ON CONFIGURATION OF ARGUESIAN SPACES

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Abstract. There are $2^{n-1}\binom{m+n+1}{n}$ arguesian points, $2^n\binom{m+n+1}{m}$ arguesian lines, ..., and 2^{m+n} arguesian $[m+1]$s associated with a pair of $(m+n)$-simplexes $(m < n-1)$[1]), each lying in a different $[m+n]$ but both lying in no space of dimension lower than $2n-2$, projective from an $[n-2]$ as may be space gathered from 3 previous papers ([3]; [6]; [7]) on the subject. The purpose of this Note is to show that this configuration of arguesian spaces can be obtained from any one $[m+1]$ by harmonic inversions [4] w.r.t. all the pairs of opposite elements of an $(m+n)$-simplex.

1. A pair of projective *n*-simplexes. If (P), (P') be a pair of n-dimensional simplexes, or briefly n-simplexes, each lying in a different n-space $[n]$ but both lying in no space of dimension lower than $2n-2$, *projective* from an $[n-2]$ t such that t is the common transversal of the $n+1$ joins of their corresponding vertices, the $n+1$ *arguesian points* common to their corresponding primes are collinear in their *arguesian line* as established in [7]. It is also shown there that there arise in all 2^n pairs of such n-simplexes, projective from the same t, thus giving us 2^n arguesian lines, one for each pair, and $2^{n-1}(n+1)$[1]) arguesian points, $n+1$ on each line and each common to 2 lines.

As observed in [6], the $2^{n-1}(n+1)$ arguesian points distribute into $n+1$ groups of 2^{n-1} each such that the points of a group form a *closed set* [5] as the vertices of the *dual of an $(n-1)$-dimensional S-configuration* [2] whose *diagonal $(n-1)$-simplex* forms a prime face of the n-simplex (P'') with vertices at the $n+1$ harmonic conjugates $P'' \equiv P - P'$ of the $n+1$ points $P + P'$ on the transversal t w.r.t. the $n+1$ pairs of corresponding vertices P_i, P'_i of (P), (P'). By definition of a closed set, called an *associated set* in [4], the figure of the 2^{n-1} arguesian points in each prime face of (P'') is *invariant* under the *group G_{n-1}* of the $2^{n-1}-1$ *harmonic inversions*

[1]) In [7] on page 318, there are a couple of errors in printing $>$ for $<$ in the 32nd line, and 2^n for 2^{n-1} in the first line. The same is corrected here.

w.r.t. all the $2^{n-1} - 1$ pairs of opposite elements of the $(n - 1)$-simplex of (P'') in the prime face considered and *identity*, and therefore remains invariant under the group G_n of the $2^n - 1$ harmonic inversions w.r.t. all the $2^n - 1$ pairs of opposite elements of (P'') and identity too. Hence, the whole figure of the $2^{n-1}(n + 1)$ arguesian points or *the configuration of the* 2^n *arguesian lines is invariant under* G_n. *That is, this configuration can be derived from any one arguesian line by the* $2^n - 1$ *harmonic inversions of* G_n as desired.

Thus for $n = 3$, we find the figure (cf. [1]) of the 8 arguesian lines associated with a pair of tetrahedra $(T = ABCD,\ T' = A'B'C'D')$, each lying in a separate solid, projective from a line t meeting the 4 joins $(AA', \ldots, DD')$ of their corresponding vertices in the 4 points $A + A', \ldots, D + D'$ [3], is invariant under the group G_3 of the 7 harmonic inversions (homologies) w.r.t. all the 7 pairs of opposite elements (3 pairs of opposite edges and 4 pairs of vertices and opposite faces) of the tetrahedron T'' and identity, where the vertices of T'' lie at the 4 harmonic conjugates $(A'' \equiv \equiv A - A', \ldots, D'' \equiv D - D')$ of the 4 points $A + A', \ldots, D + D'$ w.r.t. the 4 pairs of points $A, A'; \ldots; D, D'$ [6].

2. A pair of projective $(n + 1)$-simplexes. If $(Q), (Q')$ be a pair of $(n + 1)$-simplexes, each lying in a different $[n + 1]$ but both lying in no space of dimension lower than $2n - 2$, projective from an $[n - 2]$ t such that t meets all the $n + 2$ joins of their corresponding vertices Q_i, Q'_i in the $n + 2$ points $Q_i + Q'_i$, there arise $n + 2$ arguesian lines, one for each pair of their corresponding n-simplexes, which then lie in their *arguesian plane* as observed in [7]. Again it is pointed out there that there arise in all 2^{n+1} pairs of $(n + 1)$-simplexes, projective from the same t, thus giving us 2^{n+1} arguesian planes, one for each pair, and $2^n(n + 2)$ arguesian lines, $n + 2$ in each plane and each common to 2 planes.

Obviously enough, the arguesian lines distribute into $n + 2$ sets of 2^n each such that the lines of a set form a figure invariant under the group G_n (as established in the preceding section) of the $2^n - 1$ harmonic inversions w.r.t. all the $2^n - 1$ pairs of opposite elements of an n-simplex of the $(n + 1)$-simplex (Q'') and identity, where the vertices of (Q'') lie at the $n + 2$ harmonic conjugates $Q''_i \equiv Q_i - Q'_i$ of the $n + 2$ points $Q_i + Q'_i$ w.r.t. the $n + 2$ pairs of points Q_i, Q'_i. Consequently this fugure is invariant under the group G_{n+1} of the $2^{n+1} - 1$ harmonic inversions w.r.t. all the $2^{n+1} - 1$ pairs of opposite elements of (Q'') and identity. Hence the whole figure of $2^n(n + 2)$ arguesian lines or *the configuration of* 2^{n+1} *arguesian planes is invariant under* G_{n+1}. That is, *this configuration can be obtained from any one arguesian plane by the* $2^{n+1} - 1$ *harmonic inversions of* G_{n+1} as required.

For $n = 3$, we obtain a configuration [3] of 40 arguesian points, 40 arguesian lines and 16 arguesian planes.

3. A pair of projective $(m + n)$-simplexes. If $(R), (R')$ be a pair of $(m + n)$-simplexes $(1 < m < n - 1)$, each lying in a different $[m + n]$ but both lying in no space of dimension lower than $2n - 2$, projective from an $[n - 2]$ t such that t meets the $m + n + 1$ joins of their corresponding vertices R_i, R'_i in $m + n + 1$ points $R_i + R'_i$, there arise $\binom{m + n + 1}{n}$ arguesian points, $\binom{m + n + 1}{n + 1}$ arguesian lines, $\binom{m + n + 1}{n + 2}$ arguesian planes all lying in their *arguesian* $[m + 1]$ as observed in [7]. It is further observed there that there arise 2^{m+n} such pairs of $(m + n)$-simplexes in all, projective from the same t, thus giving us 2^{m+n} arguesian $[m + 1]$ s, one for each pair, $2^{n-1}\binom{m + n + 1}{n}$ arguesian points, $\binom{m + n + 1}{n}$ in each $[m + 1]$ and each common to $2^{m+1}[m + 1]$ s, $2^n\binom{m + n + 1}{m}$ arguesian lines, $\binom{m + n + 1}{m}$ in each $[m + 1]$ and each common to 2^m $[m + 1]$ s, and $2^{n+1}\binom{m + n + 1}{n + 2}$ planes, $\binom{m + n + 1}{n + 2}$ in each $[m + 1]$ and each common to 2^{m-1} $[m + 1]$ s.

Obviously, the arguesian lines distribute into $\binom{m + n + 1}{m}$ sets of 2^n each such that the lines of a set form a figure invariant under the group G_n (as argued in the preceding sections) of the $2^n - 1$ harmonic inversions w.r.t. all the $2^n - 1$ pairs of opposite elements of an n-simplex of the $(m + n)$-simplex (R'') and identity, where the vertices of (R'') are the $m + n + 1$ harmonic conjugates $R''_i \equiv R_i - R'_i$ of the $m + n + 1$ points $R_i + R'_i$ w.r.t. the $m + n + 1$ pairs of points R_i, R'_i. Consequently this figure is invariant under the group G_{m+n} of the $2^{m+n} - 1$ harmonic inversions w.r.t. all the $2^{m+n} - 1$ pairs of opposite elements of (R'') and identity. Hence, the whole figure of $2^n\binom{m + n + 1}{m}$ *arguesian lines or the configuration of* 2^{m+n} *arguesian* $[m + 1]$s *is invariant under* G_{m+n}. That is, *this configuration can be obtained from any one arguesian* $[m + 1]$ *by the* $2^{m+n} - 1$ *harmonic inversions of* G_{m+n} as desired.

The whole proof of the proposition is based on the fact that *a point in any face of a simplex is always transformed into a point in the same face by harmonic inversion w.r.t. any pair of opposite elements of the simplex.* It can be easily verified geometrically by simple observation or analytically by putting down the coordinates of the two points.

Again each arguesian $[m + 1]$ meets the $[n - 1]$ of every $(n - 1)$-simplex of (R'') in an arguesian point, and same will be the case for its harmonic inverse w.r.t. any pair of opposite elements of (R''). For an arguesian point in any $[n - 1]$ of (R'') goes into an arguesian point, by any such harmonic inversion, in the same $[n - 1]$.

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Výtah

O KONFIGURACI DESARGUESOVÝCH PROSTORŮ

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V článku se ukazuje, že konfigurace desarguesovských bodů, přímek atd. patřící ke dvojici projektivně sdružených simplexů lze získat harmonickou inversí z jednoho prvku a že tyto inverse tvoří grupu.

Резюме

О КОНФИГУРАЦИИ ПРОСТРАНСТВ ДЕЗАРГА

САГИБ РАМ МАНДАН (Sahib Ram Mandan), Харагпур (Индия)

В статье показано, что конфигурации дезарговых точек, прямых и т.д., принадлежащих к паре проективно сопряженных симплексов, можно получить путем гармонической инверсии из одного элемента, и что эти инверсии образуют группу.