = y_0 g_0 + \ldots + y_{n-1} g_{n-1}$$

equals

$$(u, v) = \left(\sum_k x_k f_k, \sum_j y_j g_j\right) =$$
$$= \sum_{k,j} y_j^*(f_k, g_j)\, x_k = \sum y_j^* g_{jk} x_k \,.$$

We have denoted by g_{jk} the scalar product (f_k, g_j). The matrix with elements g_{jk} will be called the Gram matrix of the n-tuples $R = (f_0, \dots, f_{n-1})$ and $S = (g_0, \dots, g_{n-1})$ and will be denoted by $G(R, S)$. If $R = S$, we write $G(R)$ instead of $G(R, R)$. If the coordinates are interpreted as column vectors in C^n

$$x = (x_0, \dots, x_{n-1})^\mathsf{T}, \quad y = (y_0, \dots, y_{n-1})^\mathsf{T}$$

the above relation may be rewritten in the form

$$(u, v) = y^* G x = (Gx, y)\,.$$

We shall represent an n-tuple of vectors as a row vector $R = (f_0, \dots, f_{n-1})$. If $S = (g_0, \dots, g_{n-1})$ is another such n-tuple we can write formally

$$G(R, S) = S^* R\,.$$

This formal relation may be justified if we introduce the following notation.

Given an n-tuple B of vector $(b_0, \dots, b_{n-1})$ we define B^* to be the column vector consisting of functionals

$$B^* = (b_0^*, \dots, b_{n-1}^*)^\mathsf{T}\,.$$

Here we take b^* to be the functional

$$b^* x = (x, b)\,.$$

If B is interpreted as a linear operator from C^n into H then B^* has also a meaning as a linear operator from H into C^n; its action can also be described as the formal multiplication of the n by 1 matrix $(b_0^*, \dots, b_{n-1}^*)^\mathsf{T}$ on 1 by 1 matrices – elements of H.

In the particular case of vectors in C^n the row vector R may be identified with the n by n matrix $(f_0, \dots, f_{n-1})$ and it is not difficult to verify that the above formula remains true even in this interpretation of S^*R.

Again, if R and S are interpreted as linear operators from C^n into H then S^*R is a linear operator in C^n and its matrix is just $G(R, S)$.

Consider now two coordinate vectors

$$x = (x_0, \dots, x_{n-1})^\mathsf{T}, \quad y = (y_0, \dots, y_{n-1})^\mathsf{T}$$

and the vectors

$$u = \sum x_j f_j\,, \quad v = \sum y_j g_j\,.$$

Writing then in the form $u = Rx$, $v = Sy$ their scalar product becomes

$$(u, v) = (Rx, Sy) = (Sy)^* Rx = y^* S^* R x = y^* G x = (Gx, y)\,.$$

Suppose now we have two matrices $A = (a_{ik})$ and $B = (b_{ik})$ and that we define new n-tuples of vectors

$$U = (u_0, \ldots, u_{n-1}) \quad V = (v_0, \ldots, v_{n-1})$$

by the relations

$$u_i = \sum a_{ri} f_r \quad v_j = \sum b_{sj} g_s$$

then $G(U, V) = B^* G(R, S)\, A$.

If $R = (f_0, \ldots, f_{n-1})$ and $S = (g_0, \ldots, g_{n-1})$ are two bases in H_n and if T and W are two operators on H_n, we write, in accordance with our convention

$$TR = (Tf_0, \ldots, Tf_{n-1})$$
$$WS = (Wg_0, \ldots, Wg_{n-1}) .$$

Then $G(TR, WS) = \mathscr{M}(W, S)^*\, G(R, S)\, \mathscr{M}(T, R)$.

The second assertion is obviously a consequence of the first one. To prove the formula $G(U, V) = B^* G(R, S)\, A$ it suffices to observe that $U = RA$, $V = SB$ whence

$$G(U, V) = V^* U = B^* S^* R A = B^* G(R, S)\, A .$$

Suppose we have two n-tuples $f_0, \ldots$ and $g_0, \ldots$ such that $G = G(R, S)$ is invertible. (The invertibility of S^*R implies the invertibility of both S and R so that the vectors f as well as the vectors g will be linearly independent.) The following simple method of constructing the inverse matrix $G(R, S)^{-1}$ will be used in the sequel. Suppose we find two operators A and B such that

$$(Af_i, Bg_k) = \delta_{ik}; \text{ let us show that } G^{-1} = \mathscr{M}(A, R)\, \mathscr{M}(B, S)^* .$$

Indeed, it follows from the above formula that

$$1 = \mathscr{M}(B, S)^*\, G\, \mathscr{M}(A, R)$$

whence

$$\mathscr{M}(B, S)^* (G\, \mathscr{M}(A, R)\, \mathscr{M}(B, S)^* - 1) =$$
$$= (\mathscr{M}(B, S)^*\, G\, \mathscr{M}(A, R))\, \mathscr{M}(B, S)^* - \mathscr{M}(B, S)^* = 0 .$$

In a similar manner

$$(\mathscr{M}(A, R)\, \mathscr{M}(B, S)^*\, G - 1)\, \mathscr{M}(A, R) =$$
$$= \mathscr{M}(A, R)\, (\mathscr{M}(B, S)^*\, G\, \mathscr{M}(A, R) - 1) = 0 .$$

If $P = (u_0, \ldots, u_{n-1})$ and $Q = (v_0, \ldots, v_{n-1})$ are two n-tuples in H_n then $G(P) = G(Q)$ if and only if there exists a unitary operator $U \in \mathscr{L}(H_n)$ such that $v_j = Uu_j$ for $j = 0, 1, \ldots, n - 1$. Indeed, if $G(P) = G(Q)$ we have

$$\left|\sum x_j v_j\right|^2 = \left|\sum x_j u_j\right|^2$$

for every n-tuple $x_0, \ldots, x_{n-1}$ so that the mapping V_0 defined by $V_0 u_j = v_j$ may be extended by linearity to the linear span $S(P)$ of the u_j to an isometry V mapping $S(P)$

onto the linear span $S(Q)$ of the v_j. It follows that $\dim S(P) = \dim S(Q)$ so that V may be extended to a unitary operator U.

(1.1) *Let T be an operator in n-dimensional Hilbert space H. Let $X = (x_0, \ldots, x_{n-1})$ be a basis of H. Then*

$$\mathscr{M}(T^*, X) = G^{-1}\, \mathscr{M}(T, X)^*\, G$$

where G is the Gram matrix of X.

Proof. Let a_{pq} and b_{pq} be the elements of $A = \mathscr{M}(T, X)$ and $B = \mathscr{M}(T^*, X)$ respectively. We have

$$(Tx_k, x_j) = \Big(\sum_s a_{sk} x_s, x_j\Big) = \sum_s a_{sk} g_{js} = (GA)_{jk}\,;$$

on the other hand

$$(Tx_k, x_j) = (x_k, T^* x_j) = \Big(x_k, \sum_t b_{tj} x_t\Big) = \sum_t (b_{tj})^*\, g_{tk} = (B^*G)_{jk}\,.$$

We have thus $GA = B^*G$ whence $B = G^{-1}A^*G$.

If there exists a unitary operator U such that $UT_1 = T_2 U$ we shall write $T_1 \sim T_2$ and call T_1 and T_2 unitarily equivalent.

(1.2) *The operators $T_1, T_2 \in \mathscr{L}(H_n)$ are unitarily equivalent if and only if there exist two bases B_1, B_2 such that*

$$\mathscr{M}(T_1, B_1) = \mathscr{M}(T_2, B_2)$$
$$G(B_1) = G(B_2)\,.$$

Proof. Suppose first that $T_1 \sim T_2$. Take any basis $B = (b_0, \ldots, b_{n-1})$ and define a new basis B' by setting $b_j' = Ub_j$, U being the unitary operator intertwining T_1 and T_2, $UT_1 = T_2 U$. Then $G(B') = G(B)$. If m_{ik} are the elements of $\mathscr{M}(T_1, B)$, we have

$$T_1 b_j = \sum_s m_{sj} b_s$$

whence

$$T_2 b_j' = T_2 U b_j = UT_1 b_j = U \sum_s m_{sj} b_s = \sum_s m_{sj} b_s'$$

so that $\mathscr{M}(T_2, B') = \mathscr{M}(T_1, B)$.

On the other hand, if $G(B_1) = G(B_2)$ there exists a unitary operator U which takes $b_j^{(1)}$ into $b_j^{(2)}$ for all j. If $\mathscr{M}(T_1, B_1) = \mathscr{M}(T_2, B_2)$ as well, denote the elements of this matrix by m_{ik} and observe that

$$T_2 U b_j^{(1)} = T_2 b_j^{(2)} = \sum_s m_{sj} b_s^{(2)} = U \sum_s m_{sj} b_s^{(1)} = UT_1 b_j^{(1)}$$

It follows that $T_2 U = UT_1$ and the proof is complete.

It is not difficult to see that the space $\mathscr{L}(H_n)$ is a Hilbert space under the scalar product

$$(A, B) = tr\,(B^*A) = tr\,(AB^*)\,.$$

Let us list here, for further reference, some of the properties of this scalar product; the following relations hold

$$(XA, B) = (A, X^*B)$$
$$(AX, B) = (A, BX^*)\,.$$

It follows that the adjoint of the operator $\mathcal{M} \in \mathcal{L}(\mathcal{L}(H_n))$

$$\mathcal{M} : X \to RXS$$

is the operator $\mathcal{M}^*$

$$\mathcal{M}^*(X) = R^*XS^*\,.$$

Consider now the tensors ab^*. Since clearly $tr\ ab^* = b^*a = (a, b)$ we have the following formulae

$$(ab^*, M) = (a, Mb)$$
$$(M, ab^*) = (Mb, a)\,.$$

Indeed,

$$(ab^*, M) = tr\ ab^*M^* = tr\ a(Mb)^* = (Mb)^*\, a = (a, Mb)\,.$$

In the particular case that $M = R^*R$ we have the useful formula

$$(xy^*, R^*R) = (Rx, Ry)$$

The scalar product (x, y) may be represented as

$$(x, y) = (xy^*, 1) = (1, yx^*)\,.$$

The following lemma will be used later.

(1.3) *Given an operator* $R \in \mathcal{L}(H_n)$ *then*

$$\sup\{(B, R^*R);\ B \geqq 0,\ (B, 1) = 1\} \quad \textit{equals} \quad |R|^2$$

and is attained on one-dimensional operators:

$$|R|^2 = \sup\{(vv^*, R^*R);\ (vv^*, 1) = 1\}\,.$$

Proof. Every $B \geqq 0$ may be written as the sum of n operators of the form $b_j b_j^*$. The condition $(B, 1) = 1$ reduces then to $\sum |b_j|^2 = 1$. Our maximum problem is thus transformed into

$$\sup\{\sum_1^n |Rb_j|^2;\ \sum_1^n |b_j|^2 = 1\}\,.$$

Since $\sum |Rb_j|^2 \leqq |R|^2 \sum |b_j|^2 \leqq |R|^2$ this supremum cannot exceed $|R|^2$; on the other hand, the value $|R|^2$ is attainable by operators B of the form bb^* with $|b| = 1$.

Let A be a linear operator on H_n with a cyclic vector z: in other words, the vectors

$$z, Az, \ldots, A^{n-1}z$$

form a basis of H_n and the minimal polynomial of A coincides with its characteristic polynomial p. Write p in the form

$$p(z) = -(a_0 + a_1 z + \dots + a_{n-1} z^{n-1}) + z^n$$

and consider the matrix

$$T = \begin{bmatrix} 0 & 0 & & 0 & a_0 \\ 1 & 0 & & 0 & a_1 \\ 0 & 1 & & 0 & a_2 \\ \dots & & & & \\ 0 & 0 & & 1 & a_{n-1} \end{bmatrix}.$$

If B stands for $(z, Az, \dots, A^{n-1}z)$ then $AB = BT$, in other words, the matrix of A in the basis B is just T. Recall the dual interpretation of the relation $AB = BT$. It represents either the equality of two row vectors if B is considered as a row of vectors or as an identity for operators if B is taken to mean a linear operator from C^n into H_n.

We shall use the Gram matrix $G = B^*B$ of the basis B to compare the length of a vector u and of its image Au.

Any vector $u \in H_n$ may be written in the form Bx for a suitable $x \in C^n$. Thus

$$|u|^2 = u^*u = x^*B^*Bx = x^*Gx = (Gx, x)$$

and

$$|Au|^2 = |ABx|^2 = |BTx|^2 = x^*T^*BTx = x^*T^*GTx = (T^*GTx, x).$$

We shall frequently work in the space l^2 of all sequences

$$x = (x_0, x_1, \dots)$$

with $\sum |x_j|^2 < \infty$; sometimes we shall identify this space with the Hardy space H^2. The space L^2 is decomposed into the direct sum of H^2 and its orthogonal complement H^2_-. The corresponding projectors will be denoted by P_+ and P_-. We denote by S the backward shift operator

$$Sx = (x_1, x_2, \dots).$$

If g is an element of H^2, $g(z) = \sum a_n z^n$, we shall denote by $\tilde{g}$ the element of H^2 defined by $\tilde{g}(z) = g(z^*)^* = \sum a_n^* z^n$.

2. THE EXTREMAL OPERATOR

In the present section we intend to give a description of the extremal operators for the first maximum problem. Let us recall that the first maximum problem (for a given polynomial p) consists in finding operators A on H_n which maximize the norm of A^r under the constraints $|A| \leqq 1$ and $p(A) = 0$. Here p is a given polynomial of degree n with all roots less than one in modulus, r may be any natural number.

The idea of the method to be used here to characterize the extremals is the same as in the author's original paper [6] except that the present version yields two valuable advantages: first of all, the proof is technically simpler and, accordingly, shorter and more transparent, secondly, a small technical modification makes it possible to prove that the solution of the first maximum problem as stated above actually maximizes the norm of $f(A)$ for any polynomial f, not only the norm of powers of A.

The basic idea of the author's original solution consists in transforming the maximum problem into a linear one. We shall first present the essential lemma in its original form and use it then to obtain a solution of the first maximum problem in its full generality.

Let n be a natural number and let p be a polynomial of degree n all roots of which are less than one in modulus. Let $\mathscr{A}(p)$ be the set of all linear operators A on H_n such that $|A| \leqq 1$ and $p(A) = 0$. We shall describe an operator $A_p \in \mathscr{A}(p)$ such that, for any polynomial f

$$|f(A_p)| = \max \{|f(A)|;\ A \in \mathscr{A}(p)\} .$$

It will be convenient to introduce some terminology and notation. A linear operator is said to be a contraction if its norm does not exceed one; possibly a warning might be in order here not to confuse the technical meaning of the word with the meaning it has in everyday language – the identity mapping is a contraction.

Let p be the polynomial

$$p(z) = -(a_0 + a_1 z + \ldots + a_{n-1} z^{n-1}) + z^n$$

and set

$$T = \begin{pmatrix} 0 & & a_0 \\ 1 & & \\ 0 & 1 & a_{n-1} \end{pmatrix}.$$

Denote by $\mathscr{S}$ the operator of congruence defined on $\mathscr{L}(C^n) = M_n$ by the formula

$$\mathscr{S}X = T^* X T .$$

Suppose now that the spectral radius of the polynomial p is less than one; then $|\mathscr{S}|_\sigma < 1$ so that, in particular, $(1 - \mathscr{S})^{-1}$ exists.

Recall that we have denoted by E_0 the matrix $e_0 e_0^*$ where $e_0 = (1, 0, \ldots, 0)^{\mathsf{T}}$ so that

$$E_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ & & \\ 0 & 0 & 0 \end{pmatrix}.$$

Kepping in mind these notations and facts let us formulate a technical lemma on which the possibility of linearization is based.

(2,1) **Lemma.** *Denote by $\mathscr{G}$ the set of all matrices of the form*

$$G(z, Az, \ldots, A^{n-1} z)$$

where A ranges over all contractions A on H_n with $p(A) = 0$ and z over all vectors of norm 1.

Let

$$\mathcal{M} = \{Z;\ (1 - \mathcal{S})\, Z \geqq 0,\ (Z, E_0) = 1\}\ ;$$

then

$$\mathcal{G} = \mathcal{M}\ .$$

Proof. Suppose first that we are given a matrix $Z \in \mathcal{G}$. Then

$$(Z, E_0) = (Z)_{00} = |z|^2 = 1$$

so that the second condition in the definition of $\mathcal{M}$ is satisfied. Furthermore

$$(1 - \mathcal{S})\, Z = Z - T^*ZT \quad \text{and}$$

$$Z - T^*ZT = G(z, Az, \ldots, A^{n-1}z) - G(Az, A^2z, \ldots, A^nz)\,.$$

Consider now a vector $x \in C^n$ and the corresponding element

$$u = Bx \in H_n \quad \text{where} \quad B = (z, Az, \ldots, A^{n-1}z)\,.$$

We have then $Z = B^*B$ and

$$((Z - T^*ZT)\, x, x) = ((B^*B - T^*B^*BT)\, x, x) =$$
$$= |Bx|^2 - |BTx|^2 = |u|^2 - |Au|^2 \geqq 0\,,$$

so that $(1 - \mathcal{S})\, Z \geqq 0$. This proves the inclusion $\mathcal{G} \subset \mathcal{M}$.

Now suppose that Z is a matrix with $Z - T^*ZT \geqq 0$ and $z_{00} = 1$. We have to prove the existence of a contraction A with $p(A) = 0$ and of a vector z such that

$$Z = G(z, Az, \ldots, A^{n-1}z)\,.$$

We shall use the fact that the spectral radius of p is less than one. Since

$$Z - T^{*k}ZT^k = \sum_{j=0}^{k-1} T^{*j}(Z - T^*ZT)\, T^j \geqq 0$$

and $\lim T^k = 0$, it follows that $Z \geqq 0$. Hence there exists a sequence of vectors $z_0, z_1, \ldots, z_{n-1}$ such that

$$Z = G(z_0, z_1, \ldots, z_{n-1})\,.$$

Now define a sequence $w_0, w_1, \ldots, w_{n-1}$ as follows

$$w_i = z_{i+1} \quad \text{for} \quad 0 \leqq i < n - 1$$

$$w_{n-1} = a_0z_0 + \cdots + a_{n-1}z_{n-1}$$

and let us prove that

$$\left|\sum \zeta_j w_j\right|^2 \leqq \left|\sum \zeta_j z_j\right|^2$$

for every sequence of complex numbers $\zeta_0, \ldots, \zeta_{n-1}$. Indeed, if we write x for the vector $(\zeta_0, \ldots, \zeta_{n-1})^{\mathsf{T}}$ and g for the row $g = (z_0, z_1, \ldots, z_{n-1})$ then

$$\left|\sum \zeta_j z_j\right|^2 = \left|gx\right|^2 = x^* g^* g x =$$
$$= \left(G(z_0, \ldots, z_{n-1})\, x, x\right) = (Zx, x) = x^* Z x\,.$$

Now $(w_0, \ldots, w_{n-1}) = (z_0, \ldots, z_{n-1})\, T = gT$ and $\left|\sum \zeta_j w_j\right|^2 = \left|gTx\right|^2 = x^* T^* g T x =$ $= x^* T^* Z T x$. It follows that

$$\left|\sum \zeta_j z_j\right|^2 - \left|\sum \zeta_j w_j\right|^2 = x^*(Z - T^* Z T)\, x \geqq 0$$

so that it is possible to define a linear contraction operator A_0 on the linear span E_0 of the vectors $z_0, z_1, \ldots, z_{n-1}$ by the relation $A_0 z_j = w_j$. Now let α be a fixed root of the polynomial p. Since $p(\alpha) = 0$ we have $|\alpha| < 1$. Let us extend A_0 to a linear operator on the whole of H by setting $Ay = \alpha y$ for $y \in E_0^\perp$. It is easy to see that both E_0 and $E_0^\perp$ are invariant with respect to A. Since every $x \in H$ may be written in the form $x = u + y$ with $u \in E_0$ and $y \in E_0^\perp$ and $A_0 u \in E_0$, we have

$$|Ax|^2 = |A_0 u + \alpha y|^2 = |A_0 u|^2 + |\alpha|^2\, |y|^2 \leqq |u|^2 + |y|^2 = |x|^2$$

so that A is a contraction.

Let us show now that $p(A) = 0$. First of all, for every $y \in E_0^\perp$ we have $p(A)\, y =$ $= p(\alpha)\, y = 0$. Since $Az_0 = z_1,\ Az_1 = z_2, \ldots, Az_{n-2} = z_{n-1}$ it is easy to see that $z_i = A^i z_0$ for $i = 0, 1, \ldots, n-1$. Hence $p(A)\, z_0 = A^n z_0 - \sum_{j=0}^{n-1} a_j A^j z_0 = A z_{n-1} -$ $- \sum_{j=0}^{n-1} a_j z_j = w_{n-1} - \sum_{j=0}^{n-1} a_j z_j = 0$. Furthermore $p(A)\, z_j = p(A)\, A^j z_0 = A^j\, p(A)\, z_0 =$ $= 0$. Thus $p(A) = 0$. The proof is complete.

We would like to emphasise that the proposition just proved represents the decisive step in the whole theory. Before proceeding further let us try to explain the main idea of the method: it is geometrically quite intuitive but might become obscured by some of the technical details further on.

We intend to sketch now in a few words the geometric idea which underlies the further reasoning – only a slight modification will be needed to make it into a rigorous proof. The heuristic reasoning which follows is made under the additional assumption that the operators considered are nonderogatory – in this manner we can work with cyclic bases; the same reasoning is, of course, valid for subspaces generated by one vector – the rigorous proof to follow deals with the general case. At the same time it is not unreasonable to expect that the operator realizing the maximum considered will have a cyclic vector – we feel somehow that otherwise the maximum would already be attained at an operator acting on a space of smaller dimension and this seems to be quite unlikely.

Consider a contraction A with $p(A) = 0$ and with a cyclic vector z; it follows that, with respect to the basis

$$z, Az, \ldots, A^{n-1} z$$

the matrix of A is T and that of $f(A)$ is $f(T)$; we abbreviate $f(T)$ to F.

The condition that A be a contraction is equivalent to

$$T^*G(z, Az, \ldots)\, T \leqq G(z, Az, \ldots)\,.$$

The norm $|f(A)\,z|^2$ is the element with indices 0, 0 in the matrix

$$G(f(A)\,z, f(A)\,Az, \ldots) = F^*G(z, Az, \ldots)\,F\,.$$

Let us denote by q the linear functional on the algebra $\mathscr{M}_n$ of all matrices of order n which assigns to each matrix M its entry with indices 0, 0.

Our task reduces thus to finding

$$\max q(F^*\mathscr{G}F)\,.$$

By our lemma (2,1) we have $\mathscr{G} = \mathscr{M}$, and $\mathscr{M}$ is a section of the cone

$$\{Z;\, (1 - \mathscr{S})\, Z \geqq 0\}\,.$$

The mapping $1 - \mathscr{S}$ establishes a linear bijection between this cone and the cone $\mathscr{P}$ of all nonnegative definite matrices.

It is to be expected that the maximum $q(F^*\mathscr{M}F)$ will be attained at a point of an extreme ray of $\mathscr{M}$, in other words of a ray of the form $(1 - \mathscr{S})^{-1}\, P$ where P lies on an extreme ray of $\mathscr{P}$.

Now the extreme rays of $\mathscr{P}$ are generated by rank one matrices of the form pp^* and

$$(1 - \mathscr{S})^{-1}\, pp^* = pp^* + T^*p(T^*p)^* + T^{*2}p(T^{*2}p)^* + \ldots.$$

The problem assumes thus the following form:

Consider the set of all matrices of the form

$$M = pp^* + T^*p(T^*p)^* + T^{*2}p(T^{*2}p)^* + \ldots$$

such that $q(M) = 1$ and compute the maximum of $q(F^*MF)$ on this set.

Now

$$q(M) = |p|^2 + |T^{n*}p|^2 + |T^{n*2}p|^2 + \ldots.$$

If we write the coordinates of $T^{n*r}p$ as

$$z_0^{(r)}, z_1^{(r)}, \ldots, z_{n-1}^{(r)} \quad (r = 0, 1, 2, \ldots)$$

and arrange them in a sequence

$$Qp = z_0^{(0)}, \ldots, z_{n-1}^{(0)}, z_0^{(1)}, \ldots, z_{n-1}^{(1)}, z_0^{(2)}, \ldots, z_{n-1}^{(2)}, \ldots;$$

it is not difficult to see that Qp is annihilated by $\tilde{p}(S)$.

If we denote by $f_0, \ldots, f_{n-1}$ the elements of Ker $\tilde{p}(S)$ with initial conditions

$$\begin{matrix} 1, & 0, & 0, & \ldots & 0 \\ 0, & 1, & 0, & \ldots & 0 \\ & & & & \\ 0, & 0, & 0, & \ldots & 1 \end{matrix}$$

then

$$Qp = z_0 f_0 + \ldots + z_{n-1} f_{n-1} .$$

Now

$$q(M) = \sum_{r=0}^{\infty} |T^{*r} p|^2 = \sum_{r=0}^{\infty} \sum_{j=0}^{n-1} |z_j^{(r)}|^2 = |Qp|^2 .$$

As p ranges over C^n the image Qp will sweep out the whole of Ker $\tilde{p}(S)$.

Since

$$SQ = QT^*$$

we have

$$\tilde{f}(S)\, Q = Q\, \tilde{f}(T^*) = QF^* .$$

Since

$$q(M) = |Qp|^2$$

we have

$$q(F^* M F) = |QF^* p|^2 = |\tilde{f}(S)\, Qp|^2 .$$

Thus

$$\max \{|f(A)|;\ A \in \mathscr{A}(p)\} = |\tilde{f}(S \mid \operatorname{Ker} \tilde{p}(S))| = |f(S \mid \operatorname{Ker} p(S)| .$$

Consider now a linear operator A on H_n and a vector z. Denote by B the row

$$B = (z, Az, \ldots, A^{n-1} z) ;$$

it will have the usual dual meaning: we can view it as a row vector with entries in H_n or as a linear operator from C^n into H_n. In both interpretations the following identity holds

$$AB = BT$$

if A is annihilated by p. If z is a cyclic vector for A, in other words, if B is a cyclic basis for A then T is the matrix of A with respect to the basis B.

Now let f be an arbitrary polynomial and consider the operator $f(A)$. In order to compute the norm $f(A)$ let us find a convenient expression for $|f(A)\, z|^2$. Introducing the abbreviation $F = f(T)$ we intend to show that

$$|f(A)\, z|^2 = (F^*\, G(B)\, F, E_0) .$$

Since $f(A)\, B = B\, f(T) = BF$ we have

$$G(f(A)\, B) = (f(A)\, B)^* f(A)\, B = F^* B^* BF = F^*\, G(B)\, F$$

whence

$$|f(A)\, z|^2 = G(f(A)\, B)_{00} = (F^*\, G(B)\, F, E_0) .$$

Now consider a fixed polynomial p whose spectral radius is less than one; recall that we denote by $\mathscr{A}(p)$ the set of all $A \in B(H_n)$ such that $|A| \leqq 1$ and $p(A) = 0$. We intend to compute

$$\max \{|f(A)|;\ A \in \mathscr{A}(p)\} .$$

According to the above formula we have

$$\max\{|f(A)|^2;\ A \in \mathscr{A}(p)\} = \max\{|f(A)\,z|^2;\ A \in \mathscr{A}(p),\ |z| = 1\} =$$
$$= \max\,(F^*\,G(B)\,F,\ E_0)$$

where $G(B)$ ranges over all matrices of the form

$$G(z, Az, \ldots, A^{n-1}z)$$

with $A \in \mathscr{A}(p)$ and $|z| = 1$. This is exactly the set which we have denoted by $\mathscr{G}$ in the preceding lemma. Using the result of this lemma, we obtain

$$\max\{|f(A)|^2;\ A \in \mathscr{A}(p)\} = \max\,(F^*ZF, E_0)\,;$$
$$(1 - \mathscr{S})\,Z \geqq 0,\ (Z, E_0) = 1\,.$$

Now $1 - \mathscr{S}$ establishes a one-to-one linear correspondence between the set of all positive semidefinite matrices and the set of all matrices Z for which

$$(1 - \mathscr{S})\,Z \geqq 0\,.$$

The maximum problem may thus be reformulated as

$$\max\,(F^*(1 - \mathscr{S})^{-1}\,MF, E_0)\,;\quad M \geqq 0,\ ((1 - \mathscr{S})^{-1}\,M, E_0) = 1\,.$$

Since F is a function of T the conjugation by F

$$\mathscr{F} : X \to F^*XF$$

and $\mathscr{S}$, the conjugation by T, are commutative, so that

$$F^*(1 - \mathscr{S})^{-1}\,MF = (1 - \mathscr{S})^{-1}\,F^*MF = (1 - \mathscr{S})^{-1}\,\mathscr{F}M\,.$$

The maximum problem transforms to

$$\max\,(M, \mathscr{F}^*(1 - \mathscr{S}^*)^{-1}\,E_0)\,;\quad M \geqq 0\,,\quad (M, (1 - \mathscr{S}^*)^{-1}\,E_0) = 1\,.$$

Now denote by K the matrix $(1 - \mathscr{S}^*)^{-1}\,E_0$ and observe that it is positive definite. Indeed,

$$(1 - \mathscr{S}^*)^{-1}\,E_0 = (1 + \mathscr{S}^* + \mathscr{S}^{*2} + \ldots)\,E_0 =$$
$$= E_0 + TE_0T^* + T^2E_0T^{*2} + \ldots \geqq E_0 + TE_0T^* + \ldots$$
$$\ldots + T^{n-1}E_0T^{*n-1} = I\,;$$

the matrix $(1 - \mathscr{S}^*)^{-1}\,E_0$ is the sum of a series of positive semidefinite matrices and the first n summands already make up a positive definite matrix.

Write $\mathscr{K}$ for the congruence

$$\mathscr{K}X = K^{1/2}XK^{1/2}$$

so that $\mathscr{K}^* = \mathscr{K}$ and $K = \mathscr{K}1$.

Thus

$$\max\{(M, \mathscr{F}^*\mathscr{K}1);\ M \geqq 0,\ (M, \mathscr{K}1) = 1\} =$$
$$= \max\{(\mathscr{K}M, \mathscr{K}^{-1}\mathscr{F}^*\mathscr{K}1);\ M \geqq 0,\ (\mathscr{K}M, 1) = 1\}\,.$$

The last reformulation of the maximum problem is obtained upon writing B for $\mathscr{K}M$ and observing that $\mathscr{K}$ is a linear automorphism of the set $\mathscr{P}$ of all positive semidefinite matrices. The quantity to be computed thus becomes

$$\max\left(B, K^{-1/2}FKF^*K^{-1/2}\right);\quad B \geqq 0\,,\quad (B, 1) = 1\,.$$

If we set $R = K^{-1/2}FK^{1/2}$

$$\max\,(B, RR^*)\,;\quad B \geqq 0\,,\quad (B, 1) = 1$$

equals $|R|^2$ by lemma (1,3). Write A_p for $K^{-1/2}TK^{1/2}$. Now

$$R = K^{-1/2} f(T)\, K^{1/2} = f(K^{-1/2}TK^{1/2}) = f(A_p)\,.$$

Since A_p is similar to T, we have $p(A_p) = 0$. Now recall that $K = (1 - \mathscr{S}^*)^{-1} E_0$ so that $K - TKT^* = E_0$. It follows that

$$A_pA_p^* = K^{-1/2}TKT^*K^{-1/2} = K^{-1/2}(K - E_0)\,K^{-1/2} =$$
$$= 1 - K^{-1/2}E_0K^{-1/2} \leqq 1$$

so that A_p is a contraction. We have thus $A_p \in \mathscr{A}(p)$.

Now denote by X the solution of

$$X - C^*XC = E_0$$

where C is the companion matrix of the polynomial p, $C^T = T$. Since the roots of p are less than one in modulus it is possible to show that the operator

$$X \mapsto X - C^*XC$$

is invertible.

Since

$$E_0 = E_0^{\mathsf{T}} = (K - TKT^*)^{\mathsf{T}} = K^{\mathsf{T}} - C^*K^{\mathsf{T}}C \quad \text{we have} \quad K^{\mathsf{T}} = X\,.$$

Thus $A_p = K^{-1/2}TK^{1/2} = (X^{1/2}CX^{-1/2})^{\mathsf{T}}$.

Now let us return to the stage where we have expressed our maximum as

$$\max\{((1 - \mathscr{S})^{-1}\, F^*MF, E_0);\ M \geqq 0,\ ((1 - \mathscr{S})^{-1}\, M, E_0) = 1\}\,.$$

The result just proved together with (1,3) shows that it is possible to limit ourselves to matrices M of the form vv^*; the supremum is attained if instead of the whole set $\{M \geqq 0, ((1 - \mathscr{S})^{-1}\, M, E_0) = 1\}$ we allow M to range only over its extreme points.

This leads to a natural infinite dimensional interpretation of the extremal operator.

First of all, we have

$$((1 - \mathscr{S})^{-1}\, vv^*, E_0) = \sum_0^\infty (T^{*k} vv^* T^k, E_0) = \sum_0^\infty (C^{*kT} vv^* C^{kT}, E_0)\,.$$

To get a neater formula, introduce an involutory mapping Q of C^n into itself as follows

$$Q(y_0, \ldots, y_{n-1})^{\mathsf{T}} = (y_0^*, \ldots, y_{n-1}^*)^{\mathsf{T}}\,.$$

This mapping – together with another involution, the transposition – make it possible to express the mapping $M \mapsto M^*$ as follows

$$M^* = QM^{\mathsf{T}}Q\,.$$

Set $w = Qv$ and observe that $vv^* = (ww^*)^{\mathsf{T}}$. Since $C^{*k\mathsf{T}} vv^* C^{k\mathsf{T}} = (C^k ww^* C^{*k})^{\mathsf{T}}$, we have

$$((1 - \mathscr{S})^{-1}\, vv^*, E_0) = \sum_0^\infty (C^k ww^* C^{*k}, E_0) =$$

$$= \sum_0^\infty |(C^k w, e_0)|^2 = |\psi(w)|^2 = |\psi Qv|^2$$

if we denote by ψ the linear mapping of C^n into l^2 defined by the formula

$$\psi y = \{(y, e_0), (Cy, e_0), (C^2 y, e_0), \ldots\}\,.$$

Observe that

$$\psi C = S\psi$$

and that ψ maps C^n onto the n-dimensional subspace of l^2

$$\operatorname{Ker} p(S)\,.$$

At the same time

$$|((1 - \mathscr{S})^{-1}\, F^* vv^* F;\, E_0)|^2 = |\psi QF^* v|^2$$

and

$$QF^* v = QQF^{\mathsf{T}} Qv = F^{\mathsf{T}} w = f(C)\, w$$

thus

$$|\psi QF^* v|^2 = |\psi f(C)\, Qv|^2 = |f(S)\, \psi Qv|^2\,.$$

It follows that the maximum is attained – for any f – at the operator

$$S \mid \operatorname{Ker} p(S)\,.$$

3. CONNECTIONS WITH COMPLEX FUNCTIONS THEORY

Suppose φ is an inner function. Then φH^2 is a closed subspace of H^2; its complement $H^2 \ominus \varphi H^2$ will be denoted by $H(\varphi)$. The orthogonal projection of H^2 onto $H(\varphi)$ will be denoted by $P(\varphi)$. The model operator $S(\varphi)$ corresponding to φ is defined as

$$S(\varphi) = P(\varphi)\, V \mid H(\varphi)$$

where V is the isometry on H^2 defined as follows

$$Vf = g \quad \text{means} \quad g(z) = zf(z).$$

Let us state now an important result of D. Sarason.

(3,1) **Theorem.** *Suppose φ is an inner function. Then $\varphi(S(\varphi)) = 0$. If $f \in H^\infty$ then $f(S(\varphi))$ is meaningful and*

$$|f(S(\varphi))| = |f + \varphi H^\infty|_\infty,$$

the norm of the class of f in H^∞ modulo the ideal φH^∞.

We shall also need an inequality due to J. von Neumann.

(3,2) **Theorem.** *Suppose A is a completely nonunitary contraction on a Hilbert space H and let $f \in H^\infty$. Then $f(A)$ is meaningful and*

$$|f(A)| \leqq |f|_\infty.$$

Combining these two results it is easy to find a solution of the first maximum problem.

(3,3) **Theorem.** *Let A be a completely nonunitary contraction on a Hilbert space H and let φ be an inner function such that $\varphi(A) = 0$.*

Then, for each $f \in H^\infty$, $f(A)$ is meaningful and

$$|f(A)| \leqq |f(S(\varphi))|.$$

Proof. Given any $g \in H^\infty$ in the residue class of f,

$$f - g \in \varphi H^\infty$$

we have $f(A) = g(A)$ and, by the von Neumann inequality,

$$|f(A)| = |g(A)| \leqq |g|_\infty$$

so that

$$|f(A)| \leqq |f + \varphi H^\infty|_\infty.$$

By the Sarason theorem the quantity on the right hand side equals the norm of $f(S(\varphi))$; this proves the theorem.

The result just proved shows that $S(\varphi)$ is a solution of our maximum problem. We shall see later that $S(\varphi)$ is unitarily equivalent to the operator $S \mid \operatorname{Ker} \varphi(S)$, so that the following theorem will follow.

(3,4) **Theorem.** *Let A be a completely nonunitary contraction on a Hilbert space H and let φ be an inner function such that $\varphi(A) = 0$. Then, for each $f \in H^\infty$, $f(A)$ and $f(S \mid \operatorname{Ker} \varphi(S))$ are meaningful and*

$$|f(A)| \leqq |f(S \mid \operatorname{Ker} \varphi(S))|.$$

It is, however, possible to obtain this theorem as an immediate consequence of the preceding one using instead of the unitary equivalence of $S \mid \operatorname{Ker} \varphi(S)$ and $S(\varphi)$ the following more modest fact

$$(S \mid \operatorname{Ker} \varphi(S))^* = S(\tilde{\varphi})$$

where $\tilde{\varphi}$ is defined as $\varphi(z^*)^*$.

We shall give a proof of this fact later; let us assume it for a moment.

To prove that $S \mid \operatorname{Ker} \varphi(S)$ is an extremal operator, in other words, that

$$|f(A)| \leqq |f(S \mid \operatorname{Ker} \varphi(S))|$$

for any completely nonunitary contraction A which satisfies $\varphi(A) = 0$ it will be sufficient to show that

$$|f(S \mid \operatorname{Ker} \varphi(S))| = |f + \varphi H^2|_\infty .$$

To see that we argue as follows

$$|f(S \mid \operatorname{Ker} \varphi(S))| = |f(S \mid \operatorname{Ker} \varphi(S))^*| =$$
$$= |\tilde{f}((S \mid \operatorname{Ker} \varphi(S))^*)| = |\tilde{f}(S(\tilde{\varphi}))|$$

and this equals $|\tilde{f} + \tilde{\varphi} H^\infty|_\infty$ by Sarason's theorem. Of course,

$$|\tilde{f} + \tilde{\varphi} H^\infty|_\infty = |f + \varphi H^\infty|_\infty .$$

The theorem is thus established.

The rest of this section will be devoted to the proof of the unitary equivalence of the operators $S \mid \operatorname{Ker} \varphi(S)$ and $S(\varphi)$. The proof will be divided into two steps

1° we prove first that

$$(S \mid \operatorname{Ker} \varphi(S))^* = S(\tilde{\varphi})$$

2° the second step consists in showing that $S(\tilde{\varphi})^*$ is unitarily equivalent to $S(\varphi)$. Together these facts yield

$$S \mid \operatorname{Ker} \varphi(S) = S(\tilde{\varphi})^* \sim S(\varphi) .$$

We begin by showing that

$$\operatorname{Ker} \varphi(S) = H^2 \ominus \tilde{\varphi} H^2 .$$

Now

$$\operatorname{Ker} \varphi(S) = (\text{Range } \varphi(S)^*)^\perp = (\text{Range } \tilde{\varphi}(V))^\perp = H(\tilde{\varphi}) .$$

Thus $S \mid \operatorname{Ker} \varphi(S) = S \mid H(\tilde{\varphi})$ and its adjoint is $P(\tilde{\varphi})\, V \mid H(\tilde{\varphi})$ which is nothing more than $S(\tilde{\varphi})$. This proves the first assertion and, at the same time, completes the proof of the preceding theorem.

As far as the second assertion is concerned the unitary equivalence of $B = S \mid \operatorname{Ker} \varphi(S)$ and $S(\varphi)$ may be obtained as an immediate consequence of a rather powerful theorem: *two operators are unitarily equivalent if they have the same characteristic function.* Now it is known that the characteristic function of $S(\varphi)$ is φ.

Denote by Θ the characteristic function of B, then $\Theta^{\sim}$ is the characteristic function of B^* which is nothing more than $S(\tilde{\varphi})$. Thus $\Theta^{\sim} = \tilde{\varphi}$ so that $\Theta = \varphi$. Thus B and $S(\varphi)$ have the same characteristic function whence $B \sim S(\varphi)$.

Short as it is, this proof has the disadvantage that it uses the fairly sophisticated notion of the characteristic function. A closer study of the spaces $H(\varphi)$ makes it possible, however, to give an explicit expression for the unitary operator intertwining B and $S(\varphi)$.

We begin by a description of the projection $P(\varphi)$. This well known description is due to N. K. Nikolskij [3] and is included for the sake of completeness.

(3,5) **Lemma.** *Let $P(\varphi)$ be the orthogonal projection of H^2 onto $H^2 \ominus \varphi H^2$. Then, for $f \in H^2$*

$$P(\varphi)f = \varphi P_- \bar{\varphi} f .$$

Proof. Let $f \in H^2$ be given, let $k \in H^2$ be the element for which $P(\varphi)f = f - \varphi k$. Thus $f - \varphi k \perp \varphi H^2$. Since multiplication by $\bar{\varphi}$ is a unitary operator on L^2 we have

$$\bar{\varphi} f - k = \bar{\varphi}(f - \varphi k) \perp H^2$$

whence

$$\bar{\varphi} f - k = P_-(\bar{\varphi} f - k) = P_- \bar{\varphi} f .$$

Multiplying by φ we obtain

$$f - \varphi k = \varphi P_- \bar{\varphi} f \quad \text{whence} \quad P(\varphi)f = f - \varphi k = \varphi P_- \bar{\varphi} f .$$

Our next task will be to describe a unitary operator W mapping $\operatorname{Ker} \varphi(S) = H(\tilde{\varphi})$ onto $H(\varphi)$ for which the diagram

$$\begin{array}{ccc} H(\tilde{\varphi}) & \xrightarrow{W} & H(\varphi) \\ S \,|\, \operatorname{Ker} \varphi(S) \Big\downarrow & & \Big\downarrow S(\varphi) \\ H(\tilde{\varphi}) & \xrightarrow{W} & H(\varphi) \end{array}$$

is commutative.

The construction of W is based on the following observation: for an element $g \in L^2$ the relation $g \perp H^2$ is equivalent to the inclusion

$$\bar{z}\, g(\bar{z}) \in H^2 .$$

The mapping F which assigns to each $f \in L^2$ the element g defined by

$$g(z) = \bar{z} f(\bar{z})$$

is obvisouly an isometry.

Also, the relation $F^2 = 1$ is immediate; thus F is onto, hence unitary so that $F = F^* = F^{-1}$. Thus F is a selfadjoint unitary involution which maps H^2 onto H^2_-. Let us consider now the space $\operatorname{Ker} \varphi(S)$. Since $Sf = P_+ \bar{z} f(z)$ for all $f \in H^2$ we have $\varphi(S)f = P_+ \varphi(\bar{z}) f(z)$ so that $\operatorname{Ker} \varphi(S)$ is the set of those $u \in H^2$ for which

$P_+ \varphi(\bar{z}) u(z) = 0$ in other words, for which $\varphi(\bar{z}) u(z) \perp H^2$. By the above observation this is equivalent to saying that the element $v(z) = F \varphi(\bar{z}) u(z)$ belongs to H^2; it is easy to see that for each $f \in L^2$

$$F \varphi(\bar{z}) f = \varphi(z) Ff .$$

If we denote by W the operator on L^2 which assigns to each $f \in L^2$ the product

$$\varphi(z) \bar{z} f(\bar{z}) = F \varphi(\bar{z}) f = \varphi(z) Ff$$

then the above remark may be restated in the following form: the operator W maps $\operatorname{Ker} \varphi(S)$ isometrically into H^2. We can do better, however: W maps $\operatorname{Ker} \varphi(S)$ into $H(\varphi)$. This may be easily seen as follows. Given $u \in \operatorname{Ker} \varphi(S)$ then $Fu \perp H^2$ so that $v = \varphi(z) Fu \perp \varphi H^2$.

To see that W is onto consider the analogous mapping $W^\sim$ obtained by replacing φ by $\tilde{\varphi}$. Thus

$$\tilde{W}x = y \quad \text{means} \quad y(z) = \bar{z} \tilde{\varphi}(z) x(\bar{z}) .$$

By what we have already proved $\tilde{W}$ maps $H(\varphi) = \operatorname{Ker} \tilde{\varphi}(S)$ isometrically into $H(\tilde{\varphi})$. It is easy to verify that

$$W\tilde{W}x = x$$

for every $x \in H(\varphi)$, so that W is onto; since it is also isometric, it is unitary.

Now we are ready to state the following

(3,6) **Proposition.** *Let φ be an inner function; then the mapping W which assigns to each $u \in \operatorname{Ker} \varphi(S)$ the function v*

$$v(z) = \bar{z} \varphi(z) u(\bar{z})$$

is an isometry of $\operatorname{Ker} \varphi(S)$ *onto $H(\varphi)$ which intertwines S and $S(\varphi)$*

$$WS \mid \operatorname{Ker} \varphi(S) = S(\varphi) W .$$

Proof. The proof is based on the relation

$$FP_+\bar{z}f = P_-zFf$$

which is easily seen to be valid for any $f \in H^2$. Now consider any arbitrary $u \in \operatorname{Ker} \varphi(S)$. Then

$$WSu = \varphi(z) FP_+\bar{z}u$$

$$S(\varphi) Wu = P(\varphi) z \varphi(z) Fu = \varphi(z) P_-zFu$$

so that $WSu = S(\varphi) Wu$.

4. CONNECTIONS WITH TOEPLITZ AND HANKEL OPERATORS

Given two polynomials φ, ψ we denote by $C(\varphi, \psi)$ the supremum of $|\psi(T)|$ where T ranges over all contractions in Hilbert space such that $\varphi(T) = 0$.

Let us remark already here that it will be possible to admit more general functions φ and ψ later on: we hope to explain, as we proceed, that such an extension is quite natural.

Using this notation, one of the results of [11] may be formulated as follows: let p be a positive number, $p < 1$. Consider the set $\mathscr{A}$ of all contractions T on n-dimensional Hilbert spaces such that $(T - p)^n = 0$. Denote by m the Möbius function

$$m(z) = \frac{z - p}{1 - pz}$$

and by S_n the n-dimensional truncated shift operator

$$S_n = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Then $T_0 = m(S_n)$ realizes the maximum of $|T^n|$ as T ranges over $\mathscr{A}$, in other words, $T_0 \in \mathscr{A}$ and

$$|T_0^n| = \max \{|T^n|;\ T \in \mathscr{A}\}.$$

Since $0 < p < 1$ the constraint $(T - p)^n = 0$ is equivalent, for a contraction T, to the equation $m(T)^n = 0$. In the notation introduced above, this result may thus be reformulated as follows.

Let φ and ψ be defined by the formulae

$$\varphi(z) = m(z)^n \quad \psi(z) = z^n.$$

Then

$$C(\varphi, \psi) = |\varphi(S_n)|.$$

Consider now the problem $C(\psi, \varphi)$, in other words, to find the maximum of $|\varphi(T)|$ provided T ranges over all nilpotent contractions T on H_n. It is easy to see that the maximum is attained for $T = S_n$. Thus

$$C(\psi, \varphi) = |\varphi(S_n)|$$

so that

$$C(\psi, \varphi) = C(\varphi, \psi)$$

for this particular pair of functions φ, ψ. This observation led N. J. Young to formulate the following conjecture:

if φ and ψ are two Blaschke products of the same length then

$$C(\varphi, \psi) = C(\psi, \varphi);$$

a proof of this conjecture may be given [12] based on the fact that two subspaces of a Hilbert space may be mapped onto each other by unitary operators if they have the same finite dimension. It is obvious that equality of the lengths of φ and ψ is essential for the validity of the conjecture. Not long ago V. V. Peller observed that the conjecture remains true in a much more general situation and that it is possible to deduce it from a fairly simple identity (see 4,1) for Hankel and Toeplitz operators. At the same time Peller's proof also provides — by exhibiting a natural condition on φ and ψ which guarantees the result — the right explanation for the condition that the Blaschke products be of the same length in the original form of the conjecture.

In the rest of this section we reproduce, with Dr. Peller's consent, a proof of a more general result as suggested by V. V. Peller.

We begin by explaining some notation. Given a $\varphi \in L^\infty$ the corresponding Toeplitz operator $T_\varphi : H^2 \to H^2$ and the Hankel operator $H_\varphi : H^2 \to H^2_-$ are defined by the formulae

$$T_\varphi f = P_+ \varphi f\,, \quad H_\varphi f = P_- \varphi f \quad \text{for} \quad f \in H^2\,.$$

We observe that the adjoint of H_φ is given by

$$H_\varphi^* g = P_+ \bar{\varphi} g \quad \text{for} \quad g \in H^2_-\,.$$

The main result is based on the following

(4,1) **Proposition.** *Suppose φ and ψ are two arbitrary elements of H^∞. The corresponding Toeplitz and Hankel operators satisfy*

$$T_{\varphi\psi} - T_\varphi T_\psi = H_{\bar{\varphi}}^* H_\psi\,.$$

If $u \in L^\infty$ is unimodular then

$$T_u H_u^* H_u = H_{\bar{u}}^* H_{\bar{u}} T_u\,.$$

Proof. Let $f \in H^2$ be given. Then

$$\begin{aligned} T_{\varphi\psi} f - T_\varphi T_\psi f &= P_+ \varphi\psi f - P_+ \varphi P_+ \psi f = \\ &= P_+ \varphi (1 - P_+)\, \psi f = P_+ \varphi P_- \psi f = H_{\bar{\varphi}}^* H_\psi f \end{aligned}$$

and this proves the first assertion.

Now suppose that $u \in H^\infty$ is unimodular. Applying the identity just proved to $\varphi = \bar{u}$, $\psi = u$ we obtain

$$1 - T_{\bar{u}} T_u = H_u^* H_u\,.$$

In a similar manner, interchanging u and $\bar{u}$,

$$1 - T_u T_{\bar{u}} = H_{\bar{u}}^* H_{\bar{u}}\,.$$

Hence

$$H_{\bar{u}}^* H_{\bar{u}} T_u = (1 - T_u T_{\bar{u}})\, T_u = T_u (1 - T_{\bar{u}} T_u) = T_u H_u^* H_u\,.$$

(4,2) **Corollary.** *If $u \in L^\infty$ is unimodular and if T_u is invertible then*

$$|H_u| = |H_{\bar{u}}|\,.$$

Proof. According to the preceding proposition we have, for unimodular u,

$$T_u H_u^* H_u = H_{\bar{u}}^* H_{\bar{u}} T_u .$$

If, in addition, T_u is invertible, then $H_u^* H_u$ and $H_{\bar{u}}^* H_{\bar{u}}$ will be similar so that their spectral radii are equal. But these spectral radii are equal to $|H_u|^2$ and $|H_{\bar{u}}|^2$ respectively. This proves the corollary.

Now we are ready to state the main result. First we recall that the norm of H_φ equals dist (φ, H^∞) for any $\varphi \in L^\infty$.

(4,3) **Theorem.** *Let f and g be two inner functions such that $f = ug$ for a continuous function u. Suppose further that the winding number with respect to the origin of the curve determined by u is zero. Then*

$$|f + gH^\infty|_\infty = |g + fH^\infty|_\infty .$$

Proof. Since f and g are inner the function u is unimodular and $\bar{u} = u^{-1} = g/f$. The assumptions about u guarantee the invertibility of T_u so that $|H_u| = |H_{\bar{u}}|$ by (4,2). Hence

$$\operatorname{dist}(f, gH^\infty) = \operatorname{dist}(ug, gH^\infty) = \operatorname{dist}(u, H^\infty) =$$
$$= |H_u| = |H_{\bar{u}}| = \operatorname{dist}(\bar{u}, H^\infty) = \operatorname{dist}(\bar{u}f, fH^\infty) = \operatorname{dist}(g, fH^\infty)$$

and the theorem is established.

Returning to the case where f and g are two Blaschke products we see that the quotient $u = fg^{-1}$ is continuous and unimodular. Of course, without supplementary conditions, T_u will not be invertible: the most trivial examples show that nothing of that sort can be expected. For $u = z^*$ the corresponding Toeplitz operator T_u is the backward shift S. In the case $u = fg^{-1}$ the operator T_u will be invertible if the winding number of the corresponding curve will be zero but this is the case if and only if f and g have the same length.

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