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PARALLEL DISPLACEMENT OF VECTORS ON RHEONOMOUS ANHOLONOMIC MANIFOLD

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We shall make use of the notations of [2]. Let us suppose that in the stationary space $r - L_n(t)$ there is defined the rheonomous anholonomic manifold $r - L_n^m(t)$ by means of the equations

$$B_a^\alpha = B_a^\alpha(x^\omega, t)\,, \quad {}_pn^\alpha = {}_pn^\alpha(x^\omega, t)\,, \quad x^\alpha = x^\alpha(u^A, t) \tag{1}$$

where $\{x^\omega, t\} \in \Omega$, $\{u^A, t\} \in \Lambda \times I$ (see [2], Equations(1,2) and (1,3)). Let the parametric equations

$$x^\alpha = x^\alpha(T)\,, \quad t = T\,, \quad T \in J \tag{2}$$

describe a trajectory which lies on the rheonomous manifold $r - L_n^m(t)$. Thus for every $T \in J$ there exist m numbers $\mathrm{d}u^a/\mathrm{d}T$ such that the equation

$$\frac{\mathrm{d}x^\alpha}{\mathrm{d}T} = \frac{\mathrm{d}u^a}{\mathrm{d}T} B_a^\alpha + B_t^\alpha \tag{3}$$

holds.

Let $v^a(t)$, $w^a(t)$ be the functions of the class C_2 defined on the interval J. We shall suppose that along the trajectory (2) a field of virtual or tangential vectors of the rheonomous manifold $r - L_n^m(t)$ is defined by means of functions

$$v^\alpha = v^a B_a^\alpha\,, \quad T \in J \tag{4}$$

or

$$w^\alpha = w^a B_a^\alpha + B_t^\alpha\,, \quad T \in J\,, \tag{5}$$

respectively. The vector field (4) or (5) will be called the *W-parallel* or *T-parallel* if for every $T \in J$

$$B_\alpha^a D_T v^\alpha = 0 \tag{6}$$

or

(7) $$B^a_\alpha D_T w^\alpha = 0\,,$$

respectively. By means of formulae (2,1) and (2,2) of [2] we shall easily calculate that

$$D_T(v^a B^\alpha_a) = B^\alpha_a(D_T v^a + w^a_b v^b) + {}_p n^\alpha \left(h^p_{ab} v^b \frac{\mathrm{d}u^a}{\mathrm{d}T} + {}^p m_a v^a \right)$$

or

$$D_T(w^a B^\alpha_a + B^\alpha_t) = B^\alpha_a \left(D_T w^a + w^a_b w^b + {}'w^a_b \frac{\mathrm{d}u^b}{\mathrm{d}T} + W^a \right) +$$
$$+ {}_p n^\alpha \left(h^p_{ab} w^b \frac{\mathrm{d}u^a}{\mathrm{d}T} + {}^p m_a w^a + {}^{p'} m_a \frac{\mathrm{d}u^a}{\mathrm{d}T} + {}^p W \right).$$

Hence and from (4) and (5) it follows that the equations (6) or (7) are equivalent to the equations

(8) $$D_T v^a + w^a_b v^b = 0$$

or

(9) $$D_T w^a + w^a_b w^b + {}'w^a_b \frac{\mathrm{d}u^b}{\mathrm{d}T} + W^a = 0\,,$$

respectively.

If for every $T \in J$

(10) $$D_T v^a = 0$$

then the vector field (4) will be called the *pseudoparallel* field.

We have defined three "parallel" displacements, let us first consider the T-parallel displacement. We shall prove the following theorem:

Let $[x^\alpha({}_0T), {}_0T]$ be a point of the trajectory (2) and $({}_0w^a{}_0B^\alpha_a + {}_0B^\alpha_t)$ the tangential vector of the rheonomous manifold $r - L^m_n(t)$ defined at that point. Then there exists along the trajectory (2) exactly one T-parallel field (5) such that $w^a({}_0T) = {}_0w^a$.

The proof is easy. The functions $w^a(T)$ are solutions of the system of m-differential equations of (9). Writing in full the system (9) we obtain the equivalent linear system of m-differential equations of m-unknown functions in Cauchy's canonical form. From this result it follows immediately that the system (9) possesses on the interval J exactly one solution for given initial conditions.

If the tangential vectors of a given trajectory of the rheonomous manifold $r - L^m_n(t)$ form the T-parallel field then we call such trajectory a T-geodesic. The following theorem holds:

Let $({}_0w^\alpha) = ({}_0w^a{}_0B^\alpha_a + {}_0B^\alpha_t)$ be the tangential vector of the rheonomous manifold defined at its point $[{}_0x^\alpha, {}_0T]$. Then there exists (locally) exactly one T-geodesic with *parametric description* (2) *such that*

$$x^\alpha({}_0T) = {}_0x^\alpha\,, \quad \frac{\mathrm{d}x^\alpha({}_0T)}{\mathrm{d}T} = {}_0v^\alpha \equiv {}_0v^a{}_0B^\alpha_a + {}_0B^\alpha_t\,. \tag{11}$$

Proof. By (9) the W-geodesic is described by the system of differential equations

$$D_T\frac{\mathrm{d}u^a}{\mathrm{d}T} + \left(w^a_b + {}'w^a_b\right)\frac{\mathrm{d}u^a}{\mathrm{d}T} + W^a = 0\,, \quad \frac{\mathrm{d}x^\alpha}{\mathrm{d}T} = B^\alpha_a\frac{\mathrm{d}u^a}{\mathrm{d}T} + B^\alpha_t\,. \tag{12}$$

If we introduce the notation $\xi^a = \mathrm{d}u^a/\mathrm{d}T$ then (12) may be easely written in Cauchy's canonical form. For initial conditions (11), now of the form $x^\alpha({}_0T) = {}_0x^\alpha$, $\xi^a({}_0T) = {}_0w^a$, the new equivalent system has (locally) exactly one solution.

Preceding consideration of the T-parallel displacement may be extended also on the cases of the W-parallel and pseudoparallel displacement. We may show that under "usual" conditions the virtual vector may undergo a W-parallel or pseudoparallel displacement along the given trajectory in exactly one way. We may also define the notion of the W-geodesic or the pseudogeodesic and show its unique (local) existence for usual initial conditions. In case when $r - L^m_n(t)$ is a stationary manifold then ${}'w^a_b = w^a_b = 0$, $W^a = 0$, $B^\alpha_t = 0$ and all three "parallel" displacement are identical.

If a metric field $(g_{\alpha\beta})$ is defined in L_n (everywhere symmetric and positively definite) then L_n is a *Riemannian* space. We shall denote it R_n. In this case we shall denote $r - L_n(t)$ by $r - R_n(t)$. In the space $r - R_n(t)$ let be given such rheonomous manifold $r - L^m_n(t)$ that for every $\{x^\omega, t\}$ $g_{\alpha\beta}B^\alpha_a\,{}_pn^\beta = 0$, $g_{\alpha\beta}\,{}_pn^\alpha{}_pn^\beta = {}_{pq}\delta$. Then we shall denote $r - L^m_n(t)$ by $r - R^m_n(t)$. By means of the equations

$$g_{ab} = g_{\alpha\beta}B^\alpha_aB^\beta_b\,, \quad g_{ab}g^{bc} = \delta^c_a \tag{13}$$

the coordinates of the metric tensor field (g_{ab}) are defined on the rheonomous manifold $r - R^m_n(t)$.

We shall show that the following equations hold at every point of the rheonomous manifold $r - R^m_n(t)$:

$$D_cg_{ba} = 0\,, \tag{14}$$

$$\Gamma_{cb}{}^a = \left\{{a \atop cb}\right\} + \Omega_{bc}{}^a + \Omega^a_{c\ b} + \Omega_b{}^a{}_c \tag{15}$$

where

$$\left\{{a \atop cb}\right\} = \tfrac{1}{2}g^{ae}\left(\partial_cg_{be} + \partial_bg_{ce} - \partial_eg_{cb}\right), \quad \Omega_{bc}{}^a = \partial_{[b}B^\alpha_{|a|}B^\alpha_{c]}\,,$$

$$D_tg_{\beta\alpha} = 0\,. \tag{16}$$

I. From the equation

$$g_{\alpha\beta} = g_{\gamma\delta}\delta^\gamma_\alpha\delta^\delta_\beta = g_{ab}B^a_\alpha B^b_\beta + {}_{pq}\delta\,{}^pn_\alpha{}^qn_\beta \tag{17}$$

and the relation $\nabla_\gamma g_{\beta\alpha} = 0$ it follows that

$$D_c g_{ba} = B^\gamma_c B^\beta_b B^\alpha_a \nabla_\gamma (B^e_\beta B^f_\alpha g_{ef}) = B^\gamma_c B^\beta_b B^\alpha_a \nabla_\gamma (g_{\beta\alpha} - {}_{pq}\delta\, {}^p n_\beta\, {}^q n_\alpha) = 0\,.$$

II. Obviously we may write

$$\Gamma_{cb}{}^a = \tfrac{1}{2} B^\gamma_c B^\beta_b B^a_\alpha [\partial_\gamma (B^e_\beta B^g_\omega g_{eg} + {}_{pq}\delta\, {}^p n_\beta\, {}^q n_\omega) +$$
$$+ \partial_\beta (B^f_\gamma B^g_\omega g_{fg} + {}_{pq}\delta\, {}^p n_\gamma\, {}^q n_\omega) + \partial_\omega (B^f_\gamma B^e_\beta g_{fe} + {}_{pq}\delta\, {}^p n_\gamma\, {}^q n_\beta)]\, g^{\alpha\omega} - B^\beta_b\, \partial_c B^a_\beta\,.$$

Applying a simple modification by means of (17) we obtain (15).

III. Let us denote by the symbol ∇_t the absolute derivate in the space R_n along the curve which is described by the parametric equations $x^\alpha = x^\alpha({}_0 u^A, t)$, $t \in I$. Evidently $D_t g_{\alpha\beta} = \nabla_t g_{\alpha\beta} = 0$.

We shall show that the following theorem holds: *Let* (u^a), (v^b) *be two fields of W-parallel vectors which are defined along the trajectory* (2). *Then*

$$D_T(g_{ab} u^a v^b) = 0 \;^{1)} \tag{18}$$

at every point of the trajectory (2).

In the proof of the theorem we shall make use of the notation $G_{ab} = D_t g_{ab}$. From (13) and (2,4) in [2] it follows that

$$G_{ab} = g_{\alpha\beta}(D_t B^\alpha_a)\, B^\beta_b + g_{\alpha\beta} B^\alpha_a D_t B^\beta_b = w_{ab} + w_{ba}\,.$$

Hence and from the equations

$$D_T u^a + w^a_b u^b = 0\,, \quad D_T v^b + w^a_b v^b = 0\,, \quad D_T g_{ab} = G_{ab}$$

we easily verify that (18) holds.

If (u^a), (v^b) are two pseudoparallel fields of virtual vectors than $D_T(g_{ab} u^a v^b) = G_{ab} u^a v^b$ and for this season the scalar function $g_{ab} u^a v^b$ is generally not a constant. Similarly, for two T-parallel fields of virtual vectors (u^α), (v^β) the scalar function $g_{\alpha\beta} u^\alpha v^\beta$ is not in a general case a constant.

Let (v^a) be a field of virtual vectors defined along the trajectory (2). If there exists the function $k = k(T)$, $T \in J$ such that

$$D_T(k v^a) = 0\,, \quad k \neq 0\,, \quad g_{ab} v^a v^b = \text{konst.} > 0 \tag{19}$$

holds for every $T \in J$, then we call (v^a) a *δ-parallel field.* The vector field (v^a) is the δ-parallel field if and only if

$$\delta_T v^a \equiv D_T v^a + \frac{1}{2} \frac{G_{bc} v^b v^c}{g_{bc} v^b v^c}\, v^a = 0 \tag{20}$$

holds for every $T \in J$.

The proof is analogical to the case of the rheonomous anholonomic manifold in [1],

[1]) I.e. the scalar product of two W-parallel "displaced" vectors is invariant.

(20) and (22). It is easy to verify the unique existence of the parallel displacement and δ-geodesic for the usual initial conditions. Let us remark that in entirely similar way as in [1] we may define the notion of *H-parallel* displacement on the anholonomic rheonomous manifold $r - R_n^m(t)$.

If the rheonomous manifold $r - R_n^m(t)$ is stationary than at every point $G_{ab} = 0$ and the δ-parallel displacement of vectors along the given trajectory is mutually identical with the T-parallel, W-parallel and pseudoparallel displacement.

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