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# ON A PROBLEM OF LINEAR ARBORICITY

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*Summary.* The linear arboricity of a graph $G$ is the minimum number of linear forests whose union is $G$. In the paper the problem of determining the linear arboricity for nonregular graphs whose maximum degree is even is studied.

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A *linear forest* is a graph in which each component is a path. The *linear arboricity* $\Xi(G)$ of a graph $G$ is the minimum number of linear forests whose union is $G$.

The concept of linear arboricity was introduced by Harary [10] in 1970 but, until now, the value of linear arboricity has been determined only for few special classes of graphs, e.g. for trees, complete graphs and complete bipartite graphs (see [1], [2]).

The conjecture which had the main influence on the development of the theory of linear arboricity was introduced in [2]:

**Conjecture 1.** *The linear arboricity of an r-regular graph is* $\lceil (r+1)/2 \rceil$.

The topic of linear arboricity has been lately studied by many mathematicians who verified Conjecture 1 for the cases of $r = 2, 3, 4, 5, 6, 8$ and $10$ (see [2], [3], [5],[6], [7], [12]).

The bounds of linear arboricity depending on the maximum degree of a graph were determined in [3] (the best possible lower bound) and in [8] (the best upper bound at this time):

**Theorem 1.** *Let G be a graph with maximum degree Δ. Then*

$$\left\lceil \frac{\Delta}{2} \right\rceil \leqq \Xi(G).$$

**Theorem 2.** *Let G be a graph with maximum degree Δ. Then*

$$\Xi(G) \leqq \left\lceil \frac{6}{5} \frac{\Delta}{2} \right\rceil \qquad \text{if } \Delta \text{ is even, and}$$

$$\Xi(G) \leqq 1 + \left\lceil \frac{6}{5} \frac{\Delta - 1}{2} \right\rceil \qquad \text{if } \Delta \text{ is odd}.$$

On the basis of the lower bound of the linear arboricity given in Theorem 1 we can proceed as follows. If Conjecture 1 were proved in general then the linear arboricity would be nearly determined for all graphs because for the graph $G$ with maximum degree $\Delta$ we should have $\Xi(G) = (1 + \Delta)/2$ for $\Delta$ odd and $\Xi(G) = \Delta/2$ or $(1 + \Delta/2)$ for $\Delta$ even. This implies that the complementary problem to Conjecture 1 is to investigate the linear arboricity of nonregular graphs whose maximum degree is even. The most important of these graphs for this aim are those that contain vertices of only two degrees $\Delta$ and $(\Delta - 1)$. The general problem in this matter was expressed by Tomasta [12] in the following form:

**Problem 1.** *Determine the maximum number of $(r + 1)$'s in a degree sequence (of a given length)*

$$(r + 1, r + 1, \ldots, r + 1, r, r, \ldots, r)$$

*of a graph $G$ with the linear arboricity $\Xi(G) = \lceil (r + 1)/2 \rceil$ for odd $r \geqq 3$.*

The aim of this paper is to present some results concerning this problem which in fact can be interpreted in two ways, thus, we actually have two independent problems. The first is to determine the maximum number of $(r + 1)$'s so that there exists a graph $G$ with this degree sequence and $\Xi(G) = \lceil (r + 1)/2 \rceil$. The second is to determine the maximum number of $(r + 1)$'s so that every graph $G$ with this degree sequence fulfils $\Xi(G) = \lceil (r + 1)/2 \rceil$.

The solution of the first interpretation of Problem 1 will be given in Theorem 3. Let us first introduce some necessary notations. In this paper we consider finite undirected simple graphs. Let us denote by $V_r(G)$ the set of vertices of degree $r$ of the graph $G$ and let $\langle M \rangle$ denote the subgraph induced by the subset $M$ of vertices. Further, we define graphs $G_1 \cup G_2 = (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$ and $G_1 - E = (V(G_1), E(G_1) - E)$ for arbitrary graphs $G_1, G_2$ and the set of edges $E$.

**Theorem 3.** *Let $x, y, r$ be nonnegative integers, $r$ odd. Then there exists a graph $G$ with $x$ vertices of degree $(r + 1)$, with $y$ vertices of degree $r$, with $|V(G)| = x + y$ and $\Xi(G) = (r + 1)/2$ if and only if $y$ is even and $y \geqq r + 1$.*

Proof. I. The number of vertices of an odd degree in any graph must be even and so it is necessary for $y$ to be even. If the graph $G$ can be decomposed into $(r + 1)/2$ linear forests then it must contain at least $(r + 1)$ vertices which are endvertices of some linear forest and so they are of degree $r$. Hence $y \geqq r + 1$.

II. Let us have $y$ even and such that $y \geqq r + 1$.

A. Let us assume first that $(x + y)$ is even, $(x + y) = 2k$. Consider the complete graph $K_{2k}$ with $V(K_{2k}) = \{a_1, \ldots, a_k, b_1, \ldots, b_k\}$. It is known (see [11]) that the complete graph $K_{2k}$ can be decomposed into $k$ hamiltonian paths $P_1, \ldots, P_k$ and, in addition, we can choose such a decomposition that $P_1 = (a_1, a_2, \ldots, a_k, b_k, \ldots, b_1)$ and the endvertices of $P_i$ are just $a_i, b_i$ for all $i$ (see Fig. 1 for $P_1, P_2$ of the case $x + y = 8$).

Let us define the graph $G_1 = \bigcup_{i=1}^{(r+1)/2} P_i$. Then we have $|V(G_1)| = x + y$, $|V_r(G_1)| = r + 1 \leqq y$, $|V_{r+1}(G_1)| = (x + y) - (r + 1) \geqq x$. Further, let us define the sequence of independent edges of $P_1$ in the following way: $e_1 = (a_{((r+1)/2)+1}, a_{((r+1)/2)+2})$, $e_2 = (b_{((r+1)/2)+1}, b_{((r+1)/2)+2})$, $e_3 = (a_{((r+1)/2)+3}, a_{((r+1)/2)+4}), \ldots$; the last edge $e_{k-(r+1)/2}$ will be either $(b_{k-1}, b_k)$ or $(a_k, b_k)$ depending on the parity of $(k - (r + 1)/2)$. Finally, we define the graph $G = G_1 - \bigcup_{i=1}^{(y-(r+1))/2} \{e_i\}$ which has just $x$ vertices of degree $(r + 1)$, $y$ vertices of degree $r$, and can bed ecomposed into $(r + 1)/2$ linear forests.

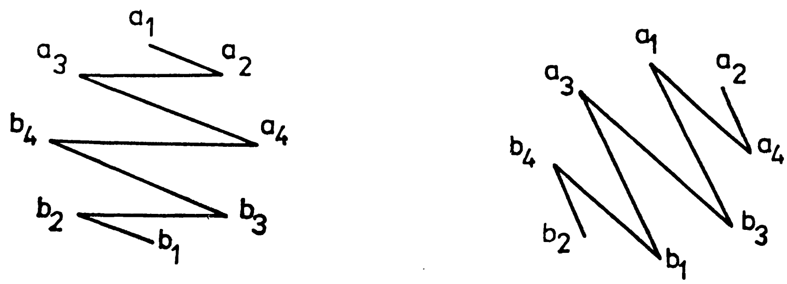


Fig. 1

B. Now, let us assume that $(x + y)$ is odd and let $x + y = 2k + 1$. Consider the complete graph $K_{2k+1}$ with $V(K_{2k+1}) = \{a_1, \ldots, a_k, b_1, \ldots, b_k, c\}$. It is known (see [11]) that the complete graph $K_{2k+1}$ can be decomposed into one matching and $k$ hamiltonian paths $P_1, P_2, \ldots, P_k$ and, in addition, we can choose such decomposition (very similar to case $A$) that $P_1 = (a_1, a_2, \ldots, a_k, c, b_k, \ldots, b_1)$ and the endvertices of $P_i$ are just $a_i, b_i$ for all $i$ (see Fig. 2 for the case $x + y = 9$). Let us define the

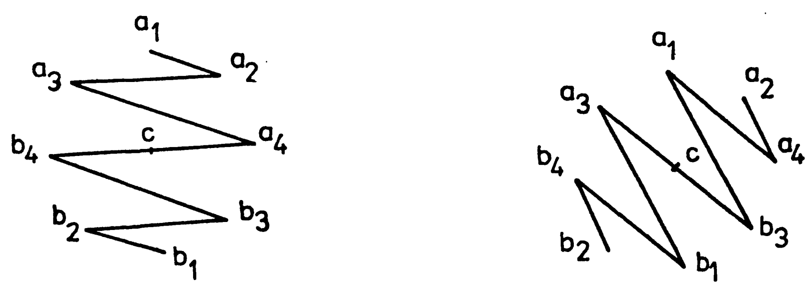


Fig. 2

graph $G_1$ and the sequence of edges $e_i$ for $i = 1, \ldots, (k - (r + 1)/2)$ in the same way as in case $A$ except for the last edge $e_{k-(r+1)/2}$ which will be either $(b_{k-1}, b_k)$or $(a_k, c)$ depending on the parity of $(k - (r + 1)/2)$. Finally, we define the graph

$G = G_1 - \bigcup_{i=1}^{(y-(r+1))/2} \{e_i\}$ which has just $x$ vertices of degree $(r + 1)$, $y$ vertices of degree $r$ and can be decomposed into $(r + 1)/2$ linear forests.

The second part of Problem 1 which is much more difficult will be partially solved in Theorem 4. Let us first introduce a necessary lemma.

**Lemma 1.** *Let $x, y, r$ be nonnegative integers, $r$ odd, $y$ even, and let $x + y \geqq r + 2$. Then there exists a graph $G$ with $|V(G)| = x + y$, $|V_r(G)| = y$ and $|V_{r+1}(G)| = x$.*

Proof. Take $(r + 1)/2$ cycles of the decomposition of the complete graph $K_{x+y}$ into hamiltonian cycles (and 1 matching if $x + y$ is even) and delete $y/2$ independent edges of the first cycle to obtain $G$.

**Theorem 4.** *Let $x, y, r$ be nonnegative integers, $r$ odd, $y$ even, and let $x \geqq 3$, $x + y \geqq 2r + 3$. Then there exists a graph $G$ with $|V(G)| = x + y$, $|V_r(G)| = y$, $|V_{r+1}(G)| = x$ and $\Xi(G) \geqq 1 + (r + 1)/2$.*

Proof. I. First, let $x + y = 2r + 3$ and $y < r + 1$. According to Lemma 1 there exists a graph $G$ with $|V(G)| = x + y$, $|V_r(G)| = y$ and $|V_{r+1}(G)| = x$ which according to Theorem 3 has $\Xi(G) \geqq 1 + (r + 1)/2$.

II. Let $x + y = 2r + 3$ and $y \geqq r + 1$. Then according to Lemma 1 there exists a graph $G_1$ with $|V(G_1)| = r + 2$, $|V_{r+1}(G_1)| = x$ and $|V_r(G_1)| = r + 2 - x \leqq r - 1 < r + 1$. Hence $\Xi(G_1) \geqq 1 + (r + 1)/2$. Let us define $G = G_1 \cup K_{r+1}$, which fulfils the conditions of Theorem 4.

III. Now, let $x + y > 2r + 3$ and let $x_1 = \min\{(2\lfloor(x - 1)/2\rfloor + 1),\ r + 2\}$, $y_1 = r + 2 - x_1$. Hence $3 \leqq x_1 \leqq x$ and $y_1$ is even, and then there exists a graph $G_1$ with $|V(G_1)| = x_1 + y_1 = r + 2$, $|V_{r+1}(G_1)| = x_1$, $|V_r(G_1)| = y_1$, which according to Theorem 3 has $\Xi(G_1) \geqq 1 + (r + 1)/2$. According to Lemma 1 there exists a graph $G_2$ with $|V(G_2)| = (x + y) - (r + 2) \geqq r + 2$, $|V_{r+1}(G_2)| = x - x_1$, $|V_r(G_2)| = y - y_1$. Finally, we define the graph $G = G_1 \cup G_2$, which fulfils the conditions of Theorem 4.

Remark. If $x \geqq 4$ then it is not difficult to construct a graph $G$ fulfilling the conditions of Theorem 4 which, moreover, is connected.

Now, let us summarize the results concerning the graphs with vertices of only two degrees $r$, $r + 1$, for $r$ odd.

I. Every graph $G$ with $|V_r(G)| = |V(G)| - |V_{r+1}(G)| < r + 1$ has $\Xi(G) \geqq 1 + (r + 1)/2$.

II. For arbitrary integers $x, y, r$ such that $x + y \geqq 2r + 3$, $x \geqq 3$, $y \geqq r + 1$, $r$ odd, $y$ even there exists graphs $G_1, G_2$ such that $|V(G_1)| = |V(G_2)| = x + y$, $|V_r(G_1)| = |V_r(G_2)| = y$, $|V_{r+1}(G_1)| = |V_{r+1}(G_2)| = x$ and $\Xi(G_1) = (r + 1)/2$, $\Xi(G_2) \geqq 1 + (r + 1)/2$.

III. For arbitrary nonnegative integers $x, y, r$; $y$ even, $r$ odd such that $y \geqq r + 1$ and

(i) $x \leqq 2$ or

(ii) $x + y < 2r + 3$

there exists a graph $G$ such that $|V(G)| = x + y$, $|V_{r+1}(G)| = x$, $|V_r(G)| = y$ and $\Xi(G) = (r + 1)/2$; however we do not know any such graph with the linear arboricity greater than $(r + 1)/2$.

On the basis of the preceding considerations let us propose an open problem and a conjecture which is a little stronger than Conjecture 1.

**Problem 2.** *Let $G$ be a graph with an odd minimum degree $r$ and an even maximum degree $r + 1$. Let $3 \leqq |V_{r+1}(G)| \leqq |V(G)| - (r + 1)$. Determine the linear arboricity of $G$ depending on the structure of the graph $G$.*

**Conjecture 2.** *Let $G$ be a graph with all vertices of degree $r$ except of at most two vertices of degree $r + 1$. Then $\Xi(G) = \lceil (r + 1)/2 \rceil$.*

This conjecture was verified up to now for the cases of $r = 1, 2, 3, 4$. The case $r = 1$ is trivial, the cases $r = 2, 4$ follow from the verification of Conjecture 1 for $r = 3$ and 5 and the case $r = 3$ follows from the nice result due to Enomoto [5], [6]:

**Theorem 5.** *Let $G$ be a graph with $\Delta(G) = 4$. Let $\Delta(\langle V_4(G) \rangle) \leqq 1$. Then $\Xi(G) = 2$.*

Another result on this topic was published in [7]:

**Theorem 6.** *Let $G$ be a graph with the degree sequence $(6, 5, \ldots, 5)$. Then $\Xi(G) = 3$.*

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Súhrn

## O JEDNOM PROBLÉME LINEÁRNEJ LESNATOSTI

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Lineárna lesnatosť grafu je minimálny počet lineárnych lesov, na ktoré sa dá rozložiť daný graf. V článku sa analyzuje problém určenia lineárnej lesnatosti nepravidelných grafov s párnym maximálnym stupňom.

Резюме

## О ПРОБЛЕМЕ ЛИНЕЙНЫХ ЛЕСОВ

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Линейная древесность графа $G$ — это минимальное число линейных лесов, объедиение которых равне $G$. В статье изучается проблема определения линейной древесности неправильных графов с четной максимальной степенью.

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