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COMPACT IMBEDDING OF WEIGHTED SOBOLEV SPACE DEFINED ON AN UNBOUNDED DOMAIN II

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Summary. This paper is a direct continuation of [1], where fundamental concepts and notation were introduced. The compactness of the imbedding of the weighted Sobolev space $W_0^{1,p}(\Omega, S)$ into the weighted Lebesgue space $L^p(\Omega, \rho)$ is investigated and this imbedding is again considered as a limit case of compact imbeddings of Sobolev spaces defined on bounded domains.

Keywords: Weighted Sobolev space, weighted Lebesgue space, compact imbedding, weight function.

AMS Classification: 46E35.

1. PRELIMINARIES

In the subsequent sections we use the following assertions:*)

1.1. Lemma. *Let $p \in \langle 1, \infty)$, $n \in N$. Then there exist functions*

$$\phi_j = \phi_j(x), \quad x \in \mathbf{R}, \quad j = n, n + 1, n + 2, \dots \tag{1.1}$$

and a positive constant K such that

a) $0 \leqq \phi_j(x) \leqq 1$, $x \in \mathbf{R}$;

b) $\phi_j^{1/p} \in C^1(\mathbf{R})$, $\operatorname{supp} \phi_j \subset (j - 1, j + 1)$;

c) $\left|\frac{\mathrm{d}}{\mathrm{d}x} [\phi_j^{1/p}(x)]\right| \leqq K$, $x \in \mathbf{R}$ (*K is independent of j*);

d) $\sum_{j=n}^{\infty} \phi_j(x) = 1$, $x \in (n, \infty)$.**)

If $0 \leqq r_1 < r_2 \leqq \infty$ we define

$$P(r_1, r_2) = \{x \in \mathbf{R}^N;\ r_1 < |x| < r_2\}. \tag{1.2}$$

1.2. Lemma. *Let $p \in \langle 1, \infty)$, $n \in N$. Then there exist functions*

$$\psi_j = \psi_j(x), \quad x \in \mathbf{R}^N, \quad j = n,\ n + 1,\ n + 2,\ \dots \tag{1.3}$$

*) Their proofs can be found in Appendix (see Section 3).

**) The sum in question contains at most two nonzero summands for each $x \in \mathbf{R}$.

and a positive constant K such that

a) $0 \leqq \psi_j(x) \leqq 1,\ x \in \boldsymbol{R}^N$;

b) $\psi_j^{1/p} \in C^1(\boldsymbol{R})$, $\operatorname{supp} \psi_j \subset P(j-1, j+1)$;

c) $\left|\dfrac{\partial}{\partial x_i}\left[\psi_j^{1/p}(x)\right]\right| \leqq K,\ x \in \boldsymbol{R}^N,\ i = 1, \dots, N$ (*K is independent of j*);

d) $\sum_{j=n}^{\infty} \psi_j(x) = 1,\ x \in P(n, \infty)$.**)

1.3. Lemma. *Let* $1 \leqq p < \infty$, $-\infty \leqq a < b \leqq +\infty$, let a_0, a_1 *be measurable, a.e. in* (a, b) *nonnegative functions. Let there exist numbers* $c \in (a, b)$, $d \in \boldsymbol{R}$ *and* $C > 0$ *such that*

$$\left|\int_c^x a_0(s)\,\mathrm{d}s + d\right| \leqq C a_0^{1/p^*}(x)\, a_1^{1/p}(x) \quad \text{for a.e.} \quad x \in (a, b). \tag{1.4}$$

Then the inequality

$$\int_a^b |f(x)|^p\, a_0(x)\,\mathrm{d}x \leqq (Cp)^p \int_a^b |f'(x)|^p\, a_1(x)\,\mathrm{d}x \tag{1.5}$$

holds for each function $f \in C_0^1((a, b))$.

1.4. Remark. (i) Let us note that the condition $f \in C_0^1((a, b))$ in Lemma 1.3 can be weakened; it suffices to assume that $f \in AC((a, b))$ and that $\operatorname{supp} f$ is a compact subset of (a, b).

(ii) In Lemma 1.3 there are no assumptions on the convergence of

$$\int_a^x a_0(s)\,\mathrm{d}s \quad \text{or} \quad \int_x^b a_0(s)\,\mathrm{d}s \quad \text{for} \quad x \in (a, b)$$

(we have $c \in (a, b)$). This is the principal difference e.g. from Theorem 5.10 in [2] where a more general class of functions f is considered.

1.5. Lemma. *Let* $\alpha \in \boldsymbol{R} \setminus \{0\}$, $N \in \boldsymbol{N}$. *For* $x \in \boldsymbol{R}$ *let us denote*

$$I_{N,\alpha}(x) = \int_0^x e^{\alpha t} t^{N-1}\,\mathrm{d}t + (-1)^{N-1} \frac{(N-1)!}{\alpha^N}. \tag{1.6}$$

Then there exists $n_1 \in \boldsymbol{N}$ *such that for* $x > n_1$,

$$|I_{N,\alpha}(x)| \leqq \frac{N}{|\alpha|} e^{\alpha x} x^{N-1}. \tag{1.7}$$

2. COMPACT IMBEDDING OF WEIGHTED SOBOLEV SPACES

2.1. Using Cartesian coordinates

For $p \in \langle 1, \infty)$ we shall consider the weighted spaces $W^{1,p}(\Omega, S)$ and $L^p(\Omega, \varrho)$ (see [1]). The points $x = (x_1, \dots, x_N) \in \boldsymbol{R}^N$ will sometimes be written in the form

$x = (x', x_N)$ where $x' = (x_1, \ldots, x_{N-1}) \in \boldsymbol{R}^{N-1}$. In this section we suppose the following two conditions to be satisfied:

C1. Ω is an unbounded domain in $\boldsymbol{R}^N$, $\Omega \subset (-a, a)^{N-1} \times (-a, \infty)$, where $a > 0$.

C2. $W^{1,p}(\Omega_n, S) \hookrightarrow\hookrightarrow L^p(\Omega_n, \varrho)$ $\forall n \in \boldsymbol{N}$,
where $\Omega_n = \{x \in \Omega;\ x_N < n\}$ for $n \in \boldsymbol{N}$.

We shall investigate under what additional assumptions

$$W_0^{1,p}(\Omega, S) \hookrightarrow\hookrightarrow L^p(\Omega, \varrho) \tag{2.1.1}$$

holds.

Let us denote by the symbol X the set $C_0^\infty(\Omega)$ with the norm $\|\cdot\|_X = \|\cdot\|_{1,p,\Omega,S}$. It is possible to prove (see [1]) that (2.1.1) is true if

$$\sup_{\|u\|_X \leq 1} \|u\|_{p,\Omega^n,\varrho} \to 0 \text{ for } n \to \infty\,, \tag{2.1.2}$$

where $\Omega^n = \{x \in \Omega;\ x_N > n\}$, $n \in \boldsymbol{N}$.

If $Q \subset \boldsymbol{R}^N$ then $P_N(Q)$ stands for the orthogonal projection of the set Q into the hyperplane $x_N = 0$.

Let us further assume that the following condition is fulfilled:

C3. There exist numbers $C > 0$, $n_0 \in \boldsymbol{N}$, $d_n \in \boldsymbol{R}$ ($n \in \boldsymbol{N}$, $n > n_0$) and nonnegative measurable functions $\mu: (n_0, \infty) \to \boldsymbol{R}$, $\nu: (n_0, \infty) \to \boldsymbol{R}$, $\varkappa: (n_0, \infty) \to \boldsymbol{R}$, $\xi: P_N(\Omega^{n_0}) \to \boldsymbol{R}$ such that

$$\varrho(x) \leqq C\, \mu(x_N)\, \xi(x') \quad \text{for a.e.} \quad x \in \Omega^{n_0}\,; \tag{2.1.3}$$

$$\nu(x_N)\, \xi(x') \leqq C \min \{w_{(0,\ldots,0)}(x),\ w_{(0,\ldots,0,1)}(x)\} \quad \text{for a.e.} \quad x \in \Omega^{n_0}\,; \tag{2.1.4}$$

(2.1.5) the function $\varkappa$ is nonincreasing on (n_0, ∞) and $\lim_{n \to \infty} \varkappa(n) = 0$;

$$\left|\int_n^t \mu(s)\, \mathrm{d}s + d_n\right| \leqq C\, \mu^{1/p^*}(t)\, \varkappa^{1/p}(t)\, \nu^{1/p}(t) \tag{2.1.6}$$

for a.e. $t \in (n-1, n+1)$ and for all $n > n_0$.

Now we are going to investigate the validity of (2.1.2). Let $u \in X$ and $n > n_0$ where n_0 is the number from the condition **C3**. We extend the function u outside Ω by zero (then, clearly, $u \in C_0^\infty(\boldsymbol{R}^N)$) and put

$$\varrho(x) = 1 \quad \text{for} \quad x \in \boldsymbol{R}^N \setminus \Omega\,, \quad \xi(x') = 1 \quad \text{for} \quad x' \in \boldsymbol{R}^{N-1} \setminus P_N(\Omega^{n_0})\,.$$

Let ϕ_j ($j = n, n+1, n+2. \ldots$) be the functions from Lemma 1.1. Then we have

$$\begin{aligned} \|u(x)\|_{p,\Omega^n,\varrho}^p &= \int_{\Omega^n} |u(x)|^p\, \varrho(x)\, \mathrm{d}x = \\ &= \int_{\boldsymbol{R}^{N-1} \times (n,\infty)} |u(x)|^p\, \varrho(x)\, \mathrm{d}x = \\ &= \int_{\boldsymbol{R}^{N-1} \times (n,\infty)} |u(x)|^p \left[\sum_{j=n}^{\infty} \phi_j(x_N)\right] \varrho(x)\, \mathrm{d}x = \end{aligned} \tag{2.1.7}$$

$$= \sum_{j=n}^{\infty} \int_{R^{N-1} \times [(n,\infty) \cap \operatorname{supp} \phi_j]} |u(x)|^p \phi_j(x_N) \varrho(x) \, dx \leqq$$

$$\leqq \sum_{j=n}^{\infty} \int_{R^{N-1} \times (j-1,j+1)} |u(x) \phi_j^{1/p}(x_N)|^p \varrho(x) \, dx =$$

$$= \sum_{j=n}^{\infty} \|u_j(x)\|^p_{p,R^{N-1} \times (j-1,j+1),\varrho} ,$$

where

$$u_j(x) = u(x) \phi_j^{1/p}(x_N) , \quad x \in R^N .$$

Using Fubini's theorem and (2.1.3), we get

$$(2.1.8) \quad \|u_j(x)\|^p_{p,R^{n-1} \times (j-1,j+1),\varrho} \leqq C \int_{R^{N-1}} \left[\int_{j-1}^{j+1} |u_j(x', x_N)|^p \mu(x_N) \, dx_N\right] \xi(x') \, dx' .$$

For $x' \in R^{N-1}$, set

$$(2.1.9) \qquad f_j(s) = u_j(x', s) , \quad s \in R .$$

Evidently, $f_j \in C_0^1((j-1, j+1))$. Applying Lemma 1.3 we obtain from (2.1.6)

$$\int_{j-1}^{j+1} |f_j(s)|^p \mu(s) \, ds \leqq (Cp)^p \int_{j-1}^{j+1} |f_j'(s)|^p \varkappa(s) \nu(s) \, ds .$$

As the function $\varkappa$ is nonincreasing on (n_0, ∞) (see (2.1.5)) we get

$$(2.1.10) \qquad \int_{j-1}^{j+1} |f_j(s)|^p \mu(s) \, ds \leqq (Cp)^p \varkappa(j-1) \int_{j-1}^{j+1} |f_j'(s)|^p \nu(s) \, ds .$$

From (2.1.8), (2.1.9) and (2.1.10) we derive

$$(2.1.11) \qquad \|u_j(x)\|^p_{p,R^{N-1} \times (j-1,j+1),\varrho} \leqq C^{p+1} p^p \varkappa(j-1)$$
$$\int_{R^{N-1}} \left[\int_{j-1}^{j+1} \left|\frac{\partial}{\partial x_N} u_j(x', x_N)\right|^p \nu(x_N) \, dx_N\right] \xi(x') \, dx' .$$

Further,

$$\left|\frac{\partial}{\partial x_N} u_j(x', x_N)\right|^p \leqq 2^{p-1} \left[\left|\frac{\partial u}{\partial x_N}(x', x_N)\right|^p + K^p |u(x', x_N)|^p\right]$$

(K is the constant from Lemma 1.1) and therefore

$$(2.1.12) \qquad \|u_j(x)\|^p_{p,R^{N-1} \times (j-1,j+1),\varrho} \leqq C^{p+1} p^p \, 2^{p-1} \varkappa(j-1)$$
$$\int_{R^{N-1}} \left[\int_{j-1}^{j+1} \left|\frac{\partial u}{\partial x_N}(x', x_N)\right|^p \nu(x_N) \, dx_N + \right.$$
$$\left. + K^p \int_{j-1}^{j+1} |u(x', x_N)|^p \nu(x_N) \, dx_N\right] \xi(x') \, dx' .$$

In view of (2.1.7), (2.1.12) and (2.1.5) we have

$$\|u(x)\|^p_{p,\Omega^n,\varrho} \leqq C^{p+1}p^p 2^{p-1} \varkappa(n-1) \left\{ \int_{R^{N-1}} \left[\sum_{j=n}^{\infty} \int_{j-1}^{j+1} \left| \frac{\partial u}{\partial x_N}(x', x_N) \right|^p v(x_N)\, \mathrm{d}x_N + \right.\right.$$

$$\left.\left. + K^p \sum_{j=n}^{\infty} \int_{j-1}^{j+1} |u(x', x_N)|^p v(x_N)\, \mathrm{d}x_N \right] \xi(x')\, \mathrm{d}x' \right\} \leqq$$

$$\leqq C^{p+1}(2p)^p \varkappa(n-1) \left\{ \int_{R^{N-1}} \left[\int_{n-1}^{\infty} \left| \frac{\partial u}{\partial x_N}(x', x_N) \right|^p v(x_N)\, \mathrm{d}x_N + \right.\right.$$

$$\left.\left. + K^p \int_{n-1}^{\infty} |u(x', x_N)|^p v(x_N)\, \mathrm{d}x_N \right] \xi(x')\, \mathrm{d}x' \right. .$$

This and (2.1.4) yield

$$\|u(x)\|^p_{p,\Omega^n,\varrho} \leqq C^{p+2}(2p)^p \varkappa(n-1) \left[\int_{\Omega^{n-1}} \left| \frac{\partial u}{\partial x_N}(x) \right|^p w_{(0,\ldots,0,1)}(x)\, \mathrm{d}x + \right.$$

$$\left. + K^p \int_{\Omega^{n-1}} |u(x)|^p w_{(0,\ldots,0)}(x)\, \mathrm{d}x \right] \leqq$$

$$\leqq C^{p+2}(2pK)^p \varkappa(n-1)\, \|u\|^p_X \quad (\text{we have } K > 1)$$

and therefore

(2.1.13) $$\sup_{\|u\|_X \leqq 1} \|u\|_{p,\Omega^n,\varrho} \leqq C_1\, \varkappa^{1/p}(n-1)\,,$$

where

$$C_1 = 2pKC^{1+2/p}\,.$$

The convergence (2.1.2) follows from (2.1.13) and (2.1.5).

From the above considerations we have:

2.1.1. Theorem. *Let the conditions* **C1—C3** *be fulfilled. Then*

(2.1.14) $$W_0^{1,p}(\Omega, S) \hookrightarrow\hookrightarrow L^p(\Omega, \varrho)\,.$$

2.1.2. Example. Let Ω satisfy the condition **C1**, $p \in \langle 1, \infty)$, and

(2.1.15) $$\beta \in R\,, \quad \alpha \neq 0\,, \quad \alpha < \beta\,, \quad \varepsilon \geqq \beta\,,$$
$$\delta_\gamma \in R \quad \text{for} \quad |\gamma| = 1\,, \quad \gamma \neq (0, \ldots, 0, 1)\,.$$

For $x \in \Omega$ we define

$$\varrho(x) = e^{\alpha x_N}\,, \quad w_{(0,\ldots,0)}(x) = e^{\varepsilon x_N}\,, \quad w_{(0,\ldots,0,1)}(x) = e^{\beta x_N}\,,$$
$$w_\gamma(x) = e^{\delta_\gamma x_N} \quad \text{for} \quad |\gamma| = 1\,, \quad \gamma \neq (0, \ldots, 0, 1)\,.$$

Let $S = \{w_\gamma;\ |\gamma| \leqq 1\}$. As

$$W^{1,p}(\Omega_n, S) \rightleftarrows W^{1,p}(\Omega_n)\,, \quad L^p(\Omega_n, \varrho) \rightleftarrows L^p(\Omega_n)\,, \quad n \in N\,,$$

we obtain from the well-known (unweighted) imbedding theorem

$$W^{1,p}(\Omega_n, S) \subset\subset L^p(\Omega_n, \varrho), \quad n \in N, ^{*})$$

and hence the condition **C2** is satisfied.

If we further choose $C = 1$, $n_0 \in N$, $\mu(s) = e^{\alpha s}$, $\nu(s) = e^{\beta s}$, $\varkappa(s) = |\alpha|^{-p} e^{(\alpha-\beta)s}$ for $s \in (n_0, \infty)$, $\xi(x') \equiv 1$ for $x' \in P_N(\Omega^{n_0})$, $d_n = \alpha^{-1} e^{\alpha n}$ for $n > n_0$, we can easily verify that the condition **C3** is satisfied, too. From Theorem 2.1.1 we obtain (2.1.14).

2.1.3. Remark. (i) Let Ω satisfy the condition **C1**, $\alpha_1, \alpha_2 \in R$, $\alpha_1 \leqq \alpha_2$. For $x \in \Omega$ let us take

$$\varrho_i(x) = e^{\alpha_i x_N}, \quad i = 1, 2.$$

Then

$$L^p(\Omega, \varrho_2) \subset L^p(\Omega, \varrho_1). \tag{2.1.16}$$

The proof is easy: For $x \in \Omega$ we have $-a < x_N$, i.e. $(\alpha_2 - \alpha_1)(x_N + a) \geqq 0$ and therefore

$$\varrho_2(x)\, e^{(\alpha_2-\alpha_1)a} \geqq \varrho_1(x), \quad x \in \Omega.$$

From this inequality (2.1.16) immediately follows.

(ii) Part (i) of this remark and Example 2.1.2 imply that *the condition* $\alpha \neq 0$ in (2.1.15) *can be omitted.* [Actually, if $\alpha = 0 < \beta$ then for any $\alpha_1 \in (0, \beta)$ we have in accordance with Example 2.1.2

$$W_0^{1,p}(\Omega, S) \subset\subset L^p(\Omega, e^{\alpha_1 x_N}).$$

However, in view of (2.1.16), $L^p(\Omega, e^{\alpha_1 x_N}) \subset L^p(\Omega)$.]

For $x \in R^N$ and $\varepsilon \in R$ let us define

$$z_\varepsilon(x) = \begin{cases} x_N^\varepsilon, & x_N > 1, \\ 1, & x_N \leqq 1. \end{cases}$$

2.1.4. Example. Let Ω satisfy the condition **C1**, $p \in \langle 1, \infty)$, and let

$$\begin{aligned} &\beta \in R, \quad \alpha \neq -1, \quad \alpha < \beta - p, \quad \varepsilon \geqq \beta, \\ &\delta_\gamma \in R \quad \text{for} \quad |\gamma| = 1, \quad \gamma \neq (0, \dots, 0, 1). \end{aligned} \tag{2.1.17}$$

For $x \in \Omega$ we put

$$\begin{aligned} \varrho(x) = z_\alpha(x), \quad w_{(0,\dots,0)}(x) = z_\varepsilon(x), \quad w_{(0,\dots,0,1)}(x) = z_\beta(x), \\ w_\gamma(x) = z_{\delta_\gamma}(x) \quad \text{for} \quad |\gamma| = 1, \quad \gamma \neq (0, \dots, 0, 1). \end{aligned}$$

Let $S = \{w_\gamma; |\gamma| \leqq 1\}$. Analogously as in Example 2.1.2 we can verify that the condition **C2** is satisfied.

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*) As we work with the "nulled space" $W_0^{k,p}(\Omega, S)$, one can assume without loss of generality that $\Omega_n \in C^{0,1}$ for each $n \in N$.

If we further choose $C = 1$, $n_0 \in N$, $\mu(s) = s^\alpha$, $\nu(s) = s^\beta$, $\varkappa(s) = |\alpha + 1|^{-p} s^{\alpha-\beta+p}$ for $s \in (n_0, \infty)$, $\xi(x') \equiv 1$ for $x \in P_N(\Omega^{n_0})$, $d_n = (\alpha + 1)^{-1} n^{\alpha+1}$ for $n > n_0$, we can see that the condition **C3** is satisfied as well. The imbedding (2.1.14) follows from Theorem 2.1.1.

2.1.5. Remark. (i) Let Ω be a domain in R^N, $\alpha_1, \alpha_2 \in R$, $\alpha_1 \leqq \alpha_2$. For $x \in \Omega$ let us take

$$\varrho_i(x) = z_{\alpha_i}(x)\,, \quad i = 1, 2\,.$$

Then

(2.1.18) $$L^p(\Omega, \varrho_2) \hookrightarrow L^p(\Omega, \varrho_1)\,.$$

[The proof follows at once from the inequality $\varrho_2(x) \geqq \varrho_1(x)$ for $x \in \Omega$.]

(ii) Part (i) of this remark and Example 2.1.4 imply that in (2.1.17) *the condition* $\alpha \neq -1$ *can be omitted*. [Namely, if $\alpha = -1 < \beta - p$ then there exists α_1 such that $-1 < \alpha_1 < \beta - p$ and from Example 2.1.4 we obtain

$$W_0^{1,p}(\Omega, S) \hookrightarrow\hookrightarrow L^p(\Omega, z_{\alpha_1})\,.$$

However, in virtue of (2.1.18), $L^p(\Omega, z_{\alpha_1}) \hookrightarrow L^p(\Omega, z_{-1})$.]

2.1.6. Remark. (i) The case when Ω is unbounded in both directions of the axis x_N and

$$\Omega \subset (-a, a)^{N-1} \times R \quad (0 < a < \infty)$$

can be investigated analogously as in Theorem 2.1.1 (see Remark 3.1.8 in [1]).

(ii) Let us note (again as in [1]) that some curvilinear coordinate can play the role of the variable x_N. In the next section we will consider the case of spherical coordinates.

(iii) Let us add the following assumption to the condition **C1**:

There exist numbers $a_i, b_i \in R$ $(i = 1, 2, \ldots, N - 1)$ and $n_0 \in N$ such that

$$\Omega^{n_0} = \{x \in R^N;\ a_i < x_i < b_i,\ i = 1, 2, \ldots, N - 1,\ x_N > n_0\}\,.^{*)}$$

Then one can show that the method proposed (with minor modifications) can be used to prove the compact imbedding

$$W^{1,p}(\Omega, S) \hookrightarrow\hookrightarrow L^p(\Omega, \varrho)\,.$$

2.2. Using spherical coordinates

We shall consider *spherical coordinates* (r, Θ) in R^N, where $r = |x|$ is the distance from the point x to the origin and $\Theta = x/|x|$ is a point on the unit sphere $E = \{x \in R^N;\ |x| = 1\}$. If $Q \subset R^N$ then $P_E(Q)$ will denote the projection of the set Q into the unit sphere E, i.e., $P_E(Q) = \{\Theta \in E;\ \exists r > 0,\ (r, \Theta) \in Q\}$.

*) I.e., the domain Ω has a special form.

Let $W^{1,p}(\Omega, S)$, $L^p(\Omega, \varrho)$ and X be as in Section 2.1. Throughout this section we consider the following two conditions:

C1*. Ω is an unbounded domain in $\boldsymbol{R}^N$.

C2*. $W^{1,p}(\Omega_n, S) \hookrightarrow\hookrightarrow L^p(\Omega_n, \varrho)$ $\forall n \in \boldsymbol{N}$, where $\Omega_n = \{x \in \Omega;\ |x| < n\}$ for $n \in \boldsymbol{N}$.

Again, we shall look for additional assumptions implying

$$W_0^{1,p}(\Omega, S) \hookrightarrow\hookrightarrow L^p(\Omega, \varrho). \tag{2.2.1}$$

It is possible to prove (see [1]) that (2.2.1) is true if

$$\sup_{\|u\|_X \leq 1} \|u\|_{p,\Omega^n,\varrho} \to 0 \quad \text{for} \quad n \to \infty, \tag{2.2.2}$$

where $\Omega^n = \{x \in \Omega;\ |x| > n\}$, $n \in \boldsymbol{N}$.

Let us further suppose that the following condition is fulfilled:

C3*. There exist numbers $C > 0$, $n_0 \in \boldsymbol{N}$, $d_n \in \boldsymbol{R}$ $(n \in \boldsymbol{N},\ n > n_0)$ and nonnegative measurable functions $\mu\colon (n_0, \infty) \to \boldsymbol{R}$, $\nu\colon (n_0, \infty) \to \boldsymbol{R}$, $\varkappa\colon (n_0, \infty) \to \boldsymbol{R}$, $\xi\colon P_E(\Omega^{n_0}) \to \boldsymbol{R}$ such that

$$\varrho(x) \leq C\, \mu(|x|)\, \xi\left(\frac{x}{|x|}\right) \quad \text{for a.e.} \quad x \in \Omega^{n_0}; \tag{2.2.3}$$

$$\nu(|x|)\, \xi\left(\frac{x}{|x|}\right) \leq C \min_{|\alpha| \leq 1} w_\alpha(x) \quad \text{for a.e.} \quad x \in \Omega^{n_0}; \tag{2.2.4}$$

$$\text{the function } \varkappa \text{ is nonincreasing on } (n_0, \infty) \text{ and } \lim_{n\to\infty} \varkappa(n) = 0; \tag{2.2.5}$$

$$\left|\int_n^t \mu(s)\, s^{N-1}\, ds + d_n\right| \leq C\, \mu^{1/p^*}(t)\, \varkappa^{1/p}(t)\, \nu^{1/p}(t)\, t^{N-1} \tag{2.2.6}$$

for a.e. $t \in (n-1, n+1)$ and for all $n > n_0$.*)

Now we shall investigate the validity of (2.2.2). Let $u \in X$ and $n > n_0$ where n_0 is the number from the condition **C3***. We extend the function u outside Ω by zero (then, clearly, $u \in C_0^\infty(\boldsymbol{R}^N)$) and take

$$\varrho(x) = 1 \quad \text{for} \quad x \in \boldsymbol{R}^N \setminus \Omega, \quad \xi(\Theta) = 1 \quad \text{for} \quad \Theta \in E \setminus P_E(\Omega^{n_0}).$$

Let ψ_j $(j = n, n+1, \ldots)$ be the functions from Lemma 1.2. Then we have

$$\|u(x)\|_{p,\Omega^n,\varrho}^p = \int_{\Omega^n} |u(x)|^p\, \varrho(x)\, dx = \tag{2.2.7}$$

$$= \int_{P(n,\infty)} |u(x)|^p\, \varrho(x)\, dx = \int_{P(n,\infty)} |u(x)|^p \left[\sum_{j=n}^{\infty} \psi_j(x)\right] \varrho(x)\, dx =$$

*) Let us remark that the inequality (2.2.6) can be written in the form $|\int_n^t \mu(s)\, s^{n-1}\, ds + d_n| \leq C[\mu(t)\, t^{N-1}]^{1/p^*}\, [\varkappa(t)\, \nu(t)\, t^{N-1}]^{1/p}$.

$$= \sum_{j=n}^{\infty} \int_{P(n,\infty)\cap \operatorname{supp}\psi_j} |u(x)|^p \psi_j(x)\, \varrho(x)\, dx \leqq$$

$$\leqq \sum_{j=n}^{\infty} \int_{P(j-1,j+1)} |u(x)\, \psi_j^{1/p}(x)|^p \varrho(x)\, dx =$$

$$= \sum_{j=n}^{\infty} \|u_j(x)\|^p_{p,P(j-1,j+1),\varrho}$$

where

$$u_j(x) = u(x)\, \psi_j^{1/p}(x)\,, \quad x \in \mathbf{R}^N\,.$$

In view of (2.2.3) we get

$$\|u_j(x)\|^p_{p,P(j-1,j+1),\varrho} \leqq C \int_E \left[\int_{j-1}^{j+1} |u_j(r,\Theta)|^p \mu(r)\, r^{N-1}\, dr\right] \xi(\Theta)\, d\Theta\,. \tag{2.2.8}$$

For a fixed $\Theta \in E$ we denote

$$f_j(r) = u_j(r,\Theta)\,, \quad r > 0\,. \tag{2.2.9}$$

Clearly, $f_j \in C_0^1((j-1, j+1))$. Applying Lemma 1.3, in view of (2.2.6) we obtain

$$\int_{j-1}^{j+1} |f_j(r)|^p \mu(r)\, r^{N-1}\, dr \leqq (Cp)^p \int_{j-1}^{j+1} |f_j'(r)|^p \varkappa(r)\, \nu(r)\, r^{N-1}\, dr\,.$$

As the function $\varkappa$ is nonincreasing on (n_0, ∞) (see (2.2.5)) we get

$$\int_{j-1}^{j+1} |f_j(r)|^p \mu(r)\, r^{N-1}\, dr \leqq (Cp)^p \varkappa(j-1) \int_{j-1}^{j+1} |f_r'(r)|^p \nu(r)\, r^{N-1}\, dr\,. \tag{2.2.10}$$

From (2.2.8), (2.2.9) and (2.2.10) the inequality

$$\|u_j(x)\|^p_{p,P(j-1,j+1),\varrho} \leqq \tag{2.2.11}$$

$$\leqq C^{p+1} p^p \varkappa(j-1) \int_E \left[\int_{j-1}^{j+1} \left|\frac{\partial}{\partial r} u_j(r,\Theta)\right|^p \nu(r)\, r^{N-1}\, dr\right] \xi(\Theta)\, d\Theta$$

follows. Further,

$$\left|\frac{\partial}{\partial r} u_j(r,\Theta)\right|^p \leqq 2^{p-1} \left[\left|\frac{\partial}{\partial r} u(r,\Theta)\right|^p + K^p |u(r,\Theta)|^p\right] \leqq C_1 \sum_{|\alpha|\leqq 1} |D^\alpha u(r,\Theta)|^p$$

(where C_1 is a constant independent of the function u and the number n) and therefore

$$\|u_j(x)\|^p_{p,P(j-1,j+1),\varrho} \leqq \tag{2.2.12}$$

$$\leqq C^{p+1} p^p C_1 \varkappa(j-1) \sum_{\alpha \leqq 1} \int_E \left[\int_{j-1}^{j+1} |D^\alpha u(r,\Theta)|^p \nu(r)\, r^{N-1}\, dr\right] \xi(\Theta)\, d\Theta\,.$$

In virtue of (2.2.7), (2.2.12) and (2.2.5) we have

$$\|u(x)\|^p_{p,\Omega^n,\varrho} \leqq$$

$$\leqq C^{p+1} C_1 p^p \varkappa(n-1) \sum_{\alpha\leqq 1} \int_E \left[\sum_{j=n}^{\infty} \int_{j-1}^{j+1} |D^\alpha u(r,\Theta)|^p \nu(r)\, r^{N-1}\, dr\right] \xi(\Theta)\, d\Theta \leqq$$

$$\leqq C^{p+1} C_1 p^p\, 2\varkappa(n-1) \sum_{\alpha\leqq 1} \int_E \left[\int_{n-1}^{\infty} |D^\alpha u(r,\Theta)|^p \nu(r)\, r^{N-1}\, dr\right] \xi(\Theta)\, d\Theta\,.$$

Using the last chain of inequalities and (2.2.4) we arrive at

$$\|u(x)\|^p_{p,\Omega^n,\varrho} \leqq C^{p+2}C_1\, 2p^p\, \varkappa(n-1) \sum_{|\alpha|\leqq 1} \int_{\Omega^{n-1}} |D^\alpha u(x)|^p\, w_\alpha(x)\, \mathrm{d}x \leqq$$
$$\leqq C^{p+2}C_1\, 2p^p\, \varkappa(n-1)\, \|u\|_X$$

and therefore

(2.2.13) $$\sup_{\|u\|_X \leqq 1} \|u\|_{p,\Omega^n,\varrho} \leqq C_2 \varkappa^{1/p}(n-1)\,,$$

where

$$C_2 = pC[2C^2C_1]^{1/p}\,.$$

Now (2.2.13) and (2.2.5) imply (2.2.2).

We have proved

2.2.1. Theorem. *Let us suppose that the conditions* **C1*—C3*** *are fulfilled. Then*

(2.2.14) $$W_0^{1,p}(\Omega, S) \hookrightarrow\hookrightarrow L^p(\Omega, \varrho)\,.$$

For $x \in \boldsymbol{R}^N$ and $\varepsilon \in \boldsymbol{R}$ we define

$$\omega_\varepsilon(x) = \begin{cases} |x|^\varepsilon\,, & |x| > 1\,, \\ 1\,, & |x| \leqq 1\,. \end{cases}$$

2.2.2. Example. Let Ω be an unbounded domain in $\boldsymbol{R}^N$, $p \in \langle 1, \infty)$,

(2.2.15) $$\beta \in \boldsymbol{R}\,, \quad \alpha \neq -N\,, \quad \alpha < \beta - p\,, \quad \varepsilon \geqq \beta\,.$$

For $x \subset \Omega$ we set

$$\varrho(x) = \omega_\alpha(x)\,, \quad w_{(0,\dots,0)}(x) = \omega_\varepsilon(x)\,,$$
$$w_\gamma(x) = \omega_\beta(x) \quad \text{for} \quad |\gamma| = 1\,.$$

Let $S = \{w_\gamma;\, |\gamma| \leqq 1\}$. Since

$$W^{1,p}(\Omega_n, S) \rightleftarrows W^{1,p}(\Omega_n)\,, \quad L^p(\Omega_n, \varrho) \rightleftarrows L^p(\Omega_n)\,, \quad n \in \boldsymbol{N}\,,$$

we obtain from the imbedding theorem for the classical Sobolev spaces

$$W^{1,p}(\Omega_n, S) \hookrightarrow\hookrightarrow L^p(\Omega_n, \varrho)\,, \quad n \in \boldsymbol{N}\,,$$

hence the condition **C2*** is satisfied.

If we further choose $C = 1$, $n_0 \in \boldsymbol{N}$, $\mu(s) = s^\alpha$, $\nu(s) = s^\beta$, $\varkappa(s) = |\alpha + N|^{-p} s^{\alpha-\beta+p}$ for $s \in (n_0, \infty)$, $\xi(\Theta) = 1$ for $\Theta \in P_E(\Omega^{n_0})$, $d_n = (\alpha + N)^{-1}\, n^{\alpha+N}$ for $n > n_0$, we can easily verify the validity of the condition **C3***. The desired imbedding (2.2.14) now follows from Theorem 2.1.1.

2.2.3. Remark. *The condition* $\alpha \neq -N$ *can be omitted in* (2.2.15). This follows at once from the last example and from the imbedding

$$L^p(\Omega, \omega_{\alpha_2}) \hookrightarrow L^p(\Omega, \omega_{\alpha_1})$$

where $\Omega \subset R^N$ and $\alpha_2 \geqq \alpha_1$.

2.2.4. Example. Let Ω be an unbounded domain in R^N, $p \in \langle 1, \infty)$,

$$(2.2.16) \qquad \beta \in R\,, \quad \alpha \neq 0\,, \quad \alpha < \beta\,, \quad \varepsilon \geqq \beta\,.$$

For $x \in \Omega$ we put

$$\varrho(x) = e^{\alpha|x|}\,, \quad w_{(0,\ldots,0)}(x) = e^{\varepsilon|x|}\,,$$
$$w_\gamma(x) = e^{\beta|x|} \quad \text{for} \quad |\gamma| = 1\,.$$

Let $S = \{w_\gamma;\ |\gamma| \leqq 1\}$. We can verify analogously as in Example 2.2.2 that the condition **C2*** is satisfied.

If we choose $C = 1$, $\mu(s) = e^{\alpha s}$, $\nu(s) = e^{\beta s}$, $\varkappa(s) = (N/|\alpha|)^p\, e^{(\alpha-\beta)s}$ for $s \in (n_0, \infty)$ where $n_0 > n_1$ (the number n_1 is from Lemma 1.5), $\xi(\Theta) = 1$ for $\Theta \in P_E(\Omega^{n_0})$, $d_n = I_{N,\alpha}(n)$ for $n > n_0$ (we use the notation from Lemma 1.5), we can see that the conditions (2.2.3)–(2.2.5) are satisfied. Let us verify (2.2.6). For $n > n_0$ and $t \in (n-1, n+1)$ we obtain by means of Lemma 1.5

$$\left|\int_n^t \mu(s)\, s^{N-1}\, ds + d_n\right| = \left|\int_n^t e^{\alpha s} s^{N-1}\, ds + d_n\right| =$$
$$= \left|I_{N,\alpha}(t) - I_{N,\alpha}(n) + d_n\right| = \left|I_{n,\alpha}(t)\right| \leqq \frac{N}{|\alpha|}\, e^{\alpha t} t^{N-1}\,.$$

Further, we easily get

$$\frac{N}{|\alpha|}\, e^{\alpha t} t^{N-1} = C \mu^{1/p^*}(t)\, \varkappa^{1/p}(t)\, \nu^{1/p}(t)\, t^{N-1}$$

for $n > n_0$ and $t \in (n-1, n+1)$, hence (2.2.6) is true. From Theorem 2.2.1 we obtain (2.2.14).

2.2.5. Remark. As

$$\varrho_2(x) \geqq \varrho_1(x)$$

holds for $\alpha_2 \geqq \alpha_1$ and for $x \in R^N$, where $\varrho_i(x) = \exp(\alpha_i|x|)$ $(i = 1, 2)$, we have

$$L^p(\Omega, \varrho_2) \hookrightarrow L^p(\Omega, \varrho_1)$$

for each domain $\Omega \subset R^N$. This and the last example yield that *the condition* $\alpha \neq 0$ *can be left out in* (2.2.16).

2.2.6. Remark. Let us suppose, in addition to **C1*—C3***, that the following conditions hold:

a) There exist a number $n_1 \in N$ and a measurable set $E' \subset E$ such that

$$\Omega^{n_1} = \left\{x \in R^N;\ |x| > n_1,\ \frac{x}{|x|} \in E'\right\}.$$

b) $$W^{1,p}(\Omega, S) = \overline{C^1(\Omega) \cap W^{1,p}(\Omega, S)}^{\|\cdot\|_{1,p,\Omega,S}}.$$

Then the given method (with a small modification) can be used for the proof of the compact imbedding

$$W^{1,p}(\Omega, S) \hookrightarrow\hookrightarrow L^p(\Omega, \varrho).$$

3. APPENDIX (Proofs of lemmas from Section 1)

Proof of Lemma 1.1. Let $\tau\colon \boldsymbol{R} \to \boldsymbol{R}$ be such a function that

$$(3.1) \qquad \tau \in C^1(\boldsymbol{R}), \quad \operatorname{supp} \tau \subset (-1, 1),$$
$$0 \leqq \tau(x) \leqq 1 \quad \text{for} \quad x \in \boldsymbol{R}, \quad \tau(x) = 1 \quad \text{for} \quad |x| \leqq \tfrac{3}{4}.$$

Define the function f by

$$(3.2) \qquad f(x) = \sum_{j=n}^{\infty} \tau^p(x - j), \quad x \in \boldsymbol{R}.$$

For each $x \in \boldsymbol{R}$ at most two summands on the right hand side of (3.2) are nonzero. Therefore, in virtue of (3.1), we can see that $f \in C^1(\boldsymbol{R})$, $0 \leqq f(x) \leqq 2$ for $x \in \boldsymbol{R}$, $f(x) \geqq 1$ for $x \geqq n - \frac{3}{4}$, there exists $\varepsilon, 0 < \varepsilon < \frac{1}{4}$ such that $f(x) = 1$ for $x \in \langle n - \frac{3}{4}, n + \varepsilon)$, and the function g,

$$g(x) = \begin{cases} f(x), & x > n, \\ 1, & x \leqq n, \end{cases}$$

possesses the following properties:

$$(3.3.) \qquad g \in C^1(\boldsymbol{R}), \quad 1 \leqq g(x) \leqq 2, \quad x \in \boldsymbol{R},$$
$$|g'(x)| \leqq 2p \max_{y \in \boldsymbol{R}} |\tau'(y)|, \quad x \in \boldsymbol{R}.$$

Putting $\phi_j(x) = \tau^p(x - j)/g(x)$ for $x \in \boldsymbol{R}$, $j \in \boldsymbol{N}$, $j \geqq n$, we easily verify that $0 \leqq \leqq \phi_j(x) \leqq 1$ for $x \in \boldsymbol{R}$, $\phi_j^{1/p} \in C^1(\boldsymbol{R})$ and $\operatorname{supp} \phi_j \subset (j - 1, j + 1)$. Further, in view of (3.1) and (3.3), for $x \in \boldsymbol{R}$ we obtain

$$\left|\frac{\mathrm{d}}{\mathrm{d}x}[\phi_j^{1/p}(x)]\right| = \left|\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\tau(x - j)}{g^{1/p}(x)}\right]\right| =$$
$$= \left|\frac{\tau'(x - j)\, g^{1/p}(x) - \tau(x - j)\, \frac{1}{p}\, g^{1/p - 1}(x)\, g'(x)}{g^{2/p}(x)}\right| \leqq$$
$$\leqq \frac{|\tau'(x - j)|}{g^{1/p}(x)} + \frac{1}{p}\,\frac{\tau(x - j)\,|g'(x)|}{g^{1 + 1/p}(x)} \leqq$$
$$\leqq |\tau'(x - j)| + \frac{1}{p}\,|g'(x)|,$$

hence

$$\left|\frac{\mathrm{d}}{\mathrm{d}x}\,[\phi_j^{1/p}(x)]\right| \leqq K\,, \quad x \in \boldsymbol{R}$$

where

$$K = 3 \max_{y \in \boldsymbol{R}} |\tau'(y)|\,.$$

Now, let $x \in (n, \infty)$. Then

$$\sum_{j=n}^{\infty} \phi_j(x) = \sum_{j=n}^{\infty} \frac{\tau^p(x-j)}{g(x)} = \sum_{j=n}^{\infty} \frac{\tau^p(x-j)}{f(x)} = 1$$

which completes the proof of Lemma 1.1.

Proof of Lemma 1.2. Lemma 1.2 immediately follows from Lemma 1.1 if we set

$$\psi_j(x) = \phi_j(|x|)\,, \quad x \in \boldsymbol{R}^N$$

where the functions ϕ_j are from Lemma 1.1.

Proof of Lemma 1.3. Let

$$f \in C_0^\infty((a, b))\,, \quad \operatorname{supp} f \subset (\alpha, \beta) \tag{3.4}$$

where $a < \alpha < \beta < b$. For $x \in (a, b)$ we denote

$$g(x) = \int_c^x a_0(s)\,\mathrm{d}s + d\,.$$

Integrating by parts, we obtain

$$\int_\alpha^\beta |f(x)|^p\, a_0(x)\,\mathrm{d}x = [|f(x)|^p\, g(x)]_\alpha^\beta - p \int_\alpha^\beta |f(x)|^{p-1}\,(|f(x)|)'\, g(x)\,\mathrm{d}x\,.$$

From this, in view of (3.4), it follows that

$$\int_a^b |f(x)|^p\, a_0(x)\,\mathrm{d}x = \int_\alpha^\beta |f(x)|^p\, a_0(x)\,\mathrm{d}x =$$
$$= -p \int_\alpha^\beta |f(x)|^{p-1}\,(|f(x)|)'\, g(x)\,\mathrm{d}x \leqq p \int_\alpha^\beta |f(x)|^{p-1}\,|f'(x)|\,|g(x)|\,\mathrm{d}x \leqq$$
$$\leqq p \int_a^b |f(x)|^{p-1}\,|f'(x)|\,|g(x)|\,\mathrm{d}x\,.$$

Then using the condition (1.4) we get

$$\int_a^b |f(x)|^p\, a_0(x)\,\mathrm{d}x \leqq Cp \int_a^b |f(x)|^{p-1}\,|f'(x)|\, a_0^{1/p^*}(x)\, a_1^{1/p}(x)\,\mathrm{d}x\,. \tag{3.5}$$

If $p = 1$ then (3.5) is the desired inequality (1.5).

Let further $p \in (1, \infty)$. Then from (3.5) by Hölder's inequality we obtain

$$\int_a^b |f(x)|^p\, a_0(x)\,\mathrm{d}x \leqq Cp[\int_\alpha^\beta |f(x)|^p\, a_0(x)\,\mathrm{d}x]^{1/p^*}\,[\int_a^b |f'(x)|^p\, a_1(x)\,\mathrm{d}x]^{1/p}$$

and this implies the inequality (1.5).

Proof of Lemma 1.5. First we shall prove that under the assumptions of Lemma 1.5

(3.6) $$I_{N,\alpha}(x) = \frac{e^{\alpha x}}{\alpha} P_{N-1,\alpha}(x)$$

where

(3.7) $$P_{N-1,\alpha}(x) = \sum_{i=0}^{N-1} (-1)^i \frac{\binom{N-1}{i} i!}{\alpha^i} x^{N-1-i}$$

(thus $P_{N-1,\alpha}(x)$ is a polynomial of the degree $N - 1$ and the coefficient at the power x^{N-1} is 1).

To prove (3.6) we use mathematical induction. We readily get

(3.8) $$I_{1,\alpha}(x) = \int_0^x e^{\alpha t}\,dt + \frac{1}{\alpha} = \frac{e^{\alpha x}}{\alpha} x^0 = \frac{e^{\alpha x}}{\alpha} P_{0,\alpha}(x),$$

so that (3.6) is true for $N = 1$.

If (3.6) holds for some $N \in \mathbf{N}$ then using the integration by parts and the induction hypothesis we obtain

(3.9) $$I_{N+1,\alpha}(x) = \frac{e^{\alpha x}}{\alpha} x^N - \frac{N}{\alpha} I_{N,\alpha}(x) =$$

$$= \frac{e^{\alpha x}}{\alpha} x^N - \frac{N}{\alpha} \frac{e^{\alpha x}}{\alpha} P_{N-1,\alpha}(x) =$$

$$= \frac{e^{\alpha x}}{\alpha} \left[x^N - \frac{N}{\alpha} \sum_{i=0}^{N-1} (-1)^i \frac{\binom{N-1}{i} i!}{\alpha^i} x^{N-1-i} \right] =$$

$$= \frac{e^{\alpha x}}{\alpha} \left[x^N + \sum_{i=0}^{N-1} (-1)^{i+1} \frac{N \binom{N-1}{i} i!}{\alpha^{i+1}} x^{N-(i+1)} \right] =$$

$$= \frac{e^{\alpha x}}{\alpha} \left[x^N + \sum_{i=0}^{N-1} (-1)^{i+1} \frac{\binom{N}{i+1} (i+1)!}{\alpha^{i+1}} x^{N-(i+1)} \right] =$$

$$= \frac{e^{\alpha x}}{\alpha} \sum_{j=0}^{N} (-1)^j \frac{\binom{N}{j} j!}{\alpha^j} x^{N-j} = \frac{e^{\alpha x}}{\alpha} P_{N,\alpha}(x),$$

hence (3.6) is verified.

Now we are able to prove Lemma 1.5. From (3.8) we get

$$|I_{1,\alpha}(x)| = |\alpha|^{-1} e^{\alpha x}, \quad x \in \mathbf{R}$$

at once and therefore (1.7) holds for $N = 1$ (it is sufficient to choose $n_1 = 1$).

It remains to prove (1.7) for $N = 2, 3, \ldots$. Writing the number N in the form $N = n + 1$ then, in view of (3.9), we have for $x > 0$

$$|I_{N,\alpha}(x)| = |I_{n+1,\alpha}(x)| \leqq \frac{e^{\alpha x}}{|\alpha|} \sum_{j=0}^{n} \frac{\binom{n}{j} j!}{|\alpha|^j} x^{n-j} = \tag{3.10}$$

$$= \frac{e^{\alpha x}}{|\alpha|} x^n \left[1 + \sum_{j=1}^{n} \frac{\binom{n}{j} j!}{|\alpha|^j} \frac{1}{x^j}\right].$$

Now, if $x > n_1$ where

$$n_1 \geqq |\alpha|^{-1} \max_{j=1,\ldots,n} \sqrt[j]{\binom{n}{j} j!}\,,$$

then (3.10) yields

$$|I_{N,\alpha}(x)| \leqq \frac{e^{\alpha x}}{|\alpha|} x^n(n+1) = \frac{N}{|\alpha|} e^{\alpha x} x^{N-1}\,,$$

which is the desired inequality (1.7).

References

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Souhrn

KOMPAKTNOST VNOŘENÍ VÁHOVÉHO SOBOLEVOVA PROSTORU DEFINOVANÉHO NA NEOMEZENÉ OBLASTI II

BOHUMÍR OPIC

Článek je přímým pokračováním práce [1], kde byly zavedeny základní pojmy a označení. Je zkoumána kompaktnost vnoření váhového Sobolevova prostoru $W_0^{k,p}(\Omega, S)$ (S je systém váhových funkcí) definovaného na neomezené oblasti do prostoru $L^p(\Omega, \varrho)$ (ϱ je váhová funkce). Dané vnoření je vyšetřováno jako limitní případ kompaktních vnoření Sobolevových prostorů definovaných na omezených oblastech.

Резюме

КОМПАКТНОЕ ВЛОЖЕНИЕ ВЕСОВОГО ПРОСТРАНСТВА СОБОЛЕВА, ОПРЕДЕЛЕННОГО В НЕОГРАНИЧЕННОЙ ОБЛАСТИ II

BOHUMÍR OPIC

Эта статья является прямым продолжением работы [1], где были приведены основные понятия и обозначения. В работе исследуется компактность вложения весового пространства Соболева $W_0^{k,p}(\Omega, S)$ (S — система весовых функций), определенного в неограниченной области, в пространство функций $L^p(\Omega, \varrho)$ (ϱ — весовая функция). Это вложение рассматривается как предельный случай компактных вложений пространств Соболева, определенных в ограниченных областях.

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