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PSEUDO-UNITARY SPACES

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In the present paper linear spaces endowed with two or more topologies will be discussed. In order to specify which of the topologies is being considered, this will be indicated in brackets in front of the corresponding symbol. E.g. the relation $(T_0)\,\bar{H}_0 = (\sigma)\,\bar{H}_0 = H_0$ states that H_0 is closed in both the topologies T_0 and σ. In the text this will be expressed by saying that H_0 is T_0-closed, σ-closed, etc. The symbol $\mathscr{L}\{A\}$ denotes the linear hull of the subset A of a linear space H.

1. DEFINITION AND SOME PROPERTIES

1.1 Let H be a linear space (module) over the field of real numbers. H will be called a pseudo-unitary space if there is given a symmetric non-degenerate bilinear form on H. This means that to any elements $x \in H$, $y \in H$ there is assigned a real number $\langle x, y\rangle$ with the following properties

S1) $\langle x, y\rangle = \langle y, x\rangle$,

S2) $\langle \lambda x_1 + \mu x_2, y\rangle = \lambda\langle x_1, y\rangle + \mu\langle x_2, y\rangle$,

S3) $\langle x_0, y\rangle = 0$ for all $y \in H$ implies $x_0 = 0$.

The number $\langle x, y\rangle$ will be called the *pseudo-scalar product of* $x, y \in H$.

For arbitrary $x_0 \in H$ the expression $\langle x_0, y\rangle$ (or, more precisely, the mapping $y \to \langle x_0, y\rangle$) determines a linear form on H. Denote by H^* the dual of H, i.e. the space of (algebraic) linear forms on H. According to S3, the linear space H is isomorphic to some linear subspace of H^*. This isomorphism is given by the relation $x \leftrightarrow (y \to \langle x, y\rangle)$. Two elements $x \in H$, $y \in H$ are termed orthogonal if $\langle x, y\rangle = 0$. An element $x \in H$ is called *isotropic* if $x \neq 0$ and $\langle x, x\rangle = 0$. We shall also sometimes write $x \perp y$ instead of $\langle x, y\rangle = 0$. The following statement is evident: if $x \in H$ is orthogonal to all elements of $E \subset H$ (i.e. $x \perp E$) then $x \perp \mathscr{L}\{E\}$. Let $H_0 \subset H$ be a linear subspace of H. The orthogonal complement $\perp(H_0)$ of the subspace H_0 in H is the set of all elements in H which are orthogonal to H_0. Then $\perp(H_0)$ is clearly a linear subspace of H.

Definition. *A linear subspace $H_0 \subset H$ is called non-isotropic if $x_0 \in H_0$ and $\langle x_0, y \rangle = 0$ for all $y \in H_0$ implies $x_0 = 0$.*

1.2 In the following we give some theorems concerning the algebraical structure of pseudo-unitary spaces. The first one is evident.

Theorem 1. *A linear subspace $H_0 \subset H$ is itself a pseudo-unitary space with the pseudo-scalar product defined naturally if an only if H_0 is non-isotropic in H.*

Theorem 2. *A linear subspace $H_0 \subset H$ is non-isotropic if and only if $H_0 \cap \cap \perp(H_0) = \{0\}$.*

Proof. Let H_0 be non-isotropic, $x \in \perp(H_0) \cap H_0$. Then $x \in H_0$ and $x \perp H_0$, which implies $x = 0$. The converse is similar.

Lemma 1. *Let H_0 be a non-trivial non-isotropic linear subspace of H. Then there exists an element $x \in H_0$ such that $\langle x, x \rangle \neq 0$.*

Proof. Let $\langle x, x \rangle = 0$ for all $x \in H_0$. Then for a fixed $0 \neq x_0 \in H_0$ and arbitrary $y \in H_0$ we have $0 = \langle x_0 + y, x_0 + y \rangle = \langle x_0, x_0 \rangle + 2\langle x_0, y \rangle + \langle y, y \rangle$ and thus $\langle x_0, y \rangle = 0$, which is in contradiction with the assumptions.

Theorem 3. *Let H_0 be a non-isotropic linear subspace of H, and let $\langle x, x \rangle \geqq 0$ for all $x \in H_0$. Then $\langle x, x \rangle > 0$ for all $0 \neq x \in H_0$.*

Proof. Let an $x_0 \in H_0$ with $\langle x_0, x_0 \rangle = 0$ be fixed and choose $y \in H_0$ arbitrarily. We then have, for each real k, $\langle x_0 + ky, x_0 + ky \rangle \geqq 0$, i.e. $k^2 \langle y, y \rangle + 2k\langle x_0, y \rangle \geqq \geqq 0$. Hence the discriminant of this form in k, i.e. $D = (\langle x_0, y \rangle)^2$, satisfies $D \geqq 0$; as y was arbitrary we conclude hence that $x_0 = 0$.

Denote by I the set of isotropic elements of H and put $\mathcal{I} = \mathcal{L}\{I\}$. Then the following holds

Theorem 4. *In a pseudo-unitary space H either $I = \varnothing$ (i.e. H is a unitary space) or $\mathcal{I} = H$.*

Proof. Let $I \neq \varnothing$. First we shall show that $\mathcal{I}$ is non-isotropic. Indeed, assume $i \in \mathcal{I}$, $i \perp \mathcal{I}$, $i \neq 0$. Then necessarily $i \in I$, and there exists an $y \in H$ such that $\langle i, y \rangle \neq 0$. Now if we put

$$z = y - \frac{\langle y, y \rangle}{2\langle i, y \rangle} \cdot i ,$$

we have $z \in I$ and $\langle i, z \rangle = \langle i, y \rangle \neq 0$; this is in contradiction with $i \perp \mathcal{I}$.

Next we shall prove that each $x_0 \in H$ can be expressed as a linear combination of elements in I. Obviously it suffices to suppose $x_0 \in H - I$ and, e.g., $\langle x_0, x_0 \rangle > 0$. There are two cases:

a) There exists an $i_0 \in I$ such that $\langle x_0, i_0 \rangle \neq 0$. Then we may write

$$x_0 = x_0' + \frac{\langle x_0, x_0 \rangle}{2\langle x_0, i_0 \rangle} \cdot i_0 ,$$

where

$$x_0' = x_0 - \frac{\langle x_0, x_0 \rangle}{2\langle x_0, i_0 \rangle} \cdot i_0 , \quad x_0' \in I ,$$

and thus $x_0 \in \mathscr{I}$.

b) Suppose now $x_0 \perp I$. We shall show that the form $\langle x, x \rangle$ is positive-definite on $\perp(\mathscr{I})$. Indeed, let $0 \neq y_0 \in (\mathscr{I})$ satisfy $\langle y_0, y_0 \rangle \leqq 0$. Then the equation $\langle x_0 + ky_0, x_0 + ky_0 \rangle = 0$ has a solution $k = k_0$, and according to Theorem 2 we obtain $x_0 + k_0 y_0 = 0$, i.e. $\langle x_0, x_0 \rangle = k_0^2 \langle y_0, y_0 \rangle$, which is in contradiction with $\langle x_0, x_0 \rangle > 0$.

As $I \neq \varnothing$, we conclude from Theorem 3 that there exists an $x_1 \in H$ satisfying $\langle x_1, x_1 \rangle < 0$. According to what has been proved above $x_1 \notin \perp(\mathscr{I})$, i.e. there exists an isotropic i_1 such that $\langle x_1, i_1 \rangle \neq 0$. From part a) of this proof we obtain $x_1 \in \mathscr{I}$. Now, since the equation $\langle x_0 + kx_1, x_0 + kx_1 \rangle = 0$ has a real solution, we obtain finally that $x_0 \in \mathscr{I}$.

The case $\langle x_0, x_0 \rangle < 0$ could be treated in a similar manner.

Theorem 1 shows that an n-dimensional non-isotropic linear subspace of a pseudo-unitary space is isomorphic in a natural way with the n-dimensional pseudo-euclidean space. Therefore the theorem below is in fact only a well-known result in the theory of these spaces.

Theorem 5. *Let H_0 be a non-isotropic n-dimensional linear subspace of a pseudo-unitary space H. Then the following holds:*

For arbitrary elements $x_1, x_2, \ldots, x_m$ $(1 \leqq m < n)$ such that $x_i \in H_0$, $\langle x_i, x_j \rangle = \pm \delta_{ij}$ $(i, j = 1, \ldots, m)$ there exist in H_0 $n - m$ elements $x_{m+1}, \ldots, x_n$ such that $\langle x_i, x_j \rangle = \pm \delta_{ij}$, where now $i, j = 1, 2, \ldots, n$. Moreover, the integer p $(0 \leqq p \leqq n)$ denoting the number of indices i for which $\langle x_i, x_i \rangle = +1$, respectively the integer q $(0 \leqq q \leqq n, p + q = n)$ denoting the number of indices i for which $\langle x_i, x_i \rangle = -1$, depend only upon the subspace $H_0 \subset H$. (The symbol δ_{ij} means here as usual the Kronecker delta.)

1.3 In a given pseudo-unitary space H let us choose arbitrarily a fixed element x_0. Then the function $p(y) = |\langle x_0, y \rangle|$ is a pseudo-norm in H. The system of these pseudo-norms obtained as x_0 varies over H determines the topology $\sigma(H, H)$ on the linear space H [1]. We shall call this topology, given by the algebraical structure of the pseudo-unitary space, briefly the σ-topology. It is the weakest topology on H in which the addition of elements, multiplication by real numbers and also the forms $y \to \langle x, y \rangle$ on H for each fixed $x \in H$ are all continuous. The space H endowed with this topology is a locally convex Hausdorff space [1, ch. IV § 1]. Moreover, each linear form $f(y)$ on H continuous in the σ-topology can be expressed as $f(y) = \langle x_0, y \rangle$ for some $x_0 \in H$ (loc. cit.).

Theorem 6. *Let H be a pseudo-unitary space, $H_0 \subset H$ a linear subspace and $E \subset H$ an arbitrary subset. Then the following hold:*

a) $x \perp E \Rightarrow x \perp (\sigma)\, \overline{\mathscr{L}\{E\}}$,

b) $x \perp E$, $(\sigma)\, \overline{\mathscr{L}\{E\}} = H \Rightarrow x = 0$,

c) $\perp(H_0)$ *is a σ-closed linear subspace of H,*

d) *if H_0 is σ-closed then $\perp(\perp(H_0)) = H_0$.*

Proof. The first three statements are evident. For the proof of the last one see Bourbaki [1, ch. IV § 1 n. 5].

Theorem 7. *Let H_0 be a non-isotropic σ-closed linear subspace of a pseudo-unitary space H. Then $\perp(H_0)$, which is again a σ-closed linear subspace of H, is also non-isotropic.*

Proof. $x \in \perp(H_0)$, $x \perp \perp(H_0)$ imply $x \in H_0$, $x \perp H_0$, and hence $x = 0$.

Theorem 8. *Let H_0 be a non-isotropic σ-closed linear subspace of H. Then the set $M = H_0 + \perp(H_0)$ is an (algebraical) direct sum of its summands, and is σ-dense in the space H.*

Proof. Let $x \in M$ and $x = x_1 + x_2 = x_1' + x_2'$, where $x_1, x_1' \in H_0$ and $x_2, x_2' \in \perp(H_0)$, i.e. $(x_1 - x_1') + (x_2 - x_2') = 0$. Then for arbitrary $y \in H_0$ we have $\langle x_1 - x_1', y \rangle = 0$, i.e. $x_1 = x_1'$. From the above equation we also conclude that $x_2 = x_2'$.

The set M is clearly a linear subspace of H. Let $x \perp M$ for some element $x \in H$. Then also $x \perp H_0$ and $x \perp \perp(H_0)$, and hence, according to Theorem 6d, we have $x \in H_0 \cap \perp(H_0)$. Theorem 2 yields $x = 0$, i.e. $\perp(M) = \{0\}$, and Theorem 6d then yields $\perp(\{0\}) = (\sigma)\, \overline{M}$, i.e. $(\sigma)\, \overline{M} = H$.

Theorem 9. *Let H_0 be a linear subspace σ-dense in H. Then H_0 is non-isotropic.*
This is an immediate consequence of Theorem 6b.

2. INTRODUCTION OF CONTINUOUS TOPOLOGY INTO PSEUDO-UNITARY SPACE

2.1 We shall say that there is given a *continuous* topology T_0 in the pseudo-unitary space H if H, with this topology T_0, is a linear topological space and the pseudo-scalar product is continuous in both its arguments under this topology. In other words, we assume that

T1) the mapping $(\lambda_1, \lambda_2, x_1, x_2) \to \lambda_1 x_1 + \lambda_2 x_2$ is a continuous mapping from $E_1 \times E_1 \times H \times H$ into H, and

T2) the mapping $(x, y) \to \langle x, y \rangle$ is a continuous mapping from $H \times H$ into E_1.

If there exists, in a pseudo-unitary space H, a topology T_0 satisfying the above conditions, then this topology is clearly stronger than the topology σ. Thus in this case the following theorem is evident.

Theorem 10. *The Theorems 6a, 6b and 9 remain valid if the σ-topology is replaced by a continuous topology.*

The existence in H of a topology T_0 with the above properties is by no means warranted by the algebraical structure of the pseudo-unitary space, and also it cannot be selected in a natural unique manner as in the case of unitary spaces. If such a topology exists it must be determined a *posteriori.* We shall illustrate this in the following examples.

Example 1. Here we shall show that in a pseudo-unitary space there need not exist any locally convex topology satisfying **T1)** and **T2)**.

Let H be the set of all sequences of real numbers $\{a_1, a_2, \ldots\}$ such that $a_i = 0$ for each sufficently large even i. Let the pseudo-scalar product on H be defined by the relation

$$\langle \{a_i\}, \{b_i\} \rangle = \sum_{i \text{ odd}} (a_i b_{i+1} + a_{i+1} b_i) \, .$$

Clearly H is a pseudo-unitary space. The non-existence of a topology with the above properties will be proved indirectly.

Let us suppose that such a topology exists and let U be a symmetric convex neighbourhood of the origin in H such that

$$\text{(1)} \qquad a \in U, \; b \in U \Rightarrow |\langle a, b \rangle| = 1 \, .$$

Denote $e_i = \{0, 0, \ldots, 1, 0, \ldots\}$, where 1 is at the i-th place. For $a \in H$ denote by $|a|$ the pseudo-norm defined by the neighbourhood U. Our purpose is to prove that $|e_i| = 0$ for sufficently large even i.

Thus, let $u = \{u_1, u_2, \ldots\}$, where $u_i = 0$ for even i, and $u_i = 2(i + 1) \, . \, |e_{i+1}|$ for odd i. First we have $u \in H$, and for each even $i + 1$ with $|e_{i+1}| > 0$ there is

$$\left\langle \frac{u}{i + 1}, \frac{e_{i+1}}{|e_{i+1}|} \right\rangle = 2 \, .$$

From $e_{i+1}/|e_{i+1}| \in U$ it follows that $u/(i + 1) \notin U$, [1]. But this cannot happen for infinitely many i; this proves the above statement.

Consequently for some $i \geqq 2$ we have $|e_i| = 0$, which means that $\alpha \,.\, e_i \in U$ for each real α. But now from

$$\left\langle \alpha e_i, \frac{2}{\alpha} e_{i-1} \right\rangle = 2$$

we obtain $2e_{i-1}/\alpha \notin U$ (cf. [1]) for each real α, which is a contradiction.

Example 2. In this example we shall show that in a pseudo-unitary space there may exist two locally convex continuous topologies although there does not exist any topology with these properties weaker than both (i.e. there does not exist a weakest topology with the required properties, in distinction with the case of unitary spaces).

Let H be the space of all sequences of real numbers $\{a_1, a_2, \ldots\}$ for which $\sum_i a_i^2 < +\infty$ and $\sum_i e_i^2 a_i^2 < +\infty$, where we write $e_{2i} = 2i$, $e_{2i-1} = 1/2i$ (i a positive integer). The pseudo-scalar product will be defined by

$$\langle \{a_i\}, \{b_i\} \rangle = \sum_{i \text{ odd}} (a_i b_{i+1} + a_{i+1} b_i)\,,$$

which may also be written as

$$\sum_{i \text{ odd}} (e_i a_i e_{i+1} b_{i+1} + e_{i+1} a_{i+1} e_i b_i)\,.$$

Let one of the topologies be given by means of the norm $|\{a_i\}| = (\sum_i a_i^2)^{\frac{1}{2}}$, and the other by means of the norm $\|\{a_i\| = (\sum_i e_i^2 a_i^2)^{\frac{1}{2}}$. It is easily seen that the pseudo-scalar product is continuous in both these topologies.

Let $A_n = \{0, 0, \ldots, 1, 1, 0, \ldots\}$ where 1 is at the $(2n-1)$-th and $2n$-th places, and let $B_n = \{0, 0, \ldots, 2n, 1/(2n), 0, \ldots,\}$ where the non-zero elements are at the same places as before. Now, the sequence $\{A_n\}_{n=1,2,\ldots}$ is bounded in the first topology, the sequence $\{B_n\}_{n=1,2,\ldots}$ is bounded in the second one; thus the sequence $\{A_n + B_n\}_{n=1,2,\ldots}$ is bounded in each locally convex topology which is weaker than both of the topologies introduced above. Nevertheless $\langle A_n + B_n, A_n + B_n \rangle > 4n \to +\infty$, and thus the pseudo-scalar product is not continuous in any such topology.

In what follows we shall suppose that there is given a pseudo-unitary space H provided with a continuous topology T_0.

2.2 A locally convex pseudo-unitary space H will be called a *pseudo-Hilbert* space if its continuous topology T_0 has the following property: If $x \to f(x)$ is any T_0-continuous linear form on H, then there exists an element $y \in H$ such that $f(x) = \langle x, y \rangle$ for each $x \in H$.

Theorem 11. *Let $H = H(T_0)$ be a pseudo-Hilbert space, f a linear form on H. Then f is T_0-continuous if and only if it is σ-continuous.*

Proof. If f is σ-continuous then it is also T_0-continuous because the weak topology σ is obviously weaker than T_0. Conversely if f is T_0-continuous, it can be written as $x \to \langle x, y \rangle$ and thus it is also σ-continuous.

Theorem 12. *Each pseudo-Hilbert space is normable.*

Proof. Let $H(T_0)$ be a pseudo-Hilbert space and U a symmetric convex neighbourhood of the origin such that $x \in U, y \in U \Rightarrow |\langle x, y \rangle| \leqq 1$. This implication asserts that the polar set of U (i.e. the set of those $y \in H$ for which $|\langle x, y \rangle| \leqq 1$ holds for all $x \in U$) contains the neighbourhood U. But as this polar set is σ-compact, the same is true about $\bar{U}$; in particular, $\bar{U}$ is σ-bounded. According to Theorem 11, the spaces $H(T_0)$ and $H(\sigma)$ satisfy trivially for instance the assumptions of the theorem of Mackey (cf. [1] ch. IV § 2 n. 4). Hence $\bar{U}$ is also T_0-bounded and one can take it for the unit ball; this concludes the proof.

Remark. The norm in pseudo-Hilbert space, where the unit ball is the set $\bar{U}$ from the above proof, satisfies the relation

$$(2) \qquad |\langle x, y \rangle| \leqq |x| \cdot |y| \, .$$

Lemma 2. *Let H be a pseudo-Hilbert space, the topology of which is defined by a norm $x \to |x|$ satisfying (2). Then the norm $x \to \|x\|$, defined in such a manner that the set*

$$M = \{x;\ |y| \leqq 1 \Rightarrow |\langle x, y \rangle| \leqq 1\}$$

(i.e. the set polar to the unit ball in the norm $x \to |x|$) is the unit ball in the norm $x \to \|x\|$, determines the same topology as the norm $x \to |x|$.

Proof. Obviously $|x| \leqq 1 \Rightarrow x \in M \Rightarrow \|x\| \leqq 1$, so that the topology given by the norm $x \to \|x\|$ is weaker. On the other hand, the set M being polar to a neighbourhood of the origin, it is bounded (cf. [1]), which proves the converse assertion.

Theorem 13. *Each pseudo-Hilbert space is complete.*

Proof. Let us choose in H a norm $x \to |x|$ satisfying (2) and let $\{x_1, x_2, \ldots\}$ be a Cauchy sequence in H. The sequence $\{x_n\}$ is also σ-Cauchy, and thus for each $y \in H$ there exists an $f(y) = \lim_{n \to \infty} \langle x_n, y \rangle$. Then f is a linear form. The preceding lemma shows that $\{\langle x_n, y \rangle\}$ is a Cauchy sequence, uniformly on the set $\{y;\ |y| \leqq 1\}$, so that its limit is bounded on this set, i.e. f is a bounded linear form. This means that there exists an $x \in H$ such that $f(y) = \langle x, y \rangle$ for all $y \in H$. Furthermore, $\{\langle x_n, y \rangle\}$ converges to $f(y)$ uniformly on the set described above. In other words, the sequence $\{\langle x_n, y \rangle\}$ converges to $\langle x, y \rangle$ uniformly on the set $\{y;\ |y| \leqq 1\}$, and applying again Lemma 2, we finally obtain that $\lim x_n = x$; this concludes the proof.

Theorem 14. *Let H be a pseudo-unitary space. Then on H there exists at the most one (locally convex) topology such that H with this topology is a pseudo-Hilbert space.*

Proof. If there exist two such topologies T_1 and T_2, then the graph of the identical mapping of $H(T_1)$ onto $H(T_2)$ is closed in $H(T_1) \times H(T_2)$, because it is obviously closed in the same cartesian product under the weaker topology $\sigma \times \sigma$ (σ is the weak topology in H). From the closed graph theorem (cf. also Theorems 12 and 13) we conclude that the identical mapping of $H(T_1)$ onto $H(T_2)$ is continuous, and for the same reason its inverse is also continuous; hence $T_1 = T_2$.

Example 3. Here we shall show that a non-isotropic subspace of a pseudo-Hilbert space may have an isotropic closure. Thus it is not true that the complete hull of a pseudo-unitary space is again a pseudo-unitary space as in the case at the unitary spaces.

Take $H = l^2$ (the space of all sequences of real numbers $\{a_n\}$ with $\sum_{n=1}^{+\infty} a_n^2 \langle +\infty)$ with the usual topology, but with the pseudo-scalar product defined as follows: For $a = \{a_n\}_{n=1,2,\ldots}$ and $b = \{b_n\}_{n=1,2,\ldots}$ let $\langle a, b\rangle = a_1 b_2 + a_2 b_1 + \sum_{n=3}^{+\infty} a_n b_n$. Let the subspace H_0 be the set of all $a \in H$ for which $a_1 = 0$, $a_n = 0$ for n sufficiently large, and $\sum_{n=1}^{+\infty} a_n = 0$ (with the notation as above).

If $a \in H_0$, $a \neq 0$, then obviously $a_k \neq 0$ for some $k \geqq 3$. If we now choose $b_2 = -1$, $b_k = 0$ for the remaining n, then $b \in H_0$, $\langle a, b\rangle \neq 0$, and therefore H_0 is non-isotropic. On the other hand H_0 is clearly the set of all such $a \in H$ for which $a_1 = 0$. Choosing $b = \{0, 1, 0, 0, \ldots\}$ we have $\langle a, b\rangle = 0$ for all $a \in \overline{H}_0$ and thus $\overline{H}_0$ is isotropic.

Example 4. Here we shall show that – in contradistinction to Hilbert spaces – if H is a pseudo-Hilbert space and $H_0 \subset H$ a T_0-closed non-isotropic linear subspace in H, then $H = H_0 \dotplus \perp(H_0)$ need not hold (this need not hold even set-theoretically). The same remains true if H_0 even has a Hilbert-space structure, i.e. if the pseudo-scalar product is positive-definite on H_0.

First we shall prove the following auxiliary statement

Lemma 3. *Let q be a real number and $q \in \langle 0, 1)$. Then for any $(x_1, x_2) \in E_2$, $(y_1, y_2) \in E_2$ the following inequality holds*

$$(3) \quad |x_1 y_1 - x_2 y_2| \leqq \sqrt{(x_1^2 + x_2^2 - 2q x_1 x_2)} \,.\, \sqrt{(y_1^2 + y_2^2 - 2q y_1 y_2)} \,.\, \frac{1}{\sqrt{(1 - q^2)}} \,.$$

Moreover, to each point (x_1, x_2) one can find a point $(y_1, y_2) \neq 0$ such that equality holds in (3).

Proof. Indeed, using the one-to-one substitution given by

$$x_1 = \frac{a_1}{\sqrt{(1-q)}} + \frac{a_2}{\sqrt{(1+q)}}, \quad y_1 = \frac{b_2}{\sqrt{(1-q)}} + \frac{b_1}{\sqrt{(1+q)}}$$

$$x_2 = \frac{a_1}{\sqrt{(1-q)}} - \frac{a_2}{\sqrt{(1+q)}}, \quad y_2 = \frac{b_2}{\sqrt{(1-q)}} - \frac{b_1}{\sqrt{(1+q)}}$$

(3) transforms into

$$\frac{2}{\sqrt{(1-q^2)}} \cdot |a_1 b_1 + a_2 b_2| \leqq \sqrt{[2(a_1^2 + a_2^2)]} \cdot \sqrt{[2(b_1^2 + b_2^2)]} \cdot \frac{1}{\sqrt{(1-q^2)}}$$

which is a consequence of the Minkowski inequality.

Returning to our example, let H be the space of all real sequences $\{a_1, a_2, \ldots\}$ such that the following series converges

$$(4) \qquad \sum_{k \text{ odd}} (a_k^2 + a_{k+1}^2 - 2(1 - 1/k)\, a_k a_{k+1}) \cdot \frac{1}{\sqrt{[1 - (1 - 1/k)^2]}}\,.$$

The norm in H is defined as the square root of (4), and the pseudo-scalar product is defined by the relation

$$\langle \{a_n\}, \{b_n\} \rangle = \sum_{k \text{ odd}} (a_k b_k - a_{k+1} b_{k+1})\,.$$

First we shall show that (2) holds, i.e.

$$|\langle \{a_n\}, \{b_n\} \rangle \leqq |\{a_n\}| \cdot |\{b_n\}|\,.$$

According to Lemma 3, the left-hand side can be estimated from above by the expression

$$\sum_{k \text{ odd}} \sqrt{[a_k^2 + a_{k+1}^2 - 2(1 - 1/k)\, a_k a_{k+1}]} \cdot \frac{1}{\sqrt[4]{[1 - (1 - 1/k)^2]}} \times$$
$$\times \sqrt{[b_k^2 + b_{k+1}^2 - 2(1 - 1/k)\, b_k b_{k+1}]} \cdot \frac{1}{\sqrt[4]{[1 - (1 - 1/k)^2]}}$$

(we have chosen $q = 1 - 1/k$), and the rest follows from the Hölder inequality in l^2.

Further we shall verify that each continuous linear form on H can be written as

$$(5) \qquad f(x) = \sum_{k \text{ odd}} (a_k x_k - a_{k+1} x_{k+1})$$

where $x = \{x_n\}_{n=1,2,\ldots}$, $\{a_n\}_{n=1,2,\ldots} \in H$. It is clear that there exists a sequence $\{a_n\}$ such that (5) holds for all x for which $x_n \neq 0$ for only a finite number n's. As f is continuous, there exists a number $A > 0$ such that for each such x we have

$$(6) \qquad \sum_{k \text{ odd}} |a_k x_k - a_{k+1} x_{k+1}| \leqq A|x|\,.$$

It suffices to prove $\{a_n\} \in H$ only for $\{a_n\} \neq 0$. In this case the number (cf. the definition of the norm)

$$P = \sum_{\text{odd}\, k < 2N} \left(a_k^2 + a_{k+1}^2 - 2(1 - 1/k)\, a_k a_{k+1}\right) \cdot \frac{1}{\sqrt{[1 - (1 - 1/k)^2]}}$$

is non-zero for sufficiently large integers N.

Let x be defined as follows: For $n > 2N$ let $x_n = 0$. For odd $k < 2N$ let x_k and x_{k+1} be defined in such a manner that

$$(7) \qquad x_k a_k - x_{k+1} a_{k+1} = \\ = \sqrt{[a_k^2 + a_{k+1}^2 - 2(1 - 1/k)\, a_k a_{k+1}]} \cdot \sqrt{[x_k^2 + x_{k+1}^2 - 2(1 - 1/k)\, x_k x_{k+1}]} \times \\ \times \frac{1}{\sqrt{[1 - (1 - 1/k)^2]}}$$

(cf. Lemma 3). Furthermore, we can always choose the pair (x_k, x_{k+1}) so that (on multiplying both the components by the same factor if needed) the first factor in the right-hand side of (7) equals the second, and thus $|\{x_n\}|^2 = P$. According to (7) we then have $\sum_{k\,\text{odd}} x_k a_k - x_{k+1} a_{k+1} = P$ and the relation (6) now yields $P \leqq A \sqrt{P}$, i.e. $P \leqq A^2$. As N was arbitrarily large, we conclude $|\{a_n\}|^2 \leqq A^2$, i.e. $\{a_n\} \in H$.

Now it is easy to establish that (6) holds for all $x_0 \in H$. Indeed, each x_0 is the limit of a sequence of elements x for which $x_n \neq 0$ holds for only finitely many n's. Hence H is a pseudo-Hilbert space.

Let now $H_0 = \{\{x_n\};\ x_n = 0 \text{ for all even } n\}$. Clearly, $\perp(H_0) = \{\{x_n\};\ x_n = 0 \text{ for all odd } n\}$. Both these spaces are closed and non-isotropic, because the pseudo-scalar product is positive or negative-definite, respectively.

The sequence $\{x_n\}$ defined by $x_k = x_{k+1} = \sqrt[4]{[1 - (1 - 1/k)^2]}$ for all odd k is an element of H, as is easily established; but the sequence $\{x_n^*\}$ where $x_n^* = x_n$ for n odd and $x_n^* = 0$ for n even is not an element of H_0 (it is not even an element of H). Thus $x \in H_0 \dotplus \perp(H_0)$ does not hold.

3. COMPLETE ORTHONORMAL SYSTEMS AND l-SEPARABLE PSEUDO-UNITARY SPACES

3.1 We shall say that a system $\{x_\alpha\}$ $(\alpha \in A)$ of elements of a pseudo-unitary space H is σ-complete (or T_0-complete) if $(\sigma)\ \overline{\mathscr{L}\{x_\alpha\}} = H$ (or $(T_0)\ \overline{\mathscr{L}\{x_\alpha\}} = H$, respectively).

The only element in H which is orthogonal to a σ-complete or to a T_0-complete system of elements in H is the zero element. This is an immediate consequence of Theorem 6b. In addition we have

Theorem 15. *The system of elements $\{x_\alpha\}$, $(\alpha \in A)$ in H is σ-complete if an only if $x \in H$, $x \perp x_\alpha$ for all $\alpha \in A$ imply $x = 0$.*

Proof. This follows from Theorem 6 since $\perp(\{0\}) = H$.

A system $\{x_\alpha\}$, $(\alpha \in A)$ of elements of a pseudo-unitary space H is called orthonormal if $\alpha, \beta \in A \Rightarrow \langle x_\alpha, x_\beta \rangle = \delta_{\alpha\beta}$ or $-\delta_{\alpha\beta}$, where $\delta_{\alpha\beta}$ is the Kronecker delta (with an obvious generalisation).

Let $K = \{x_\alpha\}$ be an orthonormal system of elements of the space H. The following notations will be used: the symbols K^+ and K^- will denote the set of those elements $x \in K$ for which $\langle x_\alpha, x_\alpha \rangle = +1$ or $\langle x_\alpha, x_\alpha \rangle = -1$ respectively; thus $K = K^+ \cup K^-$. Furthermore, let $\mathscr{K}^+ = \mathscr{L}\{K^+\}$ and $\mathscr{K}^- = \mathscr{L}\{K^-\}$. Clearly the product $(x, y) \to \langle x, y \rangle$ is positive or negative definite on $\mathscr{K}^+$ or $\mathscr{K}^-$ respectively. The spaces $\mathscr{K}^+$ and $\mathscr{K}^-$ can therefore be treated in a natural manner as usual unitary spaces, the induced topology in them being in general different from that induced by the topology T_0 in H.

3.2 It is known that a topological space T is called separable if there exists in T a countable dense subset.

We shall call a pseudo-unitary space H *separable* if it is separable in its continuous topology T_0.

Theorem 16. *Let H be a separable pseudo-unitary space. Then there exists in H a countable T_0-complete (and therefore also σ-complete) orthonormal system.*

Before proving this theorem we shall first establish the following

Lemma 4. *Let H be a pseudo-unitary space and let $x_1, \ldots, x_n$ be a system of n orthonormal elements in H. Suppose that $z \in H$ is an element such that $\mathscr{L}\{x_1, x_2, \ldots, x_n, z\}$ is isotropic. Then there exists an element $u \in H$ such that $\mathscr{L}\{x_1, \ldots, x_n, z, u\}$ is non-isotropic.*

Proof. First, z is independent of the elements $x_1, \ldots, x_n$, and thus $\dim H > n + 1$ Now we shall prove the following statement: If $y_1, \ldots, y_k$ are linearly independent the subspace $\mathscr{L}\{y_1, \ldots, y_k\}$ is isotropic if and only if $\det[\langle y_i, y_j \rangle] = 0$ $(i, j = 1, \ldots, k)$. Indeed, the existence of an element y satisfying $\langle y, y_i \rangle = 0$, $y = \sum_{i=l}^{k} \alpha_i y_i$, $\sum_{i=l}^{k} \alpha_i^2 \neq 0$, is equivalent to the existence of a non-trivial solution of the system

$$\alpha_1\langle y_1, y_1 \rangle + \alpha_2\langle y_1, y_2 \rangle + \ldots + \alpha_k\langle y_1, y_k \rangle = 0$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\alpha_1\langle y_k, y_1 \rangle + \alpha_2\langle y_k, y_2 \rangle + \ldots + \alpha_k\langle y_k, y_k \rangle = 0$$

and this proves the statement. In our case the following determinant is zero:

$$\begin{vmatrix} \pm 1, & 0, \ldots, 0, & 0, & \langle z, x_1 \rangle \\ \ldots & \ldots & \ldots & \ldots \\ 0, & 0, \ldots, 0, & \pm 1, & \langle z, x_n \rangle \\ \langle z, x_1 \rangle, & \langle z, x_2 \rangle, \ldots, & \langle z, x_n \rangle, & \langle z, z \rangle \end{vmatrix}.$$

Next we shall prove the following statement: In H exists an element $u \perp x_i$ ($i = 1, \ldots, n$) such that $\langle u, z \rangle \neq 0$. Suppose the contrary, i.e. that $u \in \perp(\mathscr{L}\{x_1, \ldots, x_n\})$ implies $\langle u, z \rangle = 0$. Then z is orthogonal to the subspace $\perp(\mathscr{L}\{x_1, \ldots, x_n\})$, and according to Theorem 6d, $z \in \mathscr{L}\{x_1, \ldots, x_n\}$ in spite of the fact that z was chosen linearly independent of $x_1, \ldots, x_n$. Hence there exists an element $u \in H$ with the above property. It remains to prove that $\mathscr{L}\{x_1, \ldots, x_n, z, u\}$ is non-isotropic. But we have

$$\begin{vmatrix} \pm 1, & 0, \ldots, 0, & 0, & \langle z, x_1 \rangle, & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0, & 0, \ldots, 0, & \pm 1, & \langle z, x_n \rangle, & 0 \\ \langle z, x_1 \rangle, & \langle z, x_2 \rangle, \ldots, & \langle z, x_n \rangle, & \langle z, z \rangle, & \langle u, z \rangle \\ 0, & 0, \ldots, & 0, & \langle u, z \rangle, & \langle u, u \rangle \end{vmatrix} = -\langle u, z \rangle .$$

$$\begin{vmatrix} \pm 1, & 0, \ldots, 0, & 0, & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0, & 0, \ldots, 0, & \pm 1, & 0 \\ \langle z, x_1 \rangle, & \langle z, x_2 \rangle, \ldots, & \langle z, x_n \rangle, & \langle u, z \rangle \end{vmatrix} + \langle u, u \rangle . 0 = \langle u, z \rangle^2 . (\pm 1) \neq 0 ,$$

and this proves the lemma.

Proof of Theorem 16. Let $D = \{z_1, z_2, \ldots\}$ be a T_0-dense set in H. First we shall prove that the case $\langle z_i, z_i \rangle = 0$ for all $i = 1, 2, \ldots$ cannot occur. Indeed, if $\langle z_i, z_i \rangle = 0$ for all i, then from the continuity of the topology T_0 it follows that $\langle x, x \rangle = 0$ for all $x \in H$, and this contradicts Lemma 1. Assume say $\langle z_1, z_1 \rangle \neq 0$, and put

$$x_1 = \frac{1}{\sqrt{(|\langle z_1, z_1 \rangle|)}} . z_1, \quad n_1 = 1 .$$

Now assume that to the elements $z_1, \ldots, z_k$ we have already constructed orthonormal elements $x_1, \ldots, x_{n_k}$ in such a way that $z_1, \ldots, z_k \in \mathscr{L}\{x_1, \ldots, x_{n_k}\}$. It is required to construct elements $x_{n_k+1}, \ldots, x_{n_{k+1}}$ such that $z_1, \ldots, z_{k+1} \in \mathscr{L}\{x_1, \ldots, x_{n_{k+1}}\}$ and that $x_1, \ldots, x_{n_{k+1}}$ are an orthonormal system. As concerns the element z_{k+1}, one has the following possibilities:

a) $z_{k+1} \in \mathscr{L}\{x_1, \ldots, x_{n_k}\}$. In this case choose $n_{k+1} = n_k$, i.e. do not add anything to the set $x_1, \ldots, x_{n_k}$.

b) $\mathscr{L}\{x_1, \ldots, x_{n_k}, z_{k+1}\}$ is a non-isotropic $(n_k + 1)$-dimensional space. Then we conclude from Theorem 5 that there exists an element $x_{n_k+1} \in \mathscr{L}\{x_1, \ldots, x_{n_k}, z_{k+1}\}$ with the property that the elements $x_1, \ldots, x_{n_k}, x_{n_k+1}$ form an orthonormal system. Thus in this case we have $n_{k+1} = n_k + 1$.

c) $\mathscr{L}\{x_1, \ldots, x_{n_k}, z_{k+1}\}$ is an isotropic $(n_k + 1)$-dimensional space. Here on applying Lemma 4 we find an element $u \in H$ such that $\mathscr{L}\{x_1, \ldots, x_{n_k}, z_{k+1}, u\}$ is again non-isotropic. Now Theorem 5 ensures the existence of elements x_{n_k+1} and

$x_{n_k+2} = x_{n_{k+1}}$ such that the system $x_1, \ldots, x_{n_k}, x_{n_k+1}, x_{n_{k+1}}$ is an orthonormal basis in the space $\mathscr{L}\{x_1, \ldots, x_{n_k}, z_{k+1}, u\}$. Thus now we have $n_{k+1} = n_k + 2$.

In this manner we can construct an orthonormal sequence of elements $x_1, \ldots,$ $\ldots, x_{n_k}, \ldots, = \{x_i\}$ satisfying $D \in \mathscr{L}\{x_i\}$; as $\bar{D} = H$ the theorem is proved.

3.3 We shall say that a pseudo-unitary space H is *l-separable* if each linear subspace of H is separable in the continuous topology T_0 of H. It is well-known that if the topology T_0 is metrisable, then H is l-separable if and only if it is separable. Now we can establish the following

Theorem 17. *Let H be an l-separable pseudo-unitary space. Then each orthonormal system $K = \{x_\alpha\}$ $(\alpha \in A)$, $K \subset H$, is at most countable.*

Proof. As mentioned above, the linear subspace $\mathscr{K}^+ = \mathscr{L}\{K^+\}$ is a unitary space with the scalar product $(x, y) \to \langle x, y\rangle$. From the l-separability of H, the set K^+ is at most countable. Similarly one shows that K^- is also at most countable.

Theorem 18. *Let H be a pseudo-unitary space and let $K = \{x_\alpha\}$ $(\alpha \in A)$ be a T_0-complete orthonormal system in H. Suppose that K^+ consists of exactly n elements $x_1, \ldots, x_n$. Then there does not exist in H an orthonormal system of $n + 1$ elements $z_1, \ldots, z_{n+1}$ satisfying $\langle z_i, z_i\rangle > 0$ for $i = 1, 2, \ldots, n + 1$.*

Proof. Clearly $\mathscr{K}^- \subset \perp(\mathscr{K}^+)$. In the linear space $\perp(\mathscr{K}^+)$ we are given the topology induced by the topology T_0; denote it by T_0'. We shall prove that in this topology $\mathscr{K}^-$ is dense in $\perp(\mathscr{K}^+)$. Thus suppose $y \in \perp(\mathscr{K}^+)$, i.e. $y \perp \mathscr{K}^+$. The set $\mathscr{L}\{K\}$ is T_0-dense in H and hence there exists a generalized sequence $u^\iota \underset{I}{\to} y(\iota \in I)$, $u^\iota \in \mathscr{L}\{K\}$. But it can be easily seen that an arbitrary element $u^\iota \in \mathscr{L}\{K\}$ can be written in a unique manner as $u^\iota = x^\iota + y^\iota$, where $x^\iota \in \mathscr{K}^+$, $y^\iota \in \mathscr{K}^-$. Put $x^\iota =$ $= a_1^\iota x^1 + a_2^\iota x^2 + \ldots + a_n^\iota x^n$. Thus we have $a_1^\iota x^1 + \ldots a_n^\iota x^n + y^\iota \underset{I}{\to} y$. On "multiplying" this relation by x^k $(k = 1, \ldots, n)$, we obtain $a_k^\iota \underset{I}{\to} 0$ $(k = 1, \ldots, n)$ and hence $x^\iota \underset{I}{\to} 0$. Finally, there follows hence the required relation (T_0') $y^\iota \underset{I}{\to} y$, $y^\iota \in \mathscr{K}^-(\iota \in I)$. Now, from the continuity of the topology, also $\langle y^\iota, y^\iota\rangle \underset{I}{\to} \langle y, y\rangle$. Thus for all $y \in \perp(\mathscr{K}^+)$ we have $\langle y, y\rangle \leqq 0$.

Let now z_i $(i = 1, \ldots, n + 1)$ be orthonormal elements in H and let us try to find an element $z_0 = \alpha_1 z_1 + \ldots + \alpha_{n+1} z_{n+1}$ which is orthogonal to $\mathscr{K}^+$. But the system of n equations

$$\alpha_1\langle z_1, x_1\rangle + \ldots + \alpha_{n+1}\langle z_{n+1}, x_1\rangle = 0$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\alpha_1\langle z_1, x_n\rangle + \ldots + \alpha_{n+1}\langle z_{n+1}, x_n\rangle = 0$$

certainly has a non-trivial solution $(\alpha_1, \ldots, \alpha_{n+1})$, and this means that there exists an element $z_0 \in \mathscr{L}\{z_1, \ldots, z_{n+1}\} \cap \perp(\mathscr{K}^+)$. But this is not possible as the pseudo-scalar product is positive definite on $\mathscr{L}\{z_1, \ldots, z_{n+1}\}$; this contradiction completes the proof.

The following Theorem 19 is actually a corollary of Theorems 16, 17 and 18. It may be interpreted as a generalisation of the well-known rule of inertia for quadratic forms in elementary algebra to the case of l-separable pseudo-unitary spaces.

Theorem 19. *Let H be an l-separable pseudo-unitary space and let K be an arbitrary T_0-complete orthonormal system of elements in H. Then the numbers $p = \operatorname{card}(K^+) \leqq +\infty$ and $q = \operatorname{card}(K^-) \leqq +\infty$ do not depend on the choice of the system K.*

The pair of these numbers (p, q) will be called the *index of the pseudo-unitary space H*. According to Theorem 1 one may also speak about the index of a non-isotropic linear subspace of a pseudo-unitary space. If the index of the space H is of the form $(p, 0)$, then H is a unitary space, its scalar product being given by $\langle x, y\rangle$. The topology connected in the natural way with this unitary space is in general weaker than the topology T_0 given a posteriori in H. On the other hand, it is not difficult to see that one can construct from each (infinite dimensional) Hilbert space a pseudo-unitary (l-separable) space H with arbitrary index (p, q) $(p + q = +\infty)$.

Indeed, let H' be some Hilbert space. The scalar product of its elements x, y will be denoted by $x \,.\, y$. Let K be a complete orthonormal system of elements in this space. Let K consist of elements $x_1, \ldots, x_i, \ldots$ Let there be given an arbitrary but fixed sequence ε_i $(i = 1, 2, \ldots)$ of numbers $+1$ or -1. Let K^+ be the set of those elements $x_i \in K$ for which $\varepsilon_i = +1$, and K^- the set of those $x_i \in K$ for which $\varepsilon_i = -1$. We define the pseudo-scalar product in H' as $\langle u, v\rangle = \sum \varepsilon_i u^i v^i$ for $u = \sum u^i x_i$, $v = \sum v^i x_i$. It is easily seen that H', with the pseudo-scalar product thus defined, is a pseudo-unitary space and its continuous topology T_0 is given by the a priori unitary space structure in H'. In this pseudo-unitary space, K is a T_0-complete orthonormal system and the index of H' is the pair (p, q) where $p = \operatorname{card}(K^+)$ and $q = \operatorname{card}(K^-)$. This space is clearly l-separable.

Theorem 20. *Let H be an l-separable pseudo-unitary space with index $(+\infty, q)$ where q is a (finite) non-negative integer. Let $K \subset H$ be an arbitrary T_0-complete orthonormal system. Then the following holds:*

a) *The space H under the topology T_0 is a topological direct sum of its subspaces $(T_0)\,\overline{\mathscr{K}^+}$ and $\mathscr{K}^-$ provided with the topologies induced by T_0, i.e.*

$$H = (T_0)\,\overline{\mathscr{K}^+} \dotplus \mathscr{K}^- .$$

b) *One can define a scalar product in H in such a manner that H becomes a unitary space with the natural topology τ in general weaker than the a priori continuous topology T_0. The system K is simultaneously a τ-complete orthonormal system in this unitary space (in other words, $(\tau)\,\overline{\mathscr{L}\{K\}} = H$).*

Proof. It is easily established that $\mathscr{L}\{K\} = \mathscr{K}^+ \dotplus \mathscr{K}^-$. There exists, to each element $x \in H$, a generalized sequence $\{z^\iota\}$ $(\iota \in I)$ such that (T_0) $z^\iota \underset{I}{\rightarrow} x$ and $z^\iota \in$

$\in \mathscr{K}^+ + \mathscr{K}^-$, i.e., one may write in a unique way $z^\iota = u^\iota + v^\iota$, $u^\iota \in \mathscr{K}^+$, $v^\iota \in \mathscr{K}^-$. Put $v^\iota = \sum_{i=1}^{q} a_i^\iota x_i$, where $K^- = \{x_1, \ldots, x_q\}$. If we now "multiply" the relation $z^\iota = u^\iota + v^\iota$ by x_i $(i = 1, \ldots, q)$, we obtain $\langle z^\iota, x_i \rangle = a_i^\iota$. From the convergence of the left-hand side in this relation we conclude $a_i^\iota \underset{I}{\to} a_i$. Hence there exists an element $v = \sum_{i=1}^{q} a_i x_i \in \mathscr{K}^-$ such that $(T_0)\, v^\iota \underset{I}{\to} v$. But then there also exists an element $u \in H$ such that $u^\iota \underset{I}{\to} u$, and it is clear that $u \in (T_0)\, \overline{\mathscr{K}^+}$. Therefore an arbitrary element $x \in H$ can be written in a unique way in the form $x = u + v$; $u \in (T_0)\, \overline{\mathscr{K}^+}$, $v \in \mathscr{K}^-$. As we have seen in the proof of Theorem 18, there is $\langle x, x \rangle \geqq 0$ for all $x \in \perp(\mathscr{K}^-)$. Now Theorem 7 states that $\perp(\mathscr{K}^-)$ is non-isotropic, and from Theorem 3 we conclude that $\langle x, x \rangle > 0$ holds for all $0 \neq x \in \perp(\mathscr{K}^-)$ and thus also for all $0 \neq x \in (T_0)\, \overline{\mathscr{K}^+}$. Hence $(T_0)\, \overline{\mathscr{K}^+}$ is a unitary space with the scalar product given by $(x, y) \to \langle x, y \rangle$. It is also not difficult to see that H is even a topological sum of the spaces $(T_0)\, \overline{\mathscr{K}^+}$ and $\mathscr{K}^-$. If

$$x = u + v;\ u \in (T_0)\, \overline{\mathscr{K}^+},\ v \in \mathscr{K}^-, \qquad x' = u' + v';\ u' \in (T_0)\, \overline{\mathscr{K}^+},\ v' \in \mathscr{K}^-,$$

we define

$$x \,.\, x' = \langle u, u' \rangle - \langle v, v' \rangle\,. \tag{8}$$

Hence the space H may be considered as a unitary space with the scalar product given by (8). The remaining parts of the theorem are now evident.

Remark. Let a pseudo-unitary space H have the following properties: 1) it is the algebraic direct sum of two subspaces, $H = H_1 \oplus H_2$; 2) the dimension of H_1 is *finite*; 3) the pseudo-scalar product is positive-definite on H_1 and negative-definite on H_2. Then on H there is an a priori given canonical structure of a linear topological space, in which the pseudo-scalar product is continuous. According to Theorem 20, each l-separable pseudo-unitary space with index $(p, +\infty)$ (p finite) has the three properties mentioned above, and the corresponding canonical structure is given by the topology τ. An another example of such a pseudo-unitary space is the space Π_κ investigated by Yokhvidoff and Krein. Here there are also required completeness of Π_κ in the canonical topology and linearity over the field of complex numbers. For the precise definitions see [2].

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Výtah

PSEUDOUNITÁRNÍ PROSTORY

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Lineární prostor H (nad tělesem reálných čísel) na němž je definována nedegenerovaná bilineární forma nazveme pseudounitárním prostorem. Hodnotu této formy pro dva prvky z H pak nazveme pseudoskalárním součinem těchto prvků.

V práci jsou vyšetřovány některé „geometrické" vlastnosti pseudounitárních prostorů a je diskutována možnost zavedení topologie na H, která je v souladu s lineární strukturou i pseudoskalárním součinem. Je-li tato topologie l-separabilní – tj. existuje-li ke každé lineární podmnožině v H její spočetná hustá část – pak lze definovat úplné ortonormované systémy prvků z H o nichž platí zákon jenž je obdobou zákona setrvačnosti kvadratických forem u lineárních prostorů s konečnou dimensí. Je zaveden pojem indexu l-separabilního pseudounitárního prostoru a je naznačena souvislost mezi zde definovanými pseudounitárními prostory a prostory studovanými v (2).

Резюме

ПСЕВДОУНИТАРНЫЕ ПРОСТРАНСТВА

ИРЖИ ЕЛИНЕК (Jiří Jelínek), Praha, ЮРАЙ ВИРСИК (Juraj Virsik), Bratislava

Линейное пространство H (над полем вещественных чисел) с определенной на нем невырожденной билинейной формой назовем псевдоунитарным пространством. Значение этой формы для двух элементов из H назовем псевдоскалярным произведением этих элементов.

В работе исследуются некоторые „геометрические" свойства псевдоунитарных пространств и обсуждается возможность введения в H топологии, согласованной с линейной структурой и псевдоскалярным произведением. Если эта топология l-сепарабельна, т.е. если существует в каждом линейном подмножестве H_0 из H подмножество, счетное и плотное в H_0, то можно определить полные ортонормированные системы элементов из H, для которых выполняется закон, аналогичный закону инерции квадратических форм для линейных пространств с конечной размерностью. Вводится понятие индекса l-сепарабельного псевдоунитарного пространства, и назначена связь определенных здесь псевдоунитарных пространств с пространствами, исследованными в [2].