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PERIODIC SOLUTIONS OF A NONLINEAR TELEGRAPH EQUATION

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We shall prove – under certain assumptions – the existence and uniqueness of the solution of an initial value problem for the weakly nonlinear telegraph equation

$$u_{tt} - u_{xx} + 2au_t + 2bu_x + cu = h(t, x) + \varepsilon f(t, x, u, u_t, u_x, \varepsilon)$$

(a, b, c being constants, $a \neq 0$, ε being a small parameter).

Further, the functions h and f being ω-periodic in variable t, it will be shown – again under certain additional assumptions – that this equation has a unique solution $u(t, x)$ which is ω-periodic in t, too.

We shall consider our problem in a halfplane $[t, x] \in \langle 0, \infty) \times (-\infty, \infty)$ and then we shall show how it is possible to transfer the obtained results to the strip $\langle 0, \infty) \times \langle 0, \pi \rangle$ under the boundary conditions

$$u(t, 0) = u(t, \pi) \doteq 0 . \tag{0.1}$$

The used method has been taken over from paper [1] by the American mathematicians F. A. FICKEN and B. A. FLEISHMAN. These authors investigated the same problem (with $b = 0$) only for the special case $f = -u^3$ and they do not mention any generalization of their results for the other functions.

Their method can be used as we shall see only in case of $a > 0$, $b^2 + c > 0$. We have not succeeded in removing these two requirements as to the solution of an initial value problem (of course, except a linear case, for which a solution of an initial value problem is well known for quite arbitrary a, b, c). As to the periodic solutions we are able to eliminate the requirement $a > 0$ (naturally, it remains $a \neq 0$), but not the other one. For the linear equation

$$u_{tt} - u_{xx} + 2au_t + 2bu_x + cu = h(t, x)$$

under the conditions (0.1), we know how to prove by a quite another method the existence of a periodic solution without these both requirements – only under the assumption $a \neq 0$. The function $h(t, x)$ must, however, satisfy more strict assumptions.

We are just interested in classical solutions. As to generalized solutions, G. PRODI has proved in [2] the existence of a unique periodic solution of a more general hyperbolic equation, namely of the equation

$$u_{tt} - \Delta u + h(t, x, u_t) = f(t, x, u_{x_1}, \ldots, u_{x_n}) \quad (x = (x_1, \ldots, x_n)\,, \quad \Delta u = \sum_{i=1}^{n} u_{x_i x_i})$$

in the class of certain generalized solutions.

1. INITIAL VALUE PROBLEM

Let us consider the equation

$$(1.1) \qquad u_{tt} - u_{xx} + 2au_t + 2bu_x + cu = \\ = h(t, x) + \varepsilon f(t, x, u, u_t, u_x, \varepsilon)\,, \quad t \in \langle 0, \infty),\ x \in (-\infty, \infty)\,,$$

with the initial conditions

$$(1.2) \qquad u(0, x) = \sigma(x)\,, \quad u_t(0, x) = \tau(x)\,, \quad x \in (-\infty, \infty)\,,$$

where a, b, c, ε are constants, $a \neq 0$.

It is to find a classical solution of the initial value problem given by (1.1) and (1.2) i.e. a function $u(t, x)$ with continuous partial derivatives of the second order on $\langle 0, \infty) \times (-\infty, \infty)$ such that (1.1) and (1.2) are fulfilled. By the solution of (1.1) and (1.2) we will always mean such a function.

The substitution

$$u(t, x) = e^{bx} v(t, x)$$

transforms (1.1) and (1.2) into the equation

$$(1.3) \qquad v_{tt} - v_{xx} + 2av_t + (b^2 + c)\, v = k(t, x) + \varepsilon g(t, x, v, v_t, v_x, \varepsilon)$$

with the conditions

$$(1.4) \qquad v(0, x) = \varphi(x)\,, \quad v_t(0, x) = \psi(x)\,,$$

where

$$(1.5) \qquad k(t, x) = e^{-bx} h(t, x)\,, \quad g(t, x, v, v_t, v_x, \varepsilon) = e^{-bx} f(t, x, u, u_t, u_x, \varepsilon)\,, \\ \varphi(x) = e^{-bx} \sigma(x)\,, \quad \psi(x) = e^{-bx} \tau(x)\,.$$

In the sequel we shall assume that there are fulfilled these conditions:

(A_1) The function $\varphi(x)$ with its derivatives of the first and second order and the function $\psi(x)$ with its derivative of the first order are bounded and continuous for $x \in (-\infty, \infty)$.

(A_2) The function $k(t, x)$ with its partial derivative $k_x(t, x)$ is bounded and continuous in both variables for $t \in \langle 0, \infty)$, $x \in (-\infty, \infty)$.

(A_3) The function $g(t, x, v, r, s, \varepsilon)$ and its partial derivatives g_x, g_v, g_r, g_s are continuous in t, x, v, r and s for

$$t \in \langle 0, \infty), \; x \in (-\infty, \infty), \; v \in (-\infty, \infty), \; r \in (-\infty, \infty), \; s \in (-\infty, \infty),$$
$$\varepsilon \in \langle -\varepsilon_0, \varepsilon_0 \rangle \quad (\varepsilon_0 > 0).$$

Further, for any $\varrho \geqq 0$ there exist constants $K(\varrho)$, $C(\varrho)$ such that for $\max(|v|, |r|, |s|) \leqq \varrho$ it holds

$$|g|, |g_x|, |g_v|, |g_r|, |g_s| \leqq K(\varrho)$$

and the functions g, g_x, g_v, g_r, g_s are Lipschitzian in v, r, s with Lipschitz constant $C(\varrho)$.

For any function $v(t, x)$ with continuous derivatives v_t, v_x, v_{tx}, v_{xx} on $\langle 0, \infty) \times (-\infty, \infty)$ define the operator $\mathscr{P}$:

$$\mathscr{P}(v)(\varphi, \psi, k)(t, x) = \tfrac{1}{2} e^{-at} \Big\{ \varphi(x + t) + \varphi(x - t) + \tag{1.6}$$
$$+ \int_{x-t}^{x+t} \Big[J_0(d^{\frac{1}{2}}(t^2 - (x - z)^2)^{\frac{1}{2}}) (\psi(z) + a\,\varphi(z)) + \frac{\partial J_0(d^{\frac{1}{2}}(t^2 - (x - z)^2)^{\frac{1}{2}})}{\partial t} \varphi(z) \Big] \mathrm{d}z +$$
$$+ \int_0^t \int_{x-t+\vartheta}^{x+t-\vartheta} J_0(d^{\frac{1}{2}}((t - \vartheta)^2 - (x - z)^2)^{\frac{1}{2}})\, e^{a\vartheta} [k(\vartheta, z) +$$
$$+ \varepsilon g(\vartheta, z, v(\vartheta, z), v_t(\vartheta, z), v_x(\vartheta, z), \varepsilon)] \,\mathrm{d}z\, \mathrm{d}\vartheta \Big\},$$

where J_0 is the Bessel function of order zero and $d = -a^2 + b^2 + c$. (We do not express the dependence of $\mathscr{P}(v)(\varphi, \psi, k)$ on ε and g, because we do not need it and if no confusion can be arised, we shall write briefly $\mathscr{P}(v)$ instead of $\mathscr{P}(v)(\varphi, \psi, k)$.)

The function $\mathscr{P}(v)$ is continuous in t and x and has continuous derivatives $[\mathscr{P}(v)]_t$, $[\mathscr{P}(v)]_x$, $[\mathscr{P}(v)]_{tx}$, $[\mathscr{P}(v)]_{xx}$ (we can verify this easily by differentiating $\mathscr{P}(v)$ and applying to J_0 and its derivatives the following property of the Bessel function J_n of order n (see [3]):

$$\lim_{\xi \to 0} \frac{J_n(\xi)}{\xi^n} = \frac{1}{n!\, 2^n}\Big).$$

Moreover, it may be seen that there exists a continuous derivative $[\mathscr{P}(v)]_{tt}$. Thus, it is a simple calculation to see that if v (having continuous derivatives v_t, v_x, v_{tx}, v_{xx}) is a solution of the equation

$$v = \mathscr{P}(v) \tag{1.7}$$

then v satisfies the equation (1.3) and the conditions (1.4). Conversely, if v is a solution of (1.3) satisfying (1.4), then (1.7) is satisfied, too. (We obtain this by the known Riemann method.) It means that the equation (1.3) with the conditions (1.4) and the equation (1.7) are equivalent to each other (in the meaning just described).

From now, let the constants a and $b^2 + c$ be positive.

Denote $\mathscr{C}$ the space of all functions $v(t, x)$ which are with their derivatives v_t, v_x, v_{tx}, v_{xx} bounded and continuous on $\langle 0, \infty) \times (-\infty, \infty)$. The space $\mathscr{C}$ with the norm defined by

$$\|v\| = \sup_{\substack{t\in\langle 0,\infty)\\ x\in(-\infty,\infty)}} \left(|v(t, x)|, |v_t(t, x)|, |v_x(t, x)|, |v_{tx}(t, x)|, |v_{xx}(t, x)|\right)$$

is the complete normed linear space.

As it will be seen later, for $a > 0$ and $b^2 + c > 0$, the function $\mathscr{P}(v)$ and its derivatives are bounded for any $v \in \mathscr{C}$. Thereby $\mathscr{P}$ maps $\mathscr{C}$ into $\mathscr{C}$.

We shall now try for any suitably chosen $\varrho > 0$ to find $\bar{\varepsilon}$, $0 < \bar{\varepsilon} \leqq \varepsilon_0$, such that for all ε, $|\varepsilon| < \bar{\varepsilon}$, there exists a unique solution $v \in \mathscr{C}$ of the equation (1.7) with the norm $\|v\| \leqq \varrho$. According to the considerations above this is as well a unique solution of the initial value problem (1.3), (1.4) with $\|v\| \leqq \varrho$ and the function $u(t, x) = e^{bx} v(t, x)$ is a unique solution of the problem (1.1), (1.2) with the property $\|e^{-bx}u\| \leqq \varrho$.

We shall make use of the fixed point theorem in the following form:

Lemma 1. *Let the operator $\mathscr{P}$ map a complete normed linear space $\mathscr{C}$ into itself and let it hold:*

(i) *for $\|v\| \leqq \varrho$ (ϱ being any positive number) there is $\|\mathscr{P}(v)\| \leqq \varrho$;*

(ii) *$\mathscr{P}$ is a contraction operator for $\|v\| \leqq \varrho$, i.e. there exists a constant γ, $0 < \gamma < 1$, such that for any v_i, $\|v_i\| \leqq \varrho (i = 1, 2)$, there is*

$$\|\mathscr{P}(v_1) - \mathscr{P}(v_2)\| \leqq \gamma \|v_1 - v_2\| .$$

Then there exists a unique $v \in \mathscr{C}$ such that $v = \mathscr{P}(v)$, $\|v\| \leqq \varrho$. (Compare the more general form in [4].)

To find when both assumptions of this lemma are satisfied we need estimates of $\mathscr{P}(v)$ and its derivatives and to this we must know some estimates of the Bessel function.

By [3] we have:

$$J_0'(\xi) = -J_1(\xi), \quad J_1'(\xi) = \frac{J_1(\xi)}{\xi} - J_2(\xi), \quad |J_n(\xi)| \leqq \frac{|\xi|^n}{2^n n!} e^{|Im \xi|}$$

(where J_n is the Bessel function of order n). In our integrals there is $\xi = d^{\frac{1}{2}}((t - \vartheta)^2 - (x - z)^2)^{\frac{1}{2}}$, where $(t - \vartheta)^2 - (x - z)^2 \geqq 0$, $t - \vartheta \geqq 0$ so that we get immediately for J_0 and its derivatives (we do not write now the argument $\xi = d^{\frac{1}{2}}((t - \vartheta)^2 - (x - z)^2)^{\frac{1}{2}}$):

$$(1.8) \quad |J_0| \leqq 1 ,$$

$$\left|\frac{\partial J_0}{\partial x}\right| = \left|J_0' d \frac{-(x - z)}{d^{\frac{1}{2}}((t - \vartheta)^2 - (x - z)^2)^{\frac{1}{2}}}\right| = \left|J_1 d \frac{x - z}{d^{\frac{1}{2}}((t - \vartheta)^2 - (x - z)^2)^{\frac{1}{2}}}\right| \leqq \frac{d}{2} |x - z| ,$$

$$\left|\frac{\partial J_0}{\partial t}\right| = \left|J_0' d \frac{t - \vartheta}{d^{\frac{1}{2}}((t - \vartheta)^2 - (x - z)^2)^{\frac{1}{2}}}\right| = \left|-J_1 d \frac{t - \vartheta}{d^{\frac{1}{2}}((t - \vartheta)^2 - (x - z)^2)^{\frac{1}{2}}}\right| \leqq \frac{d}{2}(t - \vartheta) ,$$

$$\left|\frac{\partial^2 J_0}{\partial t \partial x}\right| = \left|J_1' d^2 \frac{(t - \vartheta)(x - z)}{d((t - \vartheta)^2 - (x - z)^2)} - J_1 d^2 \frac{(t - \vartheta)(x - z)}{d^{\frac{3}{2}}((t - \vartheta)^2 - (x - z)^2)^{\frac{3}{2}}}\right| =$$

$$= \left|- J_2 d^2 \frac{(t - \vartheta)(x - z)}{d((t - \vartheta)^2 - (x - z)^2)}\right| \leqq \frac{d^2}{8}(t - \vartheta)|x - z| ,$$

$$\left|\frac{\partial^2 J_0}{\partial x^2}\right| = \left|- J_1' d^2 \frac{(x - z)^2}{d((t - \vartheta)^2 - (x - z)^2)} + J_1 d^2 \frac{(t - \vartheta)^2}{d^{\frac{3}{2}}((t - \vartheta)^2 - (x - z)^2)^{\frac{3}{2}}}\right| =$$

$$= \left|J_1 d \frac{1}{d^{\frac{1}{2}}((t - \vartheta)^2 - (x - z)^2)^{\frac{1}{2}}} + J_2 d^2 \frac{(x - z)^2}{d((t - \vartheta)^2 - (x - z)^2)}\right| \leqq$$

$$\leqq \frac{d}{2}\left(1 + \frac{d}{4}(x - z)^2\right)$$

for $d \geqq 0$. If we write $|d|$ instead of d on the right in (1.8) and multiply these right-hand sides by $e^{|d^{\frac{1}{2}}|((t-\vartheta)^2 - (x-z)^2)^{\frac{1}{2}}}$ we get estimates for $d < 0$.

Let us take $v \in \mathscr{C}$ with $\|v\| \leqq \varrho$. Substituting $x + \zeta$ for z in the integrals in (1.6) and making use of the estimates (1.8) we obtain for $d \geqq 0$:

$$|\mathscr{P}(v)(t, x)| \leqq e^{-at} \sup_x |\varphi(x)| +$$

$$+ \tfrac{1}{2} e^{-at} \left\{ \int_{-t}^{t} \left| J_0(d^{\frac{1}{2}}(t^2 - \zeta^2)^{\frac{1}{2}})(\psi(x + \zeta) + a\varphi(x + \zeta)) + \right.\right.$$

$$\left.+ \frac{\partial J_0(d^{\frac{1}{2}}(t^2 - \zeta^2)^{\frac{1}{2}})}{\partial t} \varphi(x + \zeta)\right| \mathrm{d}\zeta + \int_0^t \int_{-t+\vartheta}^{t-\vartheta} |J_0(d^{\frac{1}{2}}((t - \vartheta)^2 - \zeta^2)^{\frac{1}{2}}) e^{a\vartheta}[k(\vartheta, x + \zeta) +$$

$$+ \varepsilon g(\vartheta, x + \zeta, v(\vartheta, x + \zeta), v_t(\vartheta, x + \zeta), v_x(\vartheta, x + \zeta), \varepsilon)]| \,\mathrm{d}\zeta \,\mathrm{d}\vartheta\} \leqq$$

$$\leqq e^{-at}(1 + at) \sup_x |\varphi(x)| + e^{-at} t \sup_x |\psi(x)| +$$

$$+ [\sup_x |k(t, x)| + |\varepsilon| K(\varrho)] \int_0^t e^{-a(t-\vartheta)}(t - \vartheta) \,\mathrm{d}\vartheta .$$

Similarly, for $d < 0$ we have:

$$|\mathscr{P}(v)(t, x)| \leqq e^{-(a-|d^{\frac{1}{2}}|)t}(1 + at) \sup_x |\varphi(x)| + e^{-(a-|d^{\frac{1}{2}}|)t} t \sup_x |\psi(x)| +$$

$$+ [\sup_x |k(t, x)| + |\varepsilon| K(\varrho)] \int_0^t e^{-(a-|d^{\frac{1}{2}}|)(t-\vartheta)}(t - \vartheta) \, d\vartheta .$$

By quite similar calculations one may obtain estimates for the derivatives of $\mathscr{P}(v)$. Denoting

$$\|\varphi\|_2 = \sup_{x\in(-\infty,\infty)} (|\varphi(x)|, |\varphi'(x)|, |\varphi''(x)|) ,$$

$$\|\psi\|_1 = \sup_{x\in(-\infty,\infty)} (|\psi(x)|, |\psi'(x)|) ,$$

$$\|k\|^1 = \sup_{\substack{t\in<0,\infty) \\ x\in(-\infty,\infty)}} (|k(t, x)|, |k_x(t, x)|) ,$$

we can state finally:

$$(1.9) \qquad \sup_x \{|\mathscr{P}(v)|, |[\mathscr{P}(v)]_t|, |[\mathscr{P}(v)]_x|, |[\mathscr{P}(v)]_{tx}|, |[\mathscr{P}(v)]_{xx}|\} \leqq$$

$$\leqq E(t) [P_3(t) \|\varphi\|_2 + P_2(t) \|\psi\|_1 + P_3(t) \|k\|^1] +$$

$$+ |\varepsilon| K(\varrho) \int_0^t E(t - \vartheta) Q_3(t - \vartheta) \, d\vartheta ,$$

where $E(t) = e^{-at}$ for $d \geqq 0$, $E(t) = e^{-(a-|d^{\frac{1}{2}}|)t}$ for $d < 0$ and $P_3(t)$, $Q_3(t)$ are polynomials in t of degree 3 and $P_2(t)$ is a polynomial in t of degree 2. There are polynomials with positive coefficients. The absolute member of Q_3 depends linearly on ϱ and all other coefficients of these polynomials do not depend on ϱ.

Because of $a > 0$ and $b^2 + c > 0$ there exist positive constants $L_2, L_3, \alpha_1, \alpha_2$ (depending only on a and $b^2 + c$) such that it is

$$(1.10) \qquad \|\mathscr{P}(v)\| \leqq L_3\|\varphi\|_2 + L_2\|\psi\|_1 + L_3\|k\|^1 + |\varepsilon| K(\varrho) (\alpha_1\varrho + \alpha_2)$$

for all d.

Hence we see that the norm $\|\mathscr{P}(v)\|$ is bounded and our assertion above that $\mathscr{P}$ maps $\mathscr{C}$ into $\mathscr{C}$ is true.

Further, we take interest in the norm $\|\mathscr{P}(v_1) - \mathscr{P}(v_2)\|$ for two elements $v_1, v_2 \in \mathscr{C}$ with the norm $\|v_i\| \leqq \varrho$ $(i = 1, 2)$. To this we make use of the Lipschitzian property of the function g and their derivatives. By the same way as above we get (writing for a while $\bar{v} = v_1 - v_2$, $\hat{v} = \mathscr{P}(v_1) - \mathscr{P}(v_2)$):

$$(1.11) \qquad \sup_x (|\hat{v}|, |\hat{v}_t|, |\hat{v}_x|, |\hat{v}_{tx}|, |\hat{v}_{xx}|) \leqq |\varepsilon| C_1(\varrho) \int_0^t E(t - \vartheta) R_3(t - \vartheta) .$$

$$. \sup_z (|\bar{v}(\vartheta, z)|, |\bar{v}_t(\vartheta, z)|, |\bar{v}_x(\vartheta, z)|, |\bar{v}_{tx}(\vartheta, z)|, |\bar{v}_{xx}(\vartheta, z)|) \, d\vartheta ,$$

where $C_1(\varrho) = \max(C(\varrho), K(\varrho))$ and $R_3(t)$ is a polynomial in t of degree 3 with positive coefficients. These coefficients, except the absolute member, do not depend on ϱ. The absolute member of R_3 is equal to the absolute member of Q_3.

Again there exists a constant $\alpha_3 > 0$ (α_3 depends only on a and $b^2 + c$ and there is $\alpha_3 > \alpha_2$) such that it is

$$\|\mathscr{P}(v_1) - \mathscr{P}(v_2)\| \leqq |\varepsilon|\, C_1(\varrho)\,(\alpha_1\varrho + \alpha_3)\,\|v_1 - v_2\| \tag{1.12}$$

for all d.

Consequently, $\mathscr{P}$ will be the contraction operator for any ε satisfying the inequality $|\varepsilon| < \varepsilon_1$, where

$$\varepsilon_1 = \min\left(\varepsilon_0, \frac{1}{C_1(\varrho)\,(\alpha_1\varrho + \alpha_3)}\right). \tag{1.13}$$

Further, the operator $\mathscr{P}$ has to transform the set of functions v with $\|v\| \leqq \varrho$ into itself, i.e. we require the fulfilling of

$$L_3\|\varphi\|_2 + L_2\|\psi\|_1 + L_3\|k\|^1 + |\varepsilon|\, K(\varrho)\,(\alpha_1\varrho + \alpha_2) \leqq \varrho\,. \tag{1.14}$$

Now, for an arbitrary ϱ for which there is

$$L_3\|\varphi\|_2 + L_2\|\psi\|_1 + L_3\|k\|^1 < \varrho \tag{1.15}$$

put

$$\bar{\varepsilon} = \min\left(\varepsilon_1, \frac{\varrho - [L_3\|\varphi\|_2 + L_2\|\psi\|_1 + L_3\|k\|^1]}{K(\varrho)\,(\alpha_1\varrho + \alpha_3)}\right).$$

Then for $|\varepsilon| < \bar{\varepsilon}$ both conditions of Lemma 1 will be fulfilled.

Thus, according to Lemma 1 and returning again to our original problem the following theorem is proved:

Theorem 1. *Under the assumptions* (A_1), (A_2), (A_3) *for any* ϱ *satisfying* (1.15) *there exists* $\bar{\varepsilon}$, $0 < \bar{\varepsilon} \leqq \varepsilon_0$, *such that for all* ε *with* $|\varepsilon| < \bar{\varepsilon}$ *the equation* (1.1) *under the initial conditions* (1.2) *has a unique solution* $u(t, x)$ *with* $\|e^{-bx}u\| \leqq \varrho$.

From our estimates it also can be derived immediately

Theorem 2. *Let* $\varrho \geqq 0$ *be given and let* $\varepsilon_1 > 0$ *be defined by* (1.13). *Let* $u_n(t,x)$ $(n = 1, 2, \ldots)$, $\|e^{-bx}u_n\| \leqq \varrho$, *be the solution of the equation* (1.1) *with the right-hand side* $h_n + \varepsilon f$, $|\varepsilon| < \varepsilon_1$, *under the initial conditions given by functions* σ_n, τ_n. *Let the functions* φ_n, ψ_n, k_n *and* g *be defined by* (1.5) *and let for* k_n *and* g *the conditions* (A_2) *and* (A_3), *respectively, be fulfilled.*

If there exist functions φ, ψ *and* k *such that for* $n \to \infty$ *it holds*: $\|\varphi_n - \varphi\|_2 \to 0$, $\|\psi_n - \psi\|_1 \to 0$, $\|k_n - k\|^1 \to 0$, *then there exists a function* $u(t, x) = \lim_{n\to\infty} u_n(t, x)$

while the convergence is uniform with respect to t and derivatives of u_n converge (uniformly in t) to corresponding derivatives of u. (If $b = 0$, the convergence is uniform with respect to x, too.) The function u is a solution of (1.1) with the right-hand side $h + \varepsilon f$ and with the initial conditions σ, τ, where

$$\sigma(x) = \lim_{n\to\infty} \sigma_n(x), \quad \tau(x) = \lim_{n\to\infty} \tau_n(x), \quad h(t, x) = \lim_{n\to\infty} h_n(t, x);$$

it is the unique solution with the property $\|e^{-bx}u\| \leqq \varrho$.

Proof. Let us put again $u_n(t, x) = e^{bx} v_n(t, x)$. The function $v_n(t, x)$ is a solution of (1.3) with the right-hand side $k_n + \varepsilon g$ under the initial conditions φ_n, ψ_n. Then by (1.10) and (1.12) we have for any natural m, n:

$$\|v_m - v_n\| \leqq L_3\|\varphi_m - \varphi_n\|_2 + L_2\|\psi_m - \psi_n\|_1 + L_3\|k_m - k_n\|^1 + \\ + |\varepsilon| \, C_1(\varrho)(\alpha_1\varrho + \alpha_3)\|v_m - v_n\| .$$

In virtue of (1.13) for $|\varepsilon| < \varepsilon_1$ there is

$$1 - |\varepsilon| \, C_1(\varrho)(\alpha_1\varrho + \alpha_3) > 0$$

so that we can write

$$\|v_m - v_n\| \leqq$$

$$\leqq \frac{1}{1 - |\varepsilon| \, C_1(\varrho)(\alpha_1\varrho + \alpha_3)} \left[L_3\|\varphi_m - \varphi_n\|_2 + L_2\|\psi_m - \psi_n\|_1 + L_3\|k_m - k_n\|^1\right].$$

Here the right-hand side tends to zero as $m, n \to \infty$, because of φ_n, ψ_n and k_n are fundamental sequences. It means that v_n also form a fundamental sequence and with respect to a completeness of the space $\mathscr{C}$ this sequence is convergent in the norm of $\mathscr{C}$. Denote $v = \lim_{n\to\infty} v_n$.

We easily verify that v is a solution of the equation (1.3) with the right-hand side $k + \varepsilon g$ and with the initial conditions φ, ψ. We show that v is a solution of the equation $v = \mathscr{P}(v)(\varphi, \psi, k)$ which is equivalent to this problem. Indeed, write

$$v - \mathscr{P}(v)(\varphi, \psi, k) = v - v_n + [\mathscr{P}(v_n)(\varphi_n, \psi_n, k_n) - \mathscr{P}(v)(\varphi, \psi, k)].$$

Hence

$$\|v - \mathscr{P}(v)(\varphi, \psi, k)\| \leqq \|v - v_n\| + \|\mathscr{P}(v_n)(\varphi_n, \psi_n, k_n) - \mathscr{P}(v)(\varphi, \psi, k)\| \leqq \\ \leqq L_3\|\varphi_n - \varphi\|_2 + L_2\|\psi_n - \psi\|_1 + L_3\|k_n - k\|^1 + \\ + [1 + |\varepsilon| \, C_1(\varrho)(\alpha_1\varrho + \alpha_3)]\|v_n - v\| .$$

The limit of the last expressions for $n \to \infty$ equals zero which implies $v = \mathscr{P}(v)(\varphi, \psi, k)$.

Since $\|v_n\| \leqq \varrho$ for all $n = 1, 2, \ldots$ there is $\|v\| \leqq \varrho$, too. By (1.13) the function v is a unique solution of the problem (1.3), (1.4) with $\|v\| \leqq \varrho$.

Now, the function $u(t, x) = e^{bx} v(t, x)$ is the sought limit of u_n and it has the required properties. This completes the proof.

(We did not prove the convergence of u_{ntt} but it is a consequence of the convergence of the u_n and its other derivatives.)

Remark 1. To prove the existence of a unique solution u, $\|e^{-bx}u\| \leqq \varrho$, of (1.1) and (1.2) on $\langle 0, T\rangle \times (-\infty, \infty)$, where $0 < T < +\infty$, we do not need to require $a > 0$ and $b^2 + c > 0$. It is seen from (1.9) and (1.11) holding for any a, b, c that for any $T > 0$ there is possible to write the estimates (1.10) and (1.12) with suitable constants. It yields that for any $T > 0$ and suitably chosen ϱ there exists $\bar{\varepsilon}$, $0 < \bar{\varepsilon} \leqq \varepsilon_0$, such that for any ε, $|\varepsilon| < \bar{\varepsilon}$, there exists a unique solution u of (1.1) and (1.2) with $\|e^{-bx}u\| \leqq \varrho$.

Similarly, Theorem 2 also can be formulated for any a, b, c if we consider the solutions u_n on $\langle 0, T\rangle \times (-\infty, \infty)$, $0 < T < +\infty$.

2. PERIODIC SOLUTIONS ON $\langle 0, \infty) \times (-\infty, \infty)$

In this section we shall investigate periodic solutions of the equation (1.1). First, we shall prove the theorem which will be useful to us.

We continue to suppose $a > 0$, $b^2 + c > 0$.

Theorem 3. *Let the conditions* (A_2), (A_3) *be fulfilled. Then there exists* ε^*, $0 < \varepsilon^* \leqq \varepsilon_0$, *such that for any two solutions* u_i, $\|e^{-bx}u_i\| \leqq \varrho$ $(i = 1, 2)$, $\varrho \geqq 0$, *of the equation* (1.1), *where* $|\varepsilon| < \varepsilon^*$, *it holds: the function* $u = u_1 - u_2$ *with all its derivatives converges to zero as* $t \to \infty$. *If* $b = 0$ *this convergence is uniform with respect to* x *and with respect to all initial conditions which are bounded (with their derivatives) by the same constant.*

Proof. Denote

$$\varphi_i(x) = v_i(x, 0)\,, \quad \psi_i(x) = v_{it}(x, 0) \quad (i = 1, 2)\,, \quad v(t, x) = v_1(t, x) - v_2(t, x)\,,$$

where $v_i(t, x) = e^{-bx} u_i(t, x)$ $(i = 1, 2)$.

There exists a constant $A > 0$ $(A \leqq \varrho)$ such that

$$\|\varphi_i\|_2 \leqq A\,, \quad \|\psi_i\|_1 \leqq A\,.$$

If we again use the estimates of section 1 we have, denoting

$$\tilde{v}(t) = \sup_x \left(|v|, |v_t|, |v_x|, |v_{tx}|, |v_{xx}|\right):$$

$$\tilde{v}(t) \leqq E(t)\, P_3(t)\, \|\varphi_1 - \varphi_2\|_2 + E(t)\, P_2(t)\, \|\psi_1 - \psi_2\|_1 + |\varepsilon|\, C_1(\varrho) \int_0^t E(t - \vartheta)\, R_3(t - \vartheta)\, \tilde{v}(\vartheta)\, d\vartheta \leqq$$

$$\leqq 2A\, E(t)\, P_3(t) + 2A\, E(t)\, P_2(t) + |\varepsilon|\, C_1(\varrho) \int_0^t E(t - \vartheta)\, R_3(t - \vartheta)\, \tilde{v}(\vartheta)\, d\vartheta\,.$$

Evidently, there exist constants $B > 0$, $\beta > 0$ such that

$$\tilde{v}(t) \leqq Be^{-\beta t} + \varepsilon B \int_0^t e^{-\beta(t-\vartheta)}\, \tilde{v}(\vartheta)\, d\vartheta \tag{2.1}$$

(B, β depend on A, ϱ and of course on a and $b^2 + c$, too).

Applying this estimate to the integral in (2.1) and iterating this proceeding we obtain after k iterations (putting $\vartheta_1 = t$):

$$\tilde{v}(t) \leqq Be^{-\beta t} \sum_{n=0}^{k} \frac{(\varepsilon Bt)^n}{n!} + \tag{2.2}$$

$$+ \varepsilon^{k+1} B^{k+1} \int_0^{\vartheta_1} \int_0^{\vartheta_2} \dots \int_0^{\vartheta_{k+1}} e^{-\beta(t-\vartheta_{k+2})}\, \tilde{v}(\vartheta_{k+2})\, d\vartheta_{k+2}\, d\vartheta_{k+1} \dots d\vartheta_2 \leqq$$

$$\leqq Be^{-\beta t} \sum_{n=0}^{k} \frac{(\varepsilon Bt)^n}{n!} + 2\varrho \left(\frac{\varepsilon B}{\beta}\right)^{k+1}.$$

Taking $\varepsilon^* = \min(\varepsilon_0, \beta/B)$ and letting $n \to \infty$ in (2.2) we have for any ε with $|\varepsilon| < \varepsilon^*$:

$$\tilde{v}(t) \leqq Be^{(-\beta + \varepsilon B)t}. \tag{2.3}$$

The right-hand side in (2.3) tends to zero as $t \to \infty$ and so it also holds for $\tilde{v}(t)$. This also implies immediately the convergence to zero of v_{tt} for $t \to \infty$. Hence, for $u(t, x) = u_1(t, x) - u_2(t, x) = e^{bx} v(t, x)$ it follows readily the assertion of Theorem 3. (The uniformity of the convergence of $u(t, x)$ and its derivatives for $t \to \infty$ is obvious.)

Now, we are able to prove the theorem about the existence of a periodic solution of the equation (1.1).

Let there be $a \neq 0$ (otherwise a arbitrary), $b^2 + c > 0$.

Theorem 4. *Let the functions h and f be ω-periodic in t and let the conditions* (A_2) *and* (A_3) *be satisfied. Let us have $\varrho > L_3 \|k\|^1$. Then there exists $\tilde{\varepsilon}$, $0 < \tilde{\varepsilon} \leqq \varepsilon_0$, such that for all ε, $|\varepsilon| < \tilde{\varepsilon}$, the equation* (1.1) *has a unique ω-periodic* (*in t*) *solution $\bar{u}(t, x)$ with $\|e^{-bx}\bar{u}\| \leqq \varrho$.*

Proof. First, let us suppose $a > 0$. Then by Theorem 1 it is possible to choose $\bar{\varepsilon}$, $0 < \bar{\varepsilon} \leqq \varepsilon_0$, such that for all $|\varepsilon| < \bar{\varepsilon}$ there exist a unique solution $u(t, x)$ of the equation (1.1) with $\|e^{-bx}u\| \leqq \varrho$ which satisfies the initial conditions

$$u(0, x) = 0, \quad u_t(0, x) = 0.$$

The function $v(t, x) = e^{-bx} u(t, x)$ fulfil the equation (1.3) (where k and g are ω-periodic in t) with the same initial conditions and $\|v\| \leqq \varrho$.

Denote

$$\varphi_n(x) = v(n\omega, x), \quad \psi_n(x) = v_t(n\omega, x), \quad v_n(t, x) = v(t + n\omega, x).$$

The function $v_n(t, x)$ does solve the equation (1.3), too, and satisfies the initial conditions given by the functions φ_n and ψ_n. According to $\|v\| \leqq \varrho$ there exists constant $A > 0$ such that $\|\varphi_n\|_2 \leqq A$, $\|\psi_n\|_1 \leqq A$ for all $n = 1, 2, \ldots$

If we take $|\varepsilon| < \tilde{\varepsilon}$, where $\tilde{\varepsilon} = \min(\bar{\varepsilon}, \varepsilon^*)$ (ε^* being from Theorem 3) then by Theorem 3 the function $v_n(t, x) - v(t, x)$ with its derivatives converges to zero as $t \to \infty$ and this convergence is uniform with respect to x and to n.

Hence we have: to any $\eta > 0$ there exists $t_\eta \geqq 0$ so that for $t \geqq t_\eta$ and for all x and n there is

$$|v_n(t, x) - v(t, x)| \leqq \eta .$$

For $m\omega \geqq t_\eta$, $t \geqq 0$ it holds $|v_n(t + m\omega, x) - v(t + m\omega, x)| \leqq \eta$. Taking $p\omega \geqq \geqq t_\eta$, $q\omega \geqq t_\eta$, $p > q$ we obtain for all x and $t \geqq 0$:

$$|v_p(t, x) - v_q(t, x)| = |v_{p-q}(t + q\omega, x) - v(t + q\omega, x)| \leqq \eta .$$

It means that v_n forms the fundamental sequence, uniformly in t and x. Hence there exists the function $\bar{v}(t, x) = \lim_{n\to\infty} v_n(t, x)$. This function is bounded and continuous in t and x. Further,

$$\bar{v}(0, x) = \lim_{n\to\infty} v_n(0, x) = \lim_{n\to\infty} v(n\omega, x) = \lim_{n\to\infty} \varphi_n(x)$$

so that the function $\varphi(x) = \lim_{n\to\infty} \varphi_n(x)$ also exists and is bounded and continuous in x.

By the same way we get finally that the functions $v_n(t, x)$ converge to $\bar{v}(t, x)$ in the norm of $\mathscr{C}$, it is $\|\bar{v}\| \leqq \varrho$, there exist the function $\psi(x) = \lim_{n\to\infty} \psi_n(x)$; moreover, $\|\psi_n - \psi\|_1 \to 0$ as $n \to \infty$ and also $\|\varphi_n - \varphi\|_2 \to 0$ as $n \to \infty$.

Even as in the proof of Theorem 2 it can be shown that the function $\bar{v}$ satisfies the equation (1.3) under the initial conditions φ, ψ.

The periodicity of the function $\bar{v}$ is clear:

$$\bar{v}(t + \omega, x) = \lim_{n\to\infty} v_n(t + \omega, x) = \lim_{n\to\infty} v_{n+1}(t, x) = \bar{v}(t, x) .$$

Since we have $|\varepsilon| < \tilde{\varepsilon}$ the solution of (1.3) with the norm bounded by ϱ is uniquely determined by the initial conditions. This implies, by Theorem 3, that $\bar{v}$ is the unique ω-periodic solution of (1.3) with $\|\bar{v}\| \leqq \varrho$.

Further, these considerations yield that the function $\bar{u}(t, x) = e^{bx}\, \bar{v}(t, x)$ is the unique ω-periodic solution of the equation (1.1) with the property $\|e^{-bx}u\| \leqq \varrho$. (The initial conditions for this solution are given by $\sigma(x) = e^{bx}\, \varphi(x)$, $\tau(x) = = e^{bx}\, \psi(x)$.)

Secondly, suppose $a < 0$.

Let us continue the functions h and f for all $t \in (-\infty, \infty)$, $x \in (-\infty, \infty)$ as ω-periodic in t. The equation (1.1) is transformed by the substitution $t = -\eta$ into the equation

(2.4) $$\hat{u}_{\eta\eta} - \hat{u}_{xx} + 2\hat{a}\hat{u}_\eta + 2b\hat{u}_x + c\hat{u} = \hat{h}(\eta, x) + \varepsilon\hat{f}(\eta, x, \hat{u}, \hat{u}_\eta, \hat{u}_x, \varepsilon) ,$$

where

$\hat{u}(\eta, x) = u(t, x)$, $\hat{h}(\eta, x) = h(t, x)$, $\hat{f}(\eta, x, \hat{u}, \hat{u}_\eta, \hat{u}_x, \varepsilon) = f(t, x, u, u_t, u_x, \varepsilon)$, $\hat{a} = -a$.

Since $\hat{a} > 0$ we can prove as above the existence of $\tilde{\varepsilon}$, $0 < \tilde{\varepsilon} \leqq \varepsilon_0$, such that for all $|\varepsilon| < \tilde{\varepsilon}$ there exists a unique ω-periodic solution $\hat{u}(\eta, x)$ of the equation (2.4) with $\|e^{-bx}\hat{u}\| \leqq \varrho$ on $\langle 0, \infty) \times (-\infty, \infty)$. Continuing $\hat{u}$ for all $\eta \in (-\infty, \infty)$ as ω-periodic in η, we see that the continued function $\hat{u}$ represents the unique ω-periodic (in η) solution of (2.4) for all $\eta \in (-\infty, \infty)$ and $x \in (-\infty, \infty)$, such that it is with its derivatives bounded by ϱ.

Thus, returning to the function

$$u(t, x) = \hat{u}(\eta, x) = \hat{u}(-t, x)$$

we have that $u(t, x)$ for $t \geqq 0$ is the unique ω-periodic solution of (1.1) with $\|e^{-bx}u\| \leqq \varrho$ which completes the proof.

3. MIXED PROBLEM AND PERIODIC SOLUTIONS ON $\langle 0, \infty) \times \langle 0, \pi \rangle$

We have done all our considerations for $t \in \langle 0, \infty)$ and $x \in (-\infty, \infty)$. Now, let us investigate the mixed problem given by (1.1), (1.2) and by the boundary conditions

$$u(t, 0) = u(t, \pi) = 0\,, \quad t \in \langle 0, \infty)\,, \tag{3.1}$$

while the functions h, f, σ and τ are defined for $t \in \langle 0, \infty)$, $x \in \langle 0, \pi \rangle$, only.

Transform again (1.1) and (1.2) into (1.3) and (1.4) (the boundary conditions remain unchanged) and formulate these conditions:

(B_1) The functions φ and ψ have on $\langle 0, \pi \rangle$ continuous derivatives of the second and of the first order, respectively, and

$$\varphi(0) = \varphi(\pi) = \varphi''(0) = \varphi''(\pi) = 0\,, \quad \psi(0) = \psi(\pi) = 0\,. \tag{3.2}$$

(B_2) The function $k(t, x)$ and its derivative $k_x(t, x)$ are continuous in both variables on $\langle 0, \infty) \times \langle 0, \pi \rangle$ and it holds

$$k(t, 0) = k(t, \pi) = 0 \quad (t \in \langle 0, \infty))\,. \tag{3.3}$$

(B_3) The function $g(t, x, v, r, s, \varepsilon)$ and its derivatives g_x, g_v, g_r, g_s are continuous in t, x, v, r, s for

$$t \in \langle 0, \infty)\,, \quad x \in \langle 0, \pi \rangle\,, \quad v, r, s \in (-\infty, \infty)\,, \quad \varepsilon \in \langle -\varepsilon_0, \varepsilon_0 \rangle \quad (\varepsilon_0 > 0)$$

and to any $\varrho \geqq 0$ there exist constants $K(\varrho)$, $C(\varrho)$ such that for $\max(|v|, |r|, |s|) \leqq \varrho$ it holds

$$|g|, |g_x|, |g_v|, |g_r|, |g_s| \leqq K(\varrho)$$

and g, g_x, g_v, g_r, g_s are Lipschitzian in v, r, s with Lipschitz constant $C(\varrho)$.

Further, let

$$(3.4)\qquad g(t, 0, 0, 0, s, \varepsilon) = g(t, \pi, 0, 0, s, \varepsilon) = 0$$
$$(t \in \langle 0, \infty), s \in (-\infty, \infty), \varepsilon \in \langle -\varepsilon_0, \varepsilon_0 \rangle) .$$

We shall continue the functions φ, ψ for all $x \in (-\infty, \infty)$ and the function k for all $t \geqq 0$, $x \in (-\infty, \infty)$ as odd and 2π-periodic functions in x and the function g will be continued for all $t \geqq 0$, $x \neq \pi n$ $(n = 0, \pm 1, \pm 2, \ldots)$, $v, r, s \in (-\infty, \infty)$, $\varepsilon \in \langle -\varepsilon_0, \varepsilon_0 \rangle$ and for all $t \geqq 0$, $x = \pi n$, $v = r = 0$, $s \in (-\infty, \infty)$, $\varepsilon \in \langle -\varepsilon_0, \varepsilon_0 \rangle$ by this way:

$$g(t, -x, -v, -r, s, \varepsilon) = -g(t, x, v, r, s, \varepsilon) = g(t, x + 2\pi, v, r, s, \varepsilon) .$$

(By means of (1.5) the continuation of the original functions σ, τ, h and f is given, too.)

According to (B_1), (B_2), (B_3) these continued functions fulfil the conditions (A_1), (A_2), (A_3) in their definition domain.

Suppose again $a > 0$, $b^2 + c > 0$.

By the same way as in section 1 we get that the operator $\mathscr{P}$ maps now $\mathscr{C}$ into $\mathscr{C}$, too, and the equation $v = \mathscr{P}(v)$ has for any suitable ϱ and ε a unique solution $v(t, x)$ with $\|v\| \leqq \varrho$. Due to the way of a continuation of φ, ψ, k and g the functions $-v(t, -x)$ and $v(t, x + 2\pi)$ are also the solutions of $v = \mathscr{P}(v)$ which in connection with the uniqueness of the solution gives

$$v(t, -x) = -v(t, x) = v(t, x + 2\pi) \quad (t \geqq 0, x \in (-\infty, \infty))$$

i.e. the function v is odd and 2π-periodic in x and thus v satisfies the conditions

$$v(t, 0) = v(t, \pi) = 0 \quad (t \geqq 0) .$$

Then the function $u(t, x) = e^{bx} v(t, x)$ satisfies (3.1), too. Hence, this function u gives for $t \in \langle 0, \infty)$, $x \in \langle 0, \pi \rangle$ the solution of our mixed problem.

Thereby we obtain this result:

Theorem 1'. *Let the conditions (B_1), (B_2) and (B_3) be fulfilled. Then for any suitably chosen ϱ (being found as in Theorem 1) there exists $\bar{\varepsilon}$, $0 < \bar{\varepsilon} \leqq \varepsilon_0$, such that for all ε with $|\varepsilon| < \bar{\varepsilon}$ the equation (1.1) under the conditions (1.2) and (3.1) has a unique solution $u(t, x)$ with $\|e^{-bx}u\| \leqq \varrho$.*

Proof: It remains only to show that the function u is actually a unique solution of the given problem with the property $\|e^{-bx}u\| \leqq \varrho$ i.e. that it does not depend on the way of a continuing of the functions σ, τ, h and f.

In the case of the existence of two different solutions u_1, u_2 with $\|e^{-bx}u_i\| \leqq \varrho$ $(i = 1, 2)$ we have for the function $u = u_1 - u_2$ the equation

$$(3.5)\qquad u_{tt} - u_{xx} + 2au_t + 2bu_x + cu =$$
$$= \varepsilon[f(t, x, u_1, u_{1t}, u_{1x}, \varepsilon) - f(t, x, u_2, u_{2t}, u_{2x}, \varepsilon)]$$

under the conditions

$$u(0, x) = 0\,, \quad u_t(0, x) = 0\,, \quad u(t, 0) = u(t, \pi) = 0\,.$$

Applying the mean value theorem to the right-hand side in (3.5) and making use of the theorem from [5] about uniqueness of a solution of the mixed problem for linear equations we obtain that u is identically zero. Hence, the uniqueness of a solution of our problem is proved.

Even so as in sections 1 and 2 we may derive Theorems 2′, 3′ and 4′ analogous to Theorems 2, 3 and 4. Let us formulate only Theorem 4′ containing the main result and omit the formulation of two other Theorems.

Let us have $a \neq 0$, $b^2 + c > 0$.

Theorem 4′. *Let the functions h and f be ω-periodic in t and let the condition (B_2) and (B_3) be satisfied. Let ϱ be suitably chosen (see Th. 4). Then there exists $\tilde{\varepsilon}$, $0 < \tilde{\varepsilon} \leqq \varepsilon_0$, such that for all ε with $|\varepsilon| < \tilde{\varepsilon}$ the equation (1.1) with the conditions (3.1) has a unique ω-periodic solution $u(t, x)$ satisfying the inequality $\|e^{-bx}u(t, x)\| \leqq \varrho$.*

Remark 2. Let us briefly treat the special case when the linear equation

$$u_{tt} - u_{xx} + 2au_t + 2bu_x + cu = h(t, x)\,, \quad t \in \langle 0, \infty)\,, \quad x \in \langle 0, \pi\rangle\,, \tag{3.6}$$

is given, again with the conditions (1.2) and (3.1).

Let σ and τ have continuous derivatives of the second and the first order, respectively, and let h and its derivative $h_x(t, x)$ be continuous in t and x, while (3.2) and (3.3) holds. Continuing σ, τ and h as above, the solution of the problem (3.6), (1.2) and (3.1) is given by (for arbitrary a, b, c):

$$\begin{aligned} u(t, x) = \tfrac{1}{2}e^{-at+bx}\Big\{&\sigma(x + t)\, e^{-b(x+t)} + \sigma(x - t)\, e^{-b(x-t)} + \\ &+ \int_{x-t}^{x+t}\Big[J_0(d^{\frac{1}{2}}(t^2 - (x - z)^2)^{\frac{1}{2}})\, e^{-bz}(\tau(z) + a\,\sigma(z)) + \\ &+ \frac{\partial J_0(d^{\frac{1}{2}}(t^2 - (x - z)^2)^{\frac{1}{2}})}{\partial t}\, e^{-bz}\sigma(z)\Big]\, dz + \\ &+ \int_0^t\int_{x-t+\vartheta}^{x+t-\vartheta} J_0(d^{\frac{1}{2}}((t - \vartheta)^2 - (x - z)^2)^{\frac{1}{2}})\, e^{a\vartheta - bz}\, h(\vartheta, z)\, dz\, d\vartheta\Big\}\,. \end{aligned} \tag{3.7}$$

The necessary and sufficient condition for the ω-periodicity of a solution $u(t, x)$ of (3.6) and (3.1) reads:

$$u(0, x) - u(\omega, x) = 0\,, \quad u_t(0, x) - u_t(\omega, x) = 0\,. \tag{3.8}$$

Making use in (3.8) of the expression (3.7) of u we obtain functional equations for initial conditions σ, τ such that the function $u(t, x)$, uniquely given by them, is ω-periodic in t. If we seek σ, τ in the form

$$\sigma(x) = e^{bx} \sum_{k=1}^{\infty} a_k \sin kx\,, \quad \tau(x) = e^{bx} \sum_{k=1}^{\infty} b_k \sin kx\,,$$

(a_k, b_k being Fourier coefficients of σ and τ, respectively) we obtain the following result:

Let the function $h(t, x)$ be continuous and have the continuous derivative of the third order with respect to x. Further, let

$$h(t, 0) = h(t, \pi) = 0\,, \quad h_{xx}(t, 0) = h_{xx}(t, \pi) = 0$$

and h be ω-periodic in t. Then if there is $a \neq 0$, $b^2 + c \neq -k^2$ for all $k = 1, 2, \ldots$, the problem (3.6) and (3.1) has a unique ω-periodic solution $u(t, x)$.

If $b^2 + c = -k_0^2$ for some k_0 and if $h(t, x)$ has continuous derivative of the first order, only, it may be possible to find (by the same way) a necessary condition that the problem (3.6) and (3.1) have ω-periodic solutions. Denoting

$$H(\omega, x) = \int_0^{\omega} \int_{x-\omega+\vartheta}^{x+\omega-\vartheta} J_0(d^{\frac{1}{2}}((\omega - \vartheta)^2 - (x - z)^2)^{\frac{1}{2}})\, h(\vartheta, z)\, \mathrm{d}z\, \mathrm{d}\vartheta$$

this condition is

$$\frac{a}{k_0} \int_0^{2\pi} H_x(\omega, x) \cos k_0 x\, \mathrm{d}x + \int_0^{2\pi} H(\omega, x) \sin k_0 x\, \mathrm{d}x = 0\,.$$

If $h(t, x)$ has continuous derivative of the third order this condition becomes sufficient, too, and then there exist infinitely many ω-periodic solutions of (3.6) and (3.1).

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Výtah

PERIODICKÁ ŘEŠENÍ NELINEÁRNÍ TELEGRAFNÍ ROVNICE

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Článek se zabývá nelineární telegrafní rovnicí

$$u_{tt} - u_{xx} + 2au_t + 2bu_x + cu = h(t, x) + \varepsilon f(t, x, u, u_t, u_x, \varepsilon) \tag{1.1}$$

$(a, b, c$ — konstanty, $a \neq 0$, ε — malý parametr$)$.

V části 1 se předpokládá $a > 0$, $b^2 + c > 0$. Dokazuje se věta o existenci a jednoznačnosti řešení rovnice (1.1) pro $t \in \langle 0, \infty)$, $x \in (-\infty, \infty)$ při počátečních podmínkách

$$u(0, x) = \sigma(x), \quad u_t(0, x) = \tau(x) \tag{1.2}$$

(věta 1).

Dále je odvozena věta o spojité závislosti řešení problému (1.1), (1.2) na počátečních podmínkách σ, τ a na funkci h (věta 2).

Hlavní výsledek části 2 je formulován ve větě 4, která dává při $a \neq 0$, $b^2 + c > 0$ existenci a jednoznačnost ω-periodického (v t) řešení rovnice (1.1) pro $t \in \langle 0, \infty)$, $x \in (-\infty, \infty)$ za předpokladu ω-periodičnosti (v t) funkcí h a f. Důkaz této věty spočívá na chování omezených řešení rovnice (1.1) při $t \to \infty$, které je obsahem věty 3 (odvozené opět pro $a > 0$, $b^2 + c > 0$).

V části 3 se vyšetřuje smíšená úloha pro $t \in \langle 0, \infty)$, $x \in \langle 0, \pi \rangle$, daná rovnicí (1.1), počátečními podmínkami (1.2) a okrajovými podmínkami

$$u(t, 0) = u(t, \pi) = 0 \tag{3.1}$$

a zkoumá se existence periodického řešení rovnice (1.1) při podmínkách (3.1). Lze odvodit věty obdobné větám 1—4 (formulovány jsou jen věty 1′ a 4′).

Резюме

ПЕРИОДИЧЕСКИЕ РЕШЕНИЯ НЕЛИНЕЙНОГО ТЕЛЕГРАФНОГО УРАВНЕНИЯ

ЯНА ГАВЛОВА (Jana Havlová), Прага

Статья занимается нелинейным телеграфным уравнением

$$u_{tt} - u_{xx} + 2au_t + 2bu_x + cu = h(t, x) + \varepsilon f(t, u, u_t, u_x, \varepsilon) \tag{1.1}$$

$(a, b, c$ — постоянные, ε — малый параметр$)$.

В параграфе 1 предполагается $a > 0$, $b^2 + c > 0$. Доказывается теорема о существовании и единственности решения уравнения (1.1) для $t \in \langle 0, \infty)$, $x \in (-\infty, \infty)$ при начальных условиях

$$(1.2) \qquad u(0, x) = \sigma(x), \quad u_t(0, x) = \tau(x)$$

(Теорема 1).

Далее приводится теорема о непрерывной зависимости решения проблемы (1.1), (1.2) от начальных условий σ, τ и от функции h (Теорема 2).

Главный результат параграфа 2 содержит Теорема 4, которая утверждает при $a \neq 0$, $b^2 + c > 0$ существование и единственность ω-периодического (в t) решения уравнения (1.1) для $t \in \langle 0, \infty)$, $x \in (-\infty, \infty)$ при предположении ω-периодичности (в t) функций h и f. Доказательство этой теоремы основано на поведении ограниченных решений уравнения (1.1) при $t \to \infty$, о котором говорит Теорема 3 (выведенная опять для $a > 0$, $b^2 + c > 0$).

В параграфе 3 исследуется смешанная задача для $t \in \langle 0, \infty)$, $x \in \langle 0, \pi \rangle$, данная уравнением (1.1), начальными условиями (1.2) и краевыми условиями

$$(3.1) \qquad u(t, 0) = u(t, \pi) = 0\,,$$

и рассматривается существование периодического решения уравнения (1.1) при условиях (3.1). Можно доказать теоремы, аналогичные Теоремам 1—4 (сформулированы только Теоремы 1' и 4').