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SPACES OF FUNCTIONS ON DOMAIN Ω, WHOSE k-TH DERIVATIVES ARE MEASURES DEFINED ON $\bar{\Omega}$

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INTRODUCTION

In §1. a new kind of a functional space is defined, the space $W^1_\mu(\bar{\Omega})$ of functions, the first derivatives of which are measures on $\bar{\Omega}$ and the properties of this space are investigated. In §2. we use results from §1. to define the space $W^k_\mu(\bar{\Omega})$, the space of functions, the k-th derivatives of which are measures on $\bar{\Omega}$.

Let $\Omega \subset E_N$ be a bounded domain with the boundary of the class C^1. Generally speaking, we can say that the space $W^1_\mu(\bar{\Omega})$ is the completing of Sobolev's space $W^1_1(\Omega)$ in weak convergence, by this weak convergence we mean the weak convergence of the function together with weak convergence of their derivatives. Now, it can be seen that element of W^1_μ is not already a function on Ω in usualy sense: if two weak convergent sequences of functions from W^1_1 has the same limit function in L_μ (in the sense of weak convergence in L_μ), then their derivatives need not have the same weak limit in L_μ, these limit measures can be different on $\partial\Omega$.

The space $W^1_\mu(\bar{\Omega})$ is the space of all $(N + 1)$-couples $(\alpha_0, \alpha_1, \ldots, \alpha_N)$ of measures on $\bar{\Omega}$, for which there exists a sequence of functions $u_n \in W^1_1(\Omega)$ such that $u_n \rightharpoonup \alpha_0$ and at the same time $\partial u_n/\partial x_i \rightharpoonup \alpha_i$. It will be seen that α_0 must be absolutely continuous with respect to Lebesgue measure and hence α_0 has the density u, which is integrable on Ω. The derivatives of this function u in the sense of distributions are then the restriction $\alpha_{i|\Omega}$. Further there exists uniquelly determined measure $\beta \in L_\mu(\partial\Omega)$ (we will call it the trace of $(\alpha_0, \ldots, \alpha_N)$ such that the Green theorem holds in this form

$$\int_{\partial\Omega} \varphi \nu_i \, d\beta = \int_\Omega u \varphi_{x_i} \, dx + \int_{\bar{\Omega}} \varphi \, d\alpha_i \,, \quad \forall \varphi \in C^1(\bar{\Omega}) \,.$$

The following important assertion is true:

If we take the function $u \in L_1(\Omega)$ and if for any measures $\alpha_1, \ldots, \alpha_N \in L_\mu(\bar{\Omega})$ there exists the measure $\beta \in L_\mu(\partial\Omega)$ such that Green theorem holds, then $(u, \alpha_1, \ldots, \alpha_N) \in$

$\in W_\mu^1(\bar{\Omega})$. It will be seen that the element $(u, \alpha_i) \in W_\mu^1$ is uniquelly determined by the function u and by the trace $\beta \in L_\mu(\partial\Omega)$. In theorem 3 there are discussed necessary and sufficient condition for u and β to define an element from W_μ^1. It will then be shown that $\beta \in L_\mu(\partial\Omega)$ can be arbitrary and only some conditions must be supposed about the function u. The trace β depends continuously and weakly continuously on $(u, \alpha_i) \in W_\mu^1$. Further there are proved the theorems on imbedding into $L_q(\Omega)$ and the theorem on equivalent norms. The unit ball in W_μ^1 is weakly compact. A so called inner trace of $(u, \alpha_i) \in W_\mu^1$ is defined as the trace of the element $(u, \bar{\alpha}_i) \in W_\mu^1$, where $\bar{\alpha}_i$ is the restriction $\alpha_{i|\Omega}$, which is uniquelly determined by u. The side of an element (u, α_i) is, on the contrary, determined by the restriction $\alpha_{i|\partial\Omega}$ and equals the difference between the trace and the inner trace.

In the next sections the possibility of joining together of two functions is investigated, which are defined on the neighbouring domains. We can join together two such functions, if they have the same trace on the common part of the boundary. The function $(u, \alpha_i) \in W_\mu^1$ can be extended to the greater domain, if the trace of this function is absolutely continuous with respect to Lebesgue measure on $\partial\Omega$. By suitable extentions it is possible to define the regularisation of element $(u, \alpha_i) \in W_\mu^1$ and by this regularisations we are able to prove that for each $(u, \alpha_i) \in W_\mu^1$ there exists a sequence of the functions $u_n \in W_\mu^1(\Omega)$ such that

$$(u_n, u_{nx_1}, \ldots, u_{nx_N}) \rightharpoonup (u, \alpha_1, \ldots, \alpha_N)$$

and moreover $\lim_{n\to\infty} \|u_n\|_{W_1{}^1} = \|(u, \alpha_i)\|_{W_\mu{}^1}$.

In §2. the space $W_\mu^k(\bar{\Omega})$ is defined as the space of functions, the $(k - 1)$-th derivatives of which belong to the space W_μ^1. The analogic properties are investigated there as was done for the space W_μ^1, but the situation is more complicated, namely for extensions of elements from W_μ^k.

The reason for investigation of these spaces is following. We can consider a functional of the type of minimal surface

$$I(u, \Omega) = \int_\Omega f(x, u, u_{x_1}, \ldots, u_{x_N})\, dx\,, \quad u \in W_1^1(\Omega)\,.$$

Let $f(x, u, p)$ be a continuous and nonnegative function, which is convex in the variable $p = (p_1, \ldots, p_N)$ and which satisfied the condition

$$c_1|p| - c_2 \leqq f(x, u, p) \leqq c_3|p| + c_4\,; \quad \forall x, u, p\,; \quad c_1, \ldots, c_4 \geqq 0\,.$$

We will look for minimum of this functional on the set of all $u \in W_1^1$, $u = u'$ on $\partial\Omega$; $u' \in L_1(\partial\Omega)$ fixed. There is one great difficulty, we cannot use direct methods of the calculus of variations because the space W_1^1 does not have a weakly compact ball. But the space W_μ^1 has a weakly compact ball and we can extend the function I to the whole space W_μ^1. Theorem 6 on weak compactness of the ball in W_μ^1 together with

Theorem 2 that the trace β of element $(u, \alpha_i) \in W_\mu^1$ depends weakly continuously on (u, α_i) are then basic for using of the direct method. Also futher properties of the space W_μ^1 are very useful for investigation of this variational problem. By this method we obtain weak solution in W_μ^1 for each boundary condition $u' \in L_1(\partial\Omega)$ and for domains Ω, which does not satisfy the usual condition of convexity.

We can also consider the analogic variational problem with derivatives to the k-th order. On this problem there were no results till now. These variational problems will be investigated in next papers.

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Notation

Ω – a bounded domain in E_N with its boundary $\partial\Omega$ belonging to C^1 class,
$\nu = (\nu_1, \ldots, \nu_N)$ – exterior normal to Ω,
c – a constant which depends only on Ω,
$C(E)$ – the space of all continuous functions defined on the compact $E \subset E_N$,
$W_\mu^k(\Omega)$ – Sobolev's space of functions possessing distributive derivatives up to the k-th order in $L_p(\Omega)$,
$|\alpha|$ – total variation of the measure α,
$L_\mu(E)$ – the Banach space of all Borel measures (σ-additive, general measures) defined on the Borel set $E \subset E_N$ and satisfying $\|\alpha\|_{L_\mu(E)} = |\alpha|(E) < \infty$,
$|E|$ – N-dimensional Lebesgue measure of the measurable set $E \subset E_N$,
$\mathrm{d}S$ – $(N-1)$-dimensional Lebesgue measure,
$\int_E \varphi \, \mathrm{d}\alpha$ – Riemann-Stieltjes's integral, where E is a compact in E_N, $\varphi \in C(E)$ and $\alpha \in L_\mu(E)$,
$u_{|E}, \alpha_{|E}$ – the restriction respectively of a function u and of a measure α on the Borel set $E \subset E_N$,
$K^h(x)$ – N-dimensional mollifier.

Important agreement

Each absolutely continuous measure (with respect to the N-dimensional Lebesgue measure) will be identified with its density with respect to Lebesgue measure, i.e., $\alpha \in L_\mu(\bar{\Omega})$ will be identified with the function $u \in L_1(\Omega)$ such that

$$\int_{\bar{\Omega}} \varphi \, \mathrm{d}\alpha = \int_\Omega \varphi u \, \mathrm{d}x \,, \quad \forall \varphi \in C(\bar{\Omega}) \,.$$

Each absolutely continuous measure $\beta \in L_\mu(\partial\Omega)$ will be identified with its density with respect to the measure $\mathrm{d}S$ on $\partial\Omega$ i.e. with such function $u' \in L_1(\partial\Omega)$ that

$$\int_{\partial\Omega} \varphi \,\mathrm{d}\beta = \int_{\partial\Omega} \varphi u' \,\mathrm{d}S\,, \quad \forall\varphi \in C(\partial\Omega)\,.$$

This identification will be used throughout the whole paper.

§1. SPACE W_μ^1

1. Definition of spaces W_μ^1 and M^1

First of all let us recall some well-known notions and theorems (see [1]).

Let E be a compact in E_N. A sequence $\alpha_n \in L_\mu(E)$, $n = 1, 2, \ldots$ is said to be w^*-convergent to $\alpha \in L_\mu(E)$, if

$$\int_E \varphi \,\mathrm{d}\alpha_n \to \int_E \varphi \,\mathrm{d}\alpha\,, \quad \forall\varphi \in C(E)\,.$$

This convergence will be denoted by $\rightharpoonup$.

The following assertions hold (see [1]):

1) $\alpha_n \rightharpoonup \alpha$ in the space $L_\mu(E)$ iff
   (i) there exists $k > 0$ such that $\|\alpha_n\|_{L_\mu(E)} \leqq k$, $n = 1, 2, \ldots$
   (ii) $\int_E \varphi \,\mathrm{d}\alpha_n \to \int_E \varphi \,\mathrm{d}\alpha$ for each $\varphi \in X$, X being a dense subset of $C(E)$.
2) If $\alpha_n \in L_\mu(E)$, $n = 1, 2, \ldots$ are from a ball in $L_\mu(E)$, then there exists a subsequence $\{\alpha_{n_k}\}$, which is w^*-convergent in $L_\mu(E)$.
3) If $\alpha_n \rightharpoonup \alpha$ in $L_\mu(E)$, then $\|\alpha\|_{L_\mu(E)} \leqq \varliminf_{n\to\infty} \|\alpha_n\|_{L_\mu(E)}$.
4) The space $L_\mu(E)$ is the dual space to $C(E)$ with respect to the duality $\alpha(\varphi) = \int_E \varphi \,\mathrm{d}\alpha$, $\alpha \in L_\mu(E)$, $\varphi \in C(E)$.

Definition 1. $W_\mu^1(\bar\Omega)$ is the space of all $(N+1)$-tuples $(\alpha_0, \ldots, \alpha_N) \in [L_\mu(\bar\Omega)]^{N+1}$ for which a sequence $u_n \in W_1^1(\Omega)$, $n = 1, 2, \ldots$ exists such that

$$(1) \qquad u_n \rightharpoonup \alpha_0\,, \quad u_{nx_i} \rightharpoonup \alpha_i$$

in the space $L_\mu(\bar\Omega)$, $i = 1, \ldots, N$, where $u_{nx_i} = \partial u_n/\partial x_i$ and the functions u_n, u_{nx_i} are identified with the absolute continuous measures according to the agreement. $\mathring{W}_\mu^1(\bar\Omega)$ is the space of all $(\alpha_0, \ldots, \alpha_N) \in [L_\mu(\bar\Omega)]^{N+1}$ for which there exists $u_n \in \mathring{W}_1^1(\Omega)$, $n = 1, 2, \ldots$ satisfying (1).

If the w^*-convergence by components is introduced in the space $[L_\mu(\bar\Omega)]^{N+1}$, then W_μ^1 is the "closure" of W_1^1 with respect to the w^*-convergence and $\mathring{W}_\mu^1$ is the „closure" of $\mathring{W}_1^1$. At the same time, W_1^1 is imbedded canonically into $[L_\mu(\bar\Omega)]^{N+1}$ by: $u \in W_1^1 \to (u, u_{x_1}, \ldots, u_{x_N}) \in [L_\mu(\bar\Omega)]^{N+1}$.

Theorem 1. *Suppose* $(\alpha_0, \ldots, \alpha_N) \in W_\mu^1(\bar{\Omega})$. *Then*

(i) $\alpha_{i|\Omega} = \partial\alpha_0/\partial x_i$, $i = 1, \ldots, N$ *in the sense of distributions,*

(ii) *the measure* α_0 *is absolutely continuous with respect to the Lebesgue measure. Let us denote its density by* $u \in L_1(\Omega)$,

(iii) *there exists a unique measure* $\beta \in L_\mu(\partial\Omega)$ *such that Green's theorem holds:*

$$(2) \qquad \int_{\partial\Omega} \varphi v_i \, d\beta = \int_\Omega u\varphi_{x_i} \, dx + \int_{\bar{\Omega}} \varphi \, d\alpha_i \,, \quad \forall\varphi \in C^1(\bar{\Omega}) \,, \quad i = 1, \ldots, N \,.$$

The measures α_i *are called the derivatives of the element* (u, α_i) *and the measure* β *is called the trace of the element* (u, α_i). *Analogously to the space* $W_1^1(\Omega)$, *the elements of* $W_\mu^1(\bar{\Omega})$ *are called functions.*

Proof. According to the definition of $W_\mu^1(\bar{\Omega})$ there exists a sequence $u_n \in W_1^1(\Omega)$, $n = 1, 2, \ldots$ such that (1) is satisfied. Let $u_n' \in L_1(\partial\Omega)$ denote the traces of u_n. Considering [2], we obtain that the functions u_n satisfy Green's theorem

$$(3) \qquad \int_{\partial\Omega} u_\mu' \varphi v_i \, dS = \int_\Omega u_{nx_i} \varphi \, dx + \int_\Omega u_n \varphi_{x_i} \, dx \,, \quad \forall\varphi \in C^1(\bar{\Omega}) \,.$$

If we substitute functions from $C_0^1(\Omega)$ for φ in (3), we obtain with respect to (1)

$$\int_{\bar{\Omega}} \varphi_{x_i} \, d\alpha_0 = - \int_{\bar{\Omega}} \varphi \, d\alpha_i \,, \quad i = 1, \ldots, N$$

and assertion (i) is proved.

With regard to (1), there exists a constant $k > 0$ such that

$$(4) \qquad \|u_n\|_{W_1^1(\Omega)} \leqq k \,, \quad n = 1, 2, \ldots$$

Theorems of imbeddings imply

$$(5) \qquad \|u_n'\|_{L_1(\partial\Omega)} \leqq ck \,, \quad \|u_n\|_{L_q(\Omega)} \leqq ck \,, \quad 1/q = 1 - 1/N \,, \quad n = 1, 2, \ldots$$

There exists a suitable subsequence $\{u_{n_k}\}$, a measure $\beta \in L_\mu(\partial\Omega)$ and $u \in L_q(\Omega)$ such that

$$(6) \qquad u_{n_k}' \rightharpoonup \beta \quad \text{in} \quad L_\mu(\partial\Omega) \,, \quad u_{n_k} \rightharpoonup u \quad \text{in} \quad L_q(\Omega) \,.$$

Due to (1) $u_{n_k} \rightharpoonup \alpha_0$ in $L_\mu(\bar{\Omega})$ and if we pass to the limit with $k \to \infty$ we obtain

$$\int_{\bar{\Omega}} \varphi \, d\alpha_0 = \int_\Omega \varphi u \, dx \,, \quad \forall\varphi \in C(\bar{\Omega}) \,.$$

Thus, assertion (ii) is proved.

Now, we pass to the limit in Green's theorem (3) for u_{n_k}. With regard to (6) we obtain Green's theorem for $(u, \alpha_1, \ldots, \alpha_N)$. It can be seen from (2) that the measure β is independent of the sequence $\{u_n\}$.

The norm in the space $W^1_\mu(\bar{\Omega})$ is defined by

$$\|(u, \alpha_1, \ldots, \alpha_N)\|_{W_\mu^1(\bar{\Omega})} = \|u\|_{L\mu(\Omega)} + \sum_{i=1}^{N} \|\alpha_i\|_{L\mu(\bar{\Omega})} .$$

In the space $W^1_\mu(\bar{\Omega})$ we introduce w^*-convergence by components, i.e. $(u_n, \alpha_{ni}) \rightharpoonup (u, \alpha_i)$ in $W^1_\mu(\bar{\Omega})$ iff $u_n \rightharpoonup u$, $\alpha_{ni} \rightharpoonup \alpha_i$ in $L_\mu(\bar{\Omega})$, $i = 1, \ldots, N$. Now, we shall describe the functions of $W^1_\mu(\bar{\Omega})$ explicitely using Green's theorem (2) for the purpose.

Definition 2. $M^1(\bar{\Omega})$ is the space of all $(N + 2)$-tuples $(u, \alpha_1, \ldots, \alpha_N, \beta)$ for which

(i) $u \in L_1(\Omega)$, $\alpha_1, \ldots, \alpha_N \in L_\mu(\bar{\Omega})$, $\beta \in L_\mu(\partial\Omega)$,

(ii) (2) holds for each $\varphi \in C^1(\bar{\Omega})$.

Let us denote $\mathring{M}^1(\bar{\Omega}) = \{(u, \alpha_i, \beta) \in M^1(\bar{\Omega}); \beta = 0\}$. The norm in the space $M^1(\bar{\Omega})$ is defined by

$$\|(u, \alpha_i, \beta)\|_{M^1(\bar{\Omega})} = \|u\|_{L_1(\Omega)} + \sum_{i=1}^{N} \|\alpha_i\|_{L_\mu(\bar{\Omega})} .$$

From Theorem 2 it will be clear that $\|\beta\|_{L_\mu(\partial\Omega)}$ can be omitted in the formula for the norm it M^1. It can be seen from (2) that the measure β is uniquely determined by the $(N + 1)$-tuple (u, α_i). Therefore (u, α_i) will be written sometimes instead of (u, α_i, β). In this sense $W^1_\mu(\bar{\Omega})$ is a subset of $M^1(\bar{\Omega})$ and $\mathring{W}^1_\mu(\bar{\Omega}) \subset \mathring{M}^1(\bar{\Omega})$. One of the aims of the next section is to prove equalities in these inclusions.

Similarly, in view of (2), the measures $\alpha_1, \ldots, \alpha_N$ are uniquely determined by the pair (u, β). Thus (u, β) can be written instead of (u, α_i, β). The function u uniquely determines the measures α_i in Ω, i.e. the measures $\alpha_{i|\Omega}$. Namely, $\alpha_{i|\Omega}$ are distribution derivatives of u.

The space $W^1_1(\Omega)$ is canonically imbedded into $W^1_\mu(\bar{\Omega})$ by:

$$u \in W^1_1 \to (u, u_{x_1}, \ldots, u_{x_N}) \in W^1_\mu$$

in the sense of our agreement. We introduce w^*-convergence in the space $M^1(\bar{\Omega})$ as the w^*-convergence of the first $(N + 1)$ components.

2. Decomposition of the unit

Definition 3. By the product of $\psi \in C(E)$ ($E \subset E_N$ being a compact) and of a measure $\alpha \in L_\mu(E)$ we understand the measure $\bar{\alpha} = \psi . \alpha \in L_\mu(E)$ defined by

$$\int_E \varphi \, d\bar{\alpha} = \int_E \varphi\psi \, d\alpha , \quad \forall \varphi \in C(E) . \tag{7}$$

By the product of $\psi \in C^1(\bar{\Omega})$ and $(u, \alpha_i, \beta) \in M^1(\bar{\Omega})$ we understand a function $(\bar{u}, \bar{\alpha}_i, \bar{\beta}) = \psi(u, \alpha_i, \beta)$ for which $\bar{u} = u\psi$, $\bar{\alpha}_i = u\psi_{x_i} + \psi\alpha_i$, $\bar{\beta} = \psi_{|\partial\Omega}\beta$ with respect to (7) and to our agreement.

It can be seen easily that $\psi(u, \alpha_i, \beta)$ satisfies (2) and hence it belongs to $M^1(\bar{\Omega})$. At the same time there is

$$\|\psi(u, \alpha_i, \beta)\|_{M^1(\bar{\Omega})} \leqq (N + 1) \|\psi\|_{C^1(\bar{\Omega})} \cdot \|(u, \alpha_i, \beta)\|_{M^1(\bar{\Omega})} . \tag{8}$$

Suppose the domain Ω to be of the class C^1. There exists a finite number of open cubes K_r in E_N, $r = 1, \ldots, R$ covering the boundary $\partial\Omega$. Let us denote $\Omega_r = K_r \cap \Omega$. There exists a domain Ω_0, $\bar{\Omega}_0 \subset \Omega$ such that $\Omega = \bigcup_{r=0}^{R} \Omega_r$.

For each $r \geqq 1$ a linear orthogonal transformation can be carried out such that K_r is of the form $K_r = \{x \in E_N;\ 0 < x_i < b\}$ in the new variables, where b is the edge of the cube K_r. At the same time a part of boundary $\partial\Omega \cap K_r$ can be described by $x_N = a(x_n, \ldots, x_{N-1})$ where a is a function of the class C^1. The cubes K_r and the transformations can be chosen in such manner that

$$\nu_N \geqq c > 0 \tag{9}$$

on $\partial\Omega \cap K_r$ in the new variables.

For the decompsition $\Omega = \bigcup_{r=0}^{R} \Omega_r$ there exist functions $\gamma_r \in C^1(\bar{\Omega})$ such that $\gamma_r \geqq 0$, $\operatorname{supp} \gamma_r \subset \Omega_r \cup \partial\Omega$, $\sum_{r=0}^{R} \gamma_r = 1$ on $\bar{\Omega}$. Suppose $(u, \alpha_i, \beta) \in M^1(\bar{\Omega})$. Then $(u, \alpha_i, \beta) = \sum_{r=0}^{R} (u_r, \alpha_{ri}, \beta_r)$ where

$$(u_r, \alpha_{ri}, \beta_r) = \gamma_r(u, \alpha_i, \beta) \in M^1(\bar{\Omega}) . \tag{10}$$

Due to (8) we obtain

$$\|(u_r, \alpha_{ri}, \beta_r)\|_{M^1(\bar{\Omega})} \leqq c\|(u, \alpha_i, \beta)\|_{M^1(\bar{\Omega})} ,$$

$$\|(u, \alpha_i, \beta)\|_{M^1(\bar{\Omega})} \leqq c \sum_{r=0}^{R} \|(u_r, \alpha_{ri}, \beta_r)\|_{M^1(\bar{\Omega})} . \tag{11}$$

At the same time we obtain

$$\|\beta_r\|_{L_\mu(\partial\Omega)} \leqq c\|\beta\|_{L_\mu(\partial\Omega)} , \quad \|\beta\|_{L_\mu(\partial\Omega)} \leqq c \sum_{r=0}^{R} \|\beta_r\|_{L_\mu(\partial\Omega)} . \tag{12}$$

The function $(u_r, \alpha_{ri}, \beta_r)$ belongs after the application of a linear orthogonal transformation again to $M^1(\bar{\Omega})$ and (11), (12) are satisfied.

3. The direct and inverse theorems on imbedding into the traces

First of all, we must regularise the measure $\beta \in L_\mu(\partial\Omega)$. Let us set for $x \in E_N$

$$R^h(x) = \frac{1}{\varkappa h^{N-1}} e^{|x|^2/(|x|^2 - h^2)}, \quad |x| < h ,$$
$$R^h(x) = 0 , \qquad |x| \geqq h . \tag{13}$$

The constant $\varkappa$ is chosen so that

$$\int_{E_{N-1}} R^h(x_1, \ldots, x_{N-1}, 0)\, dx_1, \ldots, dx_{N-1} = 1$$

holds.

Suppose $\varphi \in C(\partial\Omega)$, $\beta \in L_\mu(\partial\Omega)$. Let us set

$$\psi_h(x) = \int_{\partial\Omega} R^h(x - y)\, dS(y)\,, \tag{14}$$

$$\varphi_h(x) = \int_{\partial\Omega} R^h(x - y)\, \varphi(y)\, dS(y)\,, \tag{15}$$

$$u'_h(x) = \int_{\partial\Omega} R^h(x - y)\, d\beta(y)\,. \tag{16}$$

Now, we prove a lemma which will turn out very useful.

Lemma 1. (i) *There exists a function $c(h)$, $h > 0$ depending only on the domain Ω and satisfying*

$$\max_{x \in \partial\Omega} |\psi_h(x) - 1| \leqq c(h)\,, \quad \lim_{h \to 0} c(h) = 0\,,$$

(ii) *$\varphi_h \to \varphi$ in $C(\partial\Omega)$,*

(iii) *$u'_h \rightharpoonup \beta$ in $L_\mu(\partial\Omega)$ and at the same time*

$$\|u'_h\|_{L_1(\partial\Omega)} \to \|\beta\|_{L_\mu(\partial\Omega)}\,.$$

Proof. First we prove that $\lim_{h \to 0} \min_{\partial\Omega} \psi_h \geqq 1$. Easily we find out that ψ_h is continuous on $\partial\Omega$. For each $h > 0$ there exists $x_h \in \partial\Omega$ such that $\min_{\partial\Omega} \psi_h = \psi_h(x_h)$. Let us suppose, on the contrary, that $\lim_{h \to 0} \psi_h(x_h) < 1$. Thus there exist $h_n \to 0$ and $\varepsilon_0 > 0$ such that

$$x_{h_n} \to x_0 \in \partial\Omega\,, \quad \psi_{h_n}(x_{h_n}) \leqq 1 - \varepsilon_0\,, \quad n = 1, 2, \ldots$$

There exists a cube K and a suitable linear orthogonal transformation such that $\partial\Omega \cap K$ is described in new variables by $x_N = a(x')$, $x' = (x_1, \ldots, x_{N-1})$ and at the same time

$$x_0 \in \partial\Omega \cap K\,, \quad a_{x_i}(x'_0) = 0\,, \quad i = 1, \ldots, N - 1\,. \tag{17}$$

For large n there is

$$\psi_{h_n}(x_{h_n}) = \int_{E_{N-1}} R^{h_n}(x'_{h_n} - y'\,,\ a(x'_{h_n}) - a(y'))\, \sqrt{(1 + |\nabla a(y')|^2)}\, dy'\,.$$

Applying the substitution $y' - x'_{h_n} = h_n(z' - x'_{h_n})$ we obtain

$$\text{(18)} \qquad \psi_{h_n}(x_{h_n}) = \int_{E_{N-1}} R^1\Big(z' - x'_{h_n}, \frac{1}{h_n}\big[a(x'_{h_n} + h_n(z' - x'_{h_n})) - a(x'_{h_n})\big]\Big) \cdot$$
$$\cdot (1 + |\nabla a(x'_{h_n} + h_n(z' - x'_{h_n}))|^2)^{1/2}\, dz'.$$

For z' fixed there holds with regard to (17) and to the fact that $a \in C^1$

$$\frac{1}{h_n}\big[a(x'_{h_n} + h_n(z' - x'_{h_n})) - a(x'_{h_n})\big] =$$
$$= \int_0^1 \sum_{i=1}^{N-1} a_{x_i}(x'_{h_n} + 1h_n(z' - x'_{h_n}))\,(z - x'_{h_n i})\, dt \underset{n\to\infty}{\to} 0.$$

Now, let n increase to infinity in the integral (18). Thus we obtain $\lim_{n\to\infty} \psi_{h_n}(x_{h_n}) = \int_{E_{N-1}} R^1(z' - x'_0, 0)\, dz' = 1$ which is a contradiction.

Similarly we prove the inverse inequality $\overline{\lim}_{h\to 0} \max_{\partial\Omega} \psi_h \leqq 1$. We can set $c(h) = \max_{\partial\Omega} |\psi_h - 1|$ and thus assertion (i) is proved.

$$|\varphi_h(x) - \varphi(x)| \leqq \Big|\int_{\partial\Omega} R^h(x - y)\,[\varphi(y) - \varphi(x)]\, dS(y) +$$
$$+ \Big[\int_{\partial\Omega} R^h(x - y)\, dS(y) - 1\Big]\varphi(x)\Big| \leqq$$
$$\leqq (1 + c(h))\,\delta(h) + \|\varphi\|_{C(\partial\Omega)}\, c(h)$$

where $\delta(h)$ is the modul of continuity for φ. Hence we conclude (ii).

$$\int_{\partial\Omega} u'_h \varphi\, dS = \int_{\partial\Omega}\int_{\partial\Omega} R^h(x - y)\,\varphi(x)\, d\beta(y)\, dS(x) =$$
$$= \int_{\partial\Omega} \varphi_h(y)\, d\beta(y) \underset{h\to 0}{\to} \int_{\partial\Omega} \varphi(y)\, d\beta(y).$$

This fact implies $u_h \rightharpoonup \beta$ and hence $\|\beta\|_{L_\mu(\partial\Omega)} \leqq \lim_{h\to 0} \|u'_h\|_{L_1(\partial\Omega)}$,

$$\|u'_h\|_{L_1(\partial\Omega)} = \int_{\partial\Omega} |u'_h|\, dS \leqq \int_{\partial\Omega}\int_{\partial\Omega} R^h(x - y)\, d|\beta|\,(y)\, dS(x) \leqq$$
$$\leqq \int_{\partial\Omega} \psi_h(y)\, d|\beta|\,(y) \leqq (1 + c(h))\,\|\beta\|_{L_\mu(\partial\Omega)}.$$

Lemma 1 is proved.

Theorem 2. (i) *The imbedding* $M^1(\bar{\Omega}) \to L_\mu(\partial\Omega)$ *is continuous, i.e. for each* $(u, \alpha_i, \beta) \in M^1(\bar{\Omega})$ *there is*

$$\|\beta\|_{L_\mu(\partial\Omega)} \leqq c\|(u, \alpha_i, \beta)\|_{M^1(\bar{\Omega})} \, .$$

(ii) *The imbedding* $M^1(\bar{\Omega}) \to L_\mu(\partial\Omega)$ *is w*-continuous, i.e.* $(u_n, \alpha_{n_i}, \beta_n) \rightharpoonup (u, \alpha_i, \beta)$ *in* $M^1(\bar{\Omega})$ *implies that*

$$\beta_n \rightharpoonup \beta \quad in \quad L_\mu(\partial\Omega) \, .$$

(iii) *For each measure* $\beta \in L_\mu(\partial\Omega)$ *there exists a function* $(u, \alpha_i) \in W^1_\mu(\bar{\Omega})$ *such that* β *is the trace of* (u, α_i) *and*

$$\|(u, \alpha_i)\|_{W_\mu{}^1(\bar{\Omega})} \leqq c\|\beta\|_{L_\mu(\partial\Omega)} \, .$$

Proof. Suppose $(u, \alpha_i, \beta) \in M^1(\bar{\Omega})$. We conclude from the section on decomposition of the unit that it suffices to prove assertion (i) for a domain of the form $\Omega = \Omega_r$, $r = 1, \ldots, R$ and for a function $(u, \alpha_i, \beta) = (u_r, \alpha_{ri}, \beta_r)$ whose support is in $\Omega_r \cup \cup (\partial\Omega \cap K_r)$. Let us set

$$B = \{\varphi \in C^1(\partial\Omega \cap \bar{K}_r); \; \|\varphi\|_{C(\partial\Omega \cap \bar{K}_r)} \leqq 1\} \, .$$

An arbitrary $\varphi \in B$ will be extended on Ω_r as a constant on the lines parallel to the coordinate axis x_N (in the new variables). For such φ there holds

$$\int_{\partial\Omega \cap K_r} \varphi \nu_N \, d\beta = \int_{\Omega_r} \varphi_{x_N} u \, dx + \int_{\bar{\Omega}_r} \varphi \, d\alpha_N = \int_{\bar{\Omega}_r} \varphi \, d\alpha_N \, .$$

According to (9) there holds $\nu_N \geqq c > 0$ and hence we conclude

$$\|\beta\|_{L_\mu(\partial\Omega \cap \bar{K}_r)} = \sup_{\varphi \in B} \int \varphi \, d\beta \leqq c\|(u, \alpha_i)\|_{M^1(\bar{\Omega}_r)} \, .$$

Thus, assertion (i) is proved.

Suppose $(u_n, \alpha_{ni}, \beta_n) \rightharpoonup (u, \alpha_i, \beta)$ in $M^1(\bar{\Omega})$. Let us take $\varphi \in C^1(\bar{\Omega})$ and substitute it into (2).

$$\int_{\partial\Omega} \varphi \nu_i \, d\beta_n = \int_\Omega u_n \varphi_{x_i} \, dx + \int_{\bar{\Omega}} \varphi \, d\alpha_{ni} \underset{n \to \infty}{\to} \int_\Omega u \varphi_{x_i} \, dx + \int_{\bar{\Omega}} \varphi \, d\alpha_i = \int_{\partial\Omega} \varphi \nu_i \, d\beta \, .$$

The linear hull of the set of functions possessing the form $\varphi_{|\partial\Omega} \nu_i$, $\varphi \in C^1(\bar{\Omega})$, $i = 1, \ldots, N$ is a dense set in $C(\partial\Omega)$. It is sufficient to prove that the norms of β_n, $n = 1, 2, \ldots$ are bounded. $(u_n, \alpha_{ni}) \rightharpoonup (u, \alpha_i)$ implies that $\|(u_n, \alpha_{ni})\|_{M^1(\bar{\Omega})}$ are bounded and thus assertion (ii) will be proved by using assertion (i).

Suppose $\beta \in L_\mu(\partial\Omega)$ and $u'_h \in L_1(\partial\Omega)$ is a function from (16). According to Gagliardo's work [3] there exist functions $u_h \in W^1_1(\Omega)$, $h > 0$ such that u'_h is the trace of u_h and the estimate

(19) $$\|u_h\|_{W_1{}^1(\Omega)} \leqq c\|u'_h\|_{L_1(\partial\Omega)}$$

is satisfied. Considering (iii) from Lemma 1 we conclude

(20) $$\|u_h\|_{W_1{}^1(\Omega)} \leqq c\|\beta\|_{L_\mu(\partial\Omega)}$$

for small $h > 0$.

There exists a suitable subsequence $\{u_{h_n}\}_{h_n \to 0}$ and $(u, \alpha_i) \in W^1_\mu(\bar{\Omega})$ such that $u_{h_n} \underset{n\to\infty}{\rightharpoonup} (u, \alpha_i)$ in $W^1_\mu(\bar{\Omega})$. By limiting process in Green's theorem for u_{h_n} we conclude that (u, α_i) possesses the trace β and with respect to (20)

$$\|(u, \alpha_i)\|_{W_\mu{}^1(\bar{\Omega})} \leqq \underline{\lim}_{u\to\infty} \|u_{h_n}\|_{W_1{}^1(\Omega)} \leqq c\|\beta\|_{L_\mu(\partial\Omega)}$$

holds. Theorem 2 is proved.

From the next theorem it will be seen that the trace β of a function $(u, \alpha_i, \beta) \in M^1(\bar{\Omega})$ is independent of the function u itself.

Theorem 3. *Let us set for $u \in L_1(\Omega)$*

(21) $$d[u] = \sup\left\{\left|\int_\Omega u\varphi_{x_i}\,\mathrm{d}x\right|;\quad \varphi \in C^1(\bar{\Omega}),\quad \|\varphi\|_{C(\bar{\Omega})} \leqq 1,\quad i = 1, \ldots, N\right\}.$$

A pair (u, β) is from $M^1(\bar{\Omega})$ iff $\beta \in L_\mu(\partial\Omega)$, $u \in L_1(\Omega)$ and $d[u] < \infty$. Moreover

(22) $$\|(u, \beta)\|_{M^1(\bar{\Omega})} \leqq \|u\|_{L_1(\Omega)} + N(d[u] + \|\beta\|_{L_\mu(\partial\Omega)}),$$

(23) $$d[u] \leqq c\|(u, \beta)\|_{M^1(\bar{\Omega})}$$

hold. These facts imply in particular that $(0, \beta)$ is in $M^1(\bar{\Omega})$ for an arbitrary $\beta \in L_\mu(\partial\Omega)$.

Proof. We shall construct measures $\alpha_i \in L_\mu(\bar{\Omega})$ so that $(u, \alpha_i, \beta) \in M^1(\bar{\Omega})$. For $\varphi \in C^1(\bar{\Omega})$ we set

$$\int_{\bar{\Omega}} \varphi\,\mathrm{d}\alpha_i = \int_{\partial\Omega} \varphi\nu_i\,\mathrm{d}\beta - \int_\Omega u\varphi_{x_i}\,\mathrm{d}x, \quad i = 1, \ldots, N.$$

The measure α_i is defined by this formula as a functional on $C^1(\bar{\Omega})$. In order to prove that α_i is a measure, we must prove

$$\sup\left\{\left|\int_{\bar{\Omega}} \varphi\,\mathrm{d}\alpha_i\right|;\quad \varphi \in C^1(\bar{\Omega}),\quad \|\varphi\|_{C(\bar{\Omega})} \leqq 1\right\} < \infty.$$

For $\varphi \in C^1(\bar{\Omega})$, $\|\varphi\|_{C(\bar{\Omega})} \leqq 1$ there holds

$$\left|\int_{\bar{\Omega}} \varphi\,\mathrm{d}\alpha_i\right| \leqq \|\beta\|_{L_\mu(\partial\Omega)} + d[u].$$

Hence the estimate (22) follows. On the contrary, let us suppose $(u, \alpha_i, \beta) \in M^1(\bar{\Omega})$. Then for $\varphi \in C^1(\bar{\Omega})$, $|\varphi| \leqq 1$ on $\bar{\Omega}$ there is

$$\left|\int_\Omega u\varphi_{x_i}\,\mathrm{d}x\right| = \left|\int_{\partial\Omega} \varphi\nu_i\,\mathrm{d}\beta - \int_{\bar{\Omega}} \varphi\,\mathrm{d}\alpha_i\right| \leqq$$
$$\leqq \|\beta\|_{L_\mu(\partial\Omega)} + \|\alpha_i\|_{L_\mu(\bar{\Omega})} \leqq C\|(u, \alpha_i)\|_{M^1(\bar{\Omega})} < \infty\,.$$

4. The equality $W^1_\mu = M^1$

Let us set

(24) $$K^h(x) = \frac{1}{\varkappa h^N}\,\mathrm{e}^{\,|x|^2/(|x|^2-h^2)}\,, \quad |x| < h\,,$$
$$K^h(x) = 0\,, \qquad |x| \geqq h$$

where $\varkappa$ is a constant satisfying $\int_{E_N} K^h(x)\,\mathrm{d}x = 1$. Let us denote

(25) $$S_h = \{x \in \Omega;\ \operatorname{dist}(x, \partial\Omega) < h\}\,.$$

Lemma 2. *Let us suppose $(u, \alpha_i, \beta) \in \mathring{M}^1(\bar{\Omega})$, i.e. $\beta = 0$. Then for each $h > 0$ with $h < c$ there holds*

(26) $$\int_{S_h} |u|\,\mathrm{d}x \leqq ch\|(u, \alpha_i)\|_{M^1(\bar{\Omega})}\,.$$

Proof. We decompose the function $(u, \alpha_i, 0)$ using the decomposition of the unit

$$(u, \alpha_i) = \sum_{r=0}^{R} (u_r, \alpha_{ri}).$$

From Section 2 it is clear that it suffices to prove Lemma 2 in the case when $\Omega = \Omega_r$, $(u, \alpha_i) = (u_r, \alpha_{ri})$, $r = 1, \ldots, R$ and (u, α_i) has its support in $\Omega_r \cup (\partial\Omega \cap K_r)$.

First of all we prove the following assertion:

(27) $$u \in L_1(S_h) \Rightarrow \int_{S_h} |u|\,\mathrm{d}x = \sup\left\{\int_\Omega u\psi\,\mathrm{d}x\,;\ \ \psi \in C_0^\infty(S_h)\,,\ \ \|\psi\|_{C(S_h)} \leqq 1\right\}.$$

Let ε be a positive number. Then there exists $\Omega' \subset S_h$, $\bar{\Omega}' \subset S_h$ such that $\int_{S_h - \Omega'} |u|\,\mathrm{d}x < \varepsilon$.

Let us set

$$\psi = \operatorname{sign} u \quad \text{on} \quad \Omega'\,, \quad \psi = 0 \quad \text{on} \quad E_N - \Omega'\,.$$

Obviously $\int_{\Omega'} |u|\,\mathrm{d}x = \int_{\Omega'} u\psi\,\mathrm{d}x$ holds. Let us set for $k > 0$

(28) $$\psi_k(x) = \int_{E_N} K^k(x - y)\,\psi(y)\,\mathrm{d}y\,.$$

For small K function ψ_k belongs to $C_0^\infty(S_h)$. Then $\psi_k \to \psi$ a.e. on Ω' and

$$\|\psi\|_{C(S_h)} \leqq 1\,, \quad \int_{\Omega'} u\psi_k \,\mathrm{d}x \underset{k\to 0}{\to} \int_{\Omega'} u\psi \,\mathrm{d}x\,.$$

For small $k > 0$ there holds

$$\int_{S_h} u\psi_k \,\mathrm{d}x \geqq \int_{\Omega'} u\psi_k \,\mathrm{d}x - \varepsilon \geqq \int_{\Omega'} u\psi \,\mathrm{d}x - 2\varepsilon =$$
$$= \int_{\Omega'} |u| \,\mathrm{d}x - 2\varepsilon \geqq \int_{S_h} |u| \,\mathrm{d}x - 3\varepsilon\,.$$

Now, we can prove Lemma 2. Let us take $\psi \in C^1(\bar{\Omega})$ with $\psi = 0$ on $\Omega - S_h$ and $\|\psi\|_{C(\bar{\Omega})} \leqq 1$. Let us denote

$$\varphi(x', x_N) = \int_0^{x_N} \psi(x', \xi) \,\mathrm{d}\xi\,.$$

Then $\|\varphi\|_{C(\bar{\Omega})} \leqq ch$ holds and with respect to (2) we obtain

$$\int_{S_h} u\psi \,\mathrm{d}x = \int_{S_h} u\varphi_{x_N} \,\mathrm{d}x = -\int_{S_h} \varphi \,\mathrm{d}\alpha_N \leqq ch\|(u, \alpha_i)\|_{M^1(\bar{\Omega})}\,.$$

Theorem 4. (i) *For each* $(u, \alpha_i) \in M^1(\bar{\Omega})$ *there exist* $u_n \in W_1^1(\Omega)$, $u = 1, 2, \ldots$ *such that*

(29) $$u_n \rightharpoonup (u, \alpha_i) \quad in \quad M^1(\bar{\Omega})\,,$$

(30) $$\|u_n\|_{W_1{}^1(\Omega)} \leqq c\|(u, \alpha_i)\|_{M^1(\bar{\Omega})}\,.$$

(ii) *For each* $(u, \alpha_i) \in \mathring{M}^1(\bar{\Omega})$ *there exist* $u_n \in \mathring{W}_1^1$ *such that* (29) *and* (30) *hold. This fact imply the equalities* $M^1(\bar{\Omega}) = W_\mu^1(\bar{\Omega})$ *and* $\mathring{M}^1(\bar{\Omega}) = \mathring{W}_\mu^1(\bar{\Omega})$.

Proof. We prove assertion (ii). Let us denote

(31) $$\Omega_h = \{x \in \Omega;\ \mathrm{dist}\,(x, \partial\Omega) > h\} = \Omega - \bar{S}_h\,.$$

There exist functions $\psi_h \in C_0^\infty(\Omega)$, $h > 0$ (h being small) with properties

(32) $$0 \leqq \psi_h \leqq 1 \quad \text{on} \quad \Omega\,, \qquad \psi_h = 1 \quad \text{on} \quad \Omega_{4h}\,, \qquad \psi_h = 0 \quad \text{on} \quad S_{3h}\,,$$

$$\max_\Omega |\psi_{hx_i}| \leqq \frac{c}{h}\,, \quad i = 1, \ldots, N\,.$$

Let us set

(33) $$u_h(x) = \int_\Omega K^h(x - \xi)\, u(\xi)\, \psi_h(\xi) \,\mathrm{d}\xi\,.$$

Evidently u_h belongs to $C_0^\infty(\Omega)$. We prove the inequality

$$\|u_h\|_{W_1{}^1(\Omega)} \leqq c\|(u, \alpha_i)\|_{M^1(\bar{\Omega})} . \tag{34}$$

The assertion

$$\int_\Omega |u_h| \, \mathrm{d}x \leqq \int_\Omega \int_\Omega K^h(x - \xi) \, |u(\xi)| \, \psi_h(\xi) \, \mathrm{d}\xi \, \mathrm{d}x \leqq \int_\Omega |u| \, \mathrm{d}\xi$$

holds. With respect to (2) and $(u, \alpha_i) \in \mathring{M}^1$ we obtain for $x \in \Omega$

$$u_{hx_i}(x) = - \int_\Omega K^h_{\xi_i}(x - \xi) \, u(\xi) \, \psi_h(\xi) \, \mathrm{d}\xi =$$

$$= \int_\Omega K^h(x - \xi) \, u(\xi) \, \psi_{h\xi_i}(\xi) \, \mathrm{d}\xi - \int_\Omega [K^h(x - \xi) \, \psi_h(\xi)]_{\xi_i} \, u(\xi) \, \mathrm{d}\xi =$$

$$= \int_\Omega K^h(x - \xi) \, u(\xi) \, \psi_{h\xi_i}(\xi) \, \mathrm{d}\xi + \int_{\bar{\Omega}} K^h(x - \xi) \, \psi_h(\xi) \, \mathrm{d}\alpha_i(\xi) .$$

Further, we use Lemma 2 and assumption (32):

$$\int_\Omega |u_{hx_i}| \, \mathrm{d}x \leqq \int_{\bar{\Omega}} \mathrm{d}|\alpha_i| \, (\xi) + \int_\Omega |u(\xi)| \, |\psi_{h\xi_i}(\xi)| \, \mathrm{d}\xi \leqq$$

$$\leqq \|(u, \alpha_i)\|_{M^1(\bar{\Omega})} + \frac{c}{h} \int_{S_{4h}} |u| \, \mathrm{d}\xi \leqq c\|(u, \alpha_i)\|_{M^1(\bar{\Omega})}$$

which proves the estimate (34).

Easily we find out that $u_h \to u$ in $L_1(\Omega)$. For $\varphi \in C(\bar{\Omega})$ we obtain

$$\int_\Omega u_h \varphi \, \mathrm{d}x = \int_\Omega \int_\Omega K^h(x - \xi) \, \varphi(x) \, u(\xi) \, \psi_h(\xi) \, \mathrm{d}\xi \, \mathrm{d}x ,$$

$$\int_\Omega K^h(x - \xi) \, \varphi(x) \, \mathrm{d}x \to \varphi(\xi) \quad \text{in} \quad L_1(\Omega)$$

and hence

$$\left[\int_\Omega K^h(x - \xi) \, \varphi(x) \, \mathrm{d}x\right] u(\xi) \, \psi_h(\xi) \to u(\xi) \, \varphi(\xi) \quad \text{a.e. in} \quad \Omega , \quad \text{i.e.}$$

$$\int_\Omega u_h(x) \, \varphi(x) \, \mathrm{d}x \to \int_\Omega u(x) \, \varphi(x) \, \mathrm{d}x .$$

Suppose $\varphi \in C^1(\bar{\Omega})$. On account of (2) we have

$$\int_\Omega u_{hx_i} \varphi \, \mathrm{d}x = - \int_\Omega u_h \varphi_{x_i} \, \mathrm{d}x \to - \int_\Omega u \varphi_{x_i} \, \mathrm{d}x = \int_{\bar{\Omega}} \varphi \, \mathrm{d}\alpha_i .$$

The estimate (34) implies the assertion (29) and (ii) is proved.

Suppose $(u, \alpha_i, \beta) \in M^1(\bar{\Omega})$. Assertion (iii) from Theorem 2 implies that there exists $(u^1, \alpha_i^1, \beta) \in W_\mu^1(\bar{\Omega})$. In the proof of the Theorem 2 we found $u_\mu^1 \in W_1^1(\Omega)$, $n = 1, 2, \ldots$ satisfying with respect to (20)

$$(35) \qquad u_n^1 \rightharpoonup (u^1, \alpha_i^1, \beta) \quad \text{in} \quad W_\mu^1(\bar{\Omega}), \quad \|u_n^1\|_{W_1^1(\Omega)} \leqq C\|\beta\|_{L_\mu(\partial\Omega)}.$$

Function $(u^2, \alpha_i^2, 0) = (u, \alpha_i, \beta) - (u^1, \alpha_i^1, \beta)$ belongs to the space $\mathring{M}^1(\bar{\Omega})$. Making use of Theorem 2 we obtain the estimate

$$(36) \qquad \|(u^2, \alpha_i^2)\|_{M^1(\bar{\Omega})} \leqq c\|(u, \alpha_i)\|_{M^1(\bar{\Omega})}.$$

On account of the assertion (ii) just proved there exist $u_n^2 \in \mathring{W}_1^1(\Omega)$

$$(37) \qquad u_n^2 \rightharpoonup (u^2, \alpha_i^2, 0) \quad \text{in} \quad M^1(\bar{\Omega}), \quad \|u_n^2\|_{W_1^1} \leqq c\|(u^2, \alpha_i^2)\|_{M^1(\bar{\Omega})}.$$

Relations (35), (36), (37) and Theorem 2 yield

$$u_n^1 + u_n^2 \rightharpoonup (u, \alpha_i) \quad \text{in} \quad M^1(\bar{\Omega})$$
$$\|u_n^1 + u_n^2\|_{W_1^1(\Omega)} \leqq c\|(u, \alpha_i)\|_{M^1(\Omega)}.$$

5. Theorems on imbedding and on w^*-compactness of the ball in W_μ^1

Theorems on imbedding $W_\mu^1(\bar{\Omega})$ into $L_q(\Omega)$ are the same as those for the space $W_1^1(\Omega)$.

Theorem 5. *Suppose* $(u, \alpha_i) \in W_\mu^1(\bar{\Omega})$. *Then* $u \in L_q(\Omega)$ *and the following estimate is valid*:

$$(38) \qquad \|u\|_{L_q(\Omega)} \leqq c\|(u, \alpha_i)\|_{W_\mu^1(\bar{\Omega})}, \quad \frac{1}{q} = 1 - \frac{1}{N}.$$

The imbeding $W_\mu^1(\bar{\Omega}) \to L_{q^*}(\Omega)$ *is compact for* $q^* < q$, $q^* \geqq 1$.

Proof. According to Theorem 4 there exist $u_n \in W_1^1$ such that

$$u_n \rightharpoonup (u, \alpha_i) \quad \text{in} \quad W_\mu^1(\bar{\Omega}), \quad \|u_n\|_{W_1^1} \leqq C\|(u, \alpha_i)\|_{W_\mu^1}.$$

On account of the theorem on imbedding $W_1^1(\Omega) \to L_q(\Omega)$ we obtain

$$(39) \qquad \|u_n\|_{L_q(\Omega)} \leqq c\|(u, \alpha_i)\|_{W_\mu^1(\bar{\Omega})}.$$

From the convergence $u_n \rightharpoonup u$ in $L_\mu(\bar{\Omega})$ and from (39) we have $u \in L_q$. It is sufficient to choose $u_{n_k} \rightharpoonup \bar{u}$ in $L_{q(\Omega)}$ which implies $u = \bar{u}$, $u_n \rightharpoonup u$ in $L_q(\Omega)$.

Hence

$$\|u\|_{L_q} \leqq \lim \|u_n\|_{L_q} \leqq c\|(u, \alpha_i)\|_{W_\mu^1}$$

holds.

Now we prove the compactness of the imbedding of W_μ^1 into L_{q^*}. Let us suppose that the norms of $(u_n, \alpha_{ni}) \in W_\mu^1$ are bounded. Due to Theorem 2 there exist $u_{nk} \in W_1^1$ such that

$$(40) \qquad u_{nk} \underset{k\to\infty}{\rightharpoonup} (u_n, \alpha_{ni}) \quad \text{in} \quad W_\mu^1\,, \quad \|u_{nk}\|_{W_1{}^1} \leqq c\,, \quad n, k = 1, 2, \ldots$$

For each n there exists a subsequence $\{u_{nk_m}\}_{m=1}^\infty$ which is convergent in L_{q^*}, and it must converge to $u_n \in L_{q^*}$, because $u_{nk} \underset{k\to\infty}{\rightharpoonup} u_n$ in $L_\mu(\bar\Omega)$. For each n there exists an index m such that $v_n = u_{nk_m}$ satisfies

$$(41) \qquad \|v_n - u_n\|_{L_q{}^*} \leqq \frac{1}{n}\,.$$

With respect to (40) there exists a subsequence $\{v_{n_k}\}$ converging in L_{q^*} (due to the theorem on imbedding $W_1^1 \to L_{q^*}$). This implies that the subsequence $\{u_{n_k}\}$ is convergent in L_{q^*} regarding (41).

Theorem 6. *The space $W_\mu^1(\bar\Omega)$ is closed with respect to the w*-topology as a subspace of $[L_\mu(\bar\Omega)]^{N+1}$* (see Section 1). *Any ball in the space $W_\mu^1(\bar\Omega)$ is compact in the w*-topology. The same assertions hold also for the space $\mathring{W}_\mu^1(\bar\Omega)$.*

Proof. Firstly, we show that W_μ^1 is closed with respect to w^*-convergence. Suppose $(u_n, \alpha_{ni}) \in W_\mu^1$, $(\alpha_0, \ldots, \alpha_N) \in L_\mu^{N+1}$ and

$$(42) \qquad u_n \rightharpoonup \alpha_0\,, \quad \alpha_{n_i} \rightharpoonup \alpha_i \quad \text{in} \quad L_{n(\bar\Omega)}\,, \quad i = 1, \ldots, N\,.$$

This implies that there exists a constant $K > 0$ such that

$$(43) \qquad \|(u_n, \alpha_{ni})\|_{W_\mu{}^1} \leqq K\,.$$

From Theorem 5 we conclude that there exists a subsequence $\{u_{n_k}\}$ converging to $u \in L_1(\Omega)$ in the L_1-norm. Considerating (42) we obtain $\alpha_0 = u$ (see our agreement). Let us denote by β_n the trace of (u_n, α_{n_i}). With respect to Theorem 2 we have $\|\beta_n\|_{L_\mu(\partial\Omega)} \leqq cK$ and hence there exists a subsequence $\{\beta_{n_k}\}$ such that $\beta_{n_k} \rightharpoonup \beta \in L_\mu(\partial\Omega)$.

By limiting process in Green's theorem for $(u_{n_k}, \alpha_{n_k i}, \beta_{n_k})$ we obtain Green's theorem for (u, α_i, β) on account of (42). Thus $(u, \alpha_i, \beta) \in M^1(\bar\Omega) = W_\mu^1(\bar\Omega)$. Banach's theorem ([4], section V. 4.) implies that $W_\mu^1(\bar\Omega)$ is closed in the w^*-topology. The ball in the space $[L_\mu(\bar\Omega)]^{N+1}$ is compact in the w^*-topology ([4], addition to V.) and this implies the compactness of the ball in $W_\mu^1(\bar\Omega)$. The assertions on the space $\mathring{W}_\mu^1(\bar\Omega)$ are obtained by Theorem 2.

In the space $W_\mu^1(\bar\Omega)$ the theorem on the equivalence of the norms is valid.

Theorem 7. *Suppose $(u, \alpha_i, \beta) \in W_\mu^1(\bar\Omega)$. Then the function*

$$(44) \qquad \|(u, \alpha_i, \beta)\|'_{W_\mu{}^1} = \|\beta\|_{L_\mu(\partial\Omega)} + \sum_{i=1}^N \|\alpha_i\|_{L_\mu(\bar\Omega)}$$

is an equivalent norm in the space $W_\mu^1(\bar\Omega)$.

Proof. *Inequality* $\|(u, \alpha_i)\|'_{W_\mu^1} \leqq c\|(u, \alpha_i)\|_{W_\mu^1}$ follows from Theorem 2. On the contrary, let us suppose that the inequality

$$\|(u, \alpha_i)\|_{W_\mu^1} \leqq c\|(u, \alpha_i)\|'_{W_\mu^1}$$

is not valid. The there exist functions $(u_n, \alpha_{ni}, \beta_n) \in W_\mu^1$ such that

$$\|(u_n, \alpha_{n_i}, \beta_n)\|_{W_\mu^1} = 1\,, \quad n = 1, 2, \ldots, \tag{45}$$

$$\|(u_n, \alpha_{ni}, \beta_n)\|'_{W_\mu^1} \to 0\,, \quad n \to \infty\,. \tag{46}$$

If we choose a suitable subsequence with regard to Theorems 5 and 6, we can suppose

$$(u_n, \alpha_{ni}, \beta_n) \rightharpoonup (u, \alpha_i, \beta) \quad \text{in} \quad W_\mu^1 \tag{47}$$

$$u_n \to u \quad \text{in} \quad L_1\,. \tag{48}$$

From (45) and (46) we conclude that $\|u_n\|_{L_1} \to 1$ and hence, with respect to (48), we obtain $\|u\|_{L_1} = 1$. Theorem 2 and (47) imply $\beta_n \to \beta$ in $L_\mu(\partial\Omega)$. From (46) and (47) we have

$$\|\alpha_i\|_{L_\mu} \leqq \lim \|\alpha_{n_\mu i}\|_L = 0\,, \quad \|\beta\|_{L_\mu(\partial\Omega)} \leqq \lim \|\beta_n\|_{L_\mu(\partial\Omega)} = 0\,.$$

Thus we have

$$\|u\|_{L_1} = 1\,, \quad \alpha_i = 0 \text{ on } \bar{\Omega}\,, \quad i = 1, \ldots, N\,, \quad \beta = 0 \text{ on } \partial\Omega\,. \tag{49}$$

There exists a function $\psi \in C_0^\infty(E_N)$ satisfying $\int_\Omega u\psi \, dx \neq 0$. Easily we find a function $\varphi \in C^\infty(E_N)$ such that $\varphi_{x_1} = \psi$. From Green's theorem we conclude

$$\int_\Omega u\psi \, dx = \int_\Omega u\varphi_{x_1} \, dx = \int_{\partial\Omega} \varphi v_1 \, d\beta - \int_{\bar{\Omega}} \varphi \, d\alpha_i = 0$$

and hence we obtain a contradiction.

6. The sides and the inner traces of functions from W_μ^1

Theorem 8. *Suppose $(u, \alpha_i, \beta) \in W_\mu^1(\bar{\Omega})$. Let us set*

$$\alpha'_i = \alpha_i \text{ on } \partial\Omega\,, \quad \alpha'_i = 0 \text{ on } \Omega\,, \quad \bar{\alpha}_i = \alpha_i - \alpha'_i \text{ on } \bar{\Omega}\,. \tag{50}$$

Then $(u, \bar{\alpha}_i)$, $(0, \alpha'_i) \in W_\mu^1(\bar{\Omega})$.

Proof. Let us suppose $\beta = 0$ and let us denote

$$u_h(x) = \int_\Omega K^h(x - y)\, u(y) \, dy\,, \quad x \in \Omega\,. \tag{51}$$

On the ground of Green's theorem we obtain

$$u_{hx_i}(x) = \int_{\bar{\Omega}} K^h(x - y)\, \mathrm{d}\alpha_i(y)\,, \quad x \in \Omega\,, \quad i = 1, \ldots, N\,. \tag{52}$$

Let us consider $\varphi \in C(\bar{\Omega})$, extending it continuously on E_N. Denote

$$\psi_h(x) = \int_\Omega K^h(x - y)\, \mathrm{d}y\,, \qquad x \in \partial\Omega\,, \tag{53}$$

$$\varphi_h(x) = \int_\Omega K^h(x - y)\, \varphi(y)\, \mathrm{d}y\,, \quad x \in E_N\,, \tag{54}$$

$$\bar{\varphi}_h(x) = \int_{E_N} K^h(x - y)\, \varphi(y)\, \mathrm{d}y\,, \quad x \in E_N\,. \tag{55}$$

Evidently $\bar{\varphi}_h \to \varphi$ in $C(\bar{\Omega})$. From the fact that the domain Ω belongs to the class C^1 we obtain (see Remark below)

$$\psi_h \to \tfrac{1}{2} \quad \text{in} \quad C(\partial\Omega)\,. \tag{56}$$

By the same method as in the proof of Lemma 1 (ii) we obtain

$$\varphi_h \to \tfrac{1}{2}\varphi \quad \text{in} \quad C(\partial\Omega)\,. \tag{57}$$

Now we prove

$$u_h \to (u, \bar{\alpha}_i + \tfrac{1}{2}\alpha_i') \quad \text{in} \quad W_\mu^1(\bar{\Omega})\,. \tag{58}$$

Evidently $u_h \to u$ in $L_1(\Omega)$. Making use of (52) and (57) we conclude

$$\int_\Omega \varphi u_{hx_i}\, \mathrm{d}x = \iint_{\substack{x \in \Omega \\ y \in \bar{\Omega}}} \varphi(x)\, K^h(x - y)\, \mathrm{d}\alpha_i(y)\, \mathrm{d}x =$$

$$= \int_{\partial\Omega} \varphi_h\, \mathrm{d}\alpha_i + \int_\Omega \bar{\varphi}_h\, \mathrm{d}\alpha_i + \int_{S_h} (\varphi_h - \bar{\varphi}_h)\, \mathrm{d}\alpha_i \to$$

$$\to \int_{\partial\Omega} \tfrac{1}{2}\varphi\, \mathrm{d}\alpha_i + \int_\Omega \varphi\, \mathrm{d}\alpha_i = \int_{\bar{\Omega}} \varphi\, \mathrm{d}(\bar{\alpha}_i + \tfrac{1}{2}\alpha_i')\,,$$

because $\varphi_h = \bar{\varphi}_h$ on Ω_h and for small h there is

$$\left| \int_{S_h} (\varphi - \bar{\varphi}_h)\, \mathrm{d}\alpha_i \right| \leqq c \|\varphi\|_{C(\bar{\Omega})} \cdot |\alpha_i|\,(S_h) \to 0\,.$$

Successively we obtain from (58)

$$(u, \bar{\alpha}_i + \tfrac{1}{2}\alpha_i')\,, \quad (0, \tfrac{1}{2}\alpha_i')\,, \quad (0, \alpha_i') \in W_\mu^1$$

and hence

$$(u, \bar{\alpha}_i) = (u, \alpha_i) - (0, \alpha_i') \in W_\mu^1 .$$

In the case $\beta \neq 0$, $(u, \alpha_i, \beta) \in W_\mu^1$ Theorem 3 implies that there exist measures $\tilde{\alpha}_i \in L_\mu(\bar{\Omega})$ such that $(u, \tilde{\alpha}_i, 0) \in W_\mu^1$. At the same time $\tilde{\alpha}_i = \alpha_i$ in Ω. From the facts just proved we conclude $(u, \bar{\alpha}_i) \in W_\mu^1$, where $\bar{\alpha}_i = \tilde{\alpha}_i = \alpha_i$ in Ω, $\bar{\alpha}_i = 0$ on $\partial\Omega$.

Remark. We prove assertion (56). The normal ν is uniformly continuous on $\partial\Omega$. Suppose $x_0 \in \partial\Omega$. We realise a linear orthogonal transformation of coordinate variables such that x_0 will be mapped into point 0 and $x_N = 0$ will be tangent hyperplane with respect to $\partial\Omega$ at the point 0.

For $|x|$ "small" $\partial\Omega$ will be described by $x_N = a(x')$. The normal ν can be expressed by

$$\nu(x', a(x')) = \left(\frac{-a_{x_1}(x')}{(1 + |\nabla a(x')|^2)^{1/2}}, \ldots, \frac{-a_{x_{N-1}}(x')}{(1 + |\nabla a(x')|^2)^{1/2}}, \frac{1}{(1 + |\nabla a(x')|^2)^{1/2}}\right).$$

On account of the uniform continuity of the normal ν we conclude that for "small" $|x|$ even $|\nabla a(x')|$ is "small". From

$$a(x') = \int_0^1 a_{x_i}(tx')\, x_i' \, dt$$

we obtain $-c|x'| \leqq a(x') \leqq c|x'|$ where c is sufficiently "small".

Definition 4. Suppose $(u, \alpha_i, \beta) \in W_\mu^1(\bar{\Omega})$. The measure $\alpha_\nu \in L_\mu(\partial\Omega)$ satisfying

$$(59) \qquad \alpha_\nu = \sum_{i=1}^N \nu_i \alpha_{i|\partial\Omega} \quad \text{i.e.} \quad \int_{\partial\Omega} \varphi \, d\alpha_\nu = \sum_{i=1}^N \int_{\partial\Omega} \varphi \nu_i \, d\alpha_i , \quad \varphi \in C(\partial\Omega),$$

is called the side of the function (u, α_i, β) on $\partial\Omega$.

The trace β^0 of the function $(u, \bar{\alpha}_i) \in W_\mu^1$ from Theorem 8 is called the inner trace of the function (u, α_i, β). It is evident that the measure β^0 is uniquely determined by the function u.

If $\beta = \beta^0$, then $\alpha_i = \bar{\alpha}_i$ must hold, i.e. $\alpha_i = 0$ on $\partial\Omega$ and hence $\alpha_\nu = 0$.

Theorem 9. *Suppose $(u, \alpha_i, \beta) \in W_\mu^1(\bar{\Omega})$, let β^0 be the inner trace of (u, α_i, β) and $\bar{\alpha}_i, \alpha_i'$ the measures from (50), i.e. $(u, \bar{\alpha}_i, \beta^0) \in W_\mu^1$.*

Then $\beta = \beta^0 + \alpha_\nu$ and $\alpha_i = \nu_i \alpha_\nu$ on $\partial\Omega$, i.e.

$$\int_{\partial\Omega} \varphi \, d\alpha_i = \int_{\partial\Omega} \varphi \nu_i \, d\alpha_\nu , \quad \forall \varphi \in C(\partial\Omega), \quad i = 1, \ldots, N .$$

Proof. On account of Theorem 8, $(0, \alpha_i') = (u, \alpha_i) - (u, \bar{\alpha}_i)$ belongs to the space W_μ^1 and hence the function $(0, \alpha_i')$ possesses the trace $\beta - \beta^0$, i.e.

$$(60) \qquad \int_{\partial\Omega} \varphi \nu_i \, d(\beta - \beta^0) = \int_{\partial\Omega} \varphi \, d\alpha_i , \quad \varphi \in C(\partial\Omega)$$

holds with respect to (2). We substitute the function φ by φv_i and then we add (60) for $i = 1, \ldots, N$. Thus we obtain

$$\sum_{i=1}^{N} \int_{\partial\Omega} \varphi v_i^2 \, \mathrm{d}(\beta - \beta^0) = \sum_{i=1}^{N} \int_{\partial\Omega} \varphi v_i \, \mathrm{d}\alpha_i \quad \text{i.e.} \quad \beta - \beta^0 = \alpha_v \, .$$

Formula (60) implies

$$\int_{\partial\Omega} \varphi v_i \, \mathrm{d}\alpha_v = \int_{\partial\Omega} \varphi \, \mathrm{d}\alpha_i \, .$$

Theorem 10. *The inner trace of a function from $W_\mu^1(\bar{\Omega})$ is absolutely continuous with respect to the Lebesgue area measure* $\mathrm{d}S$ *on* $\partial\Omega$.

Proof. According to the definition we can suppose

$$(u, \alpha_i, \beta^0) \in W_\mu^1 \, , \quad \alpha_i = 0 \quad \text{on} \quad \partial\Omega \, , \quad i = 1, \ldots, N \, .$$

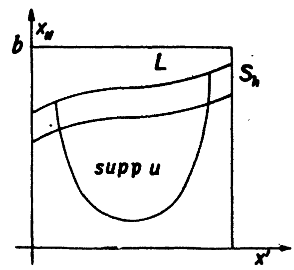

Fig. 1

From the Section 2 on the decomposition of the unit it is evident that it is sufficient to prove the theorem in the case when Ω has the shape as suggested in the figure.

$$L = \partial\Omega \cap K \, , \quad S_h = \{(x', x_N) \in \Omega ; \; a(x') - h < x_N < a(x')\} \, .$$

Let $\varepsilon > 0$ be an arbitrary number. We prove that there exists a $\delta > 0$ such that

$$\text{(61)} \qquad \varphi \in C^1(L) \, , \quad \|\varphi\|_{C(L)} \leqq 1 \, , \quad \int_{\mathrm{supp}\varphi} \mathrm{d}S \leqq \delta \Rightarrow \left| \int_L \varphi \, \mathrm{d}\beta \right| \leqq \varepsilon \, .$$

Let h be a positive number such that

$$\text{(62)} \qquad \int_{S_h} \mathrm{d}|\alpha_N| < \varepsilon \, .$$

There exists $\delta > 0$ such that

$$\text{(63)} \qquad M \subset \Omega \, , \quad |M| < \delta \, b \Rightarrow \int_M |u| \, \mathrm{d}x < h \, \varepsilon \, .$$

Suppose that φ satisfied assumptions from (61). We extend φ on Ω so that $\varphi \in C^1(\bar{\Omega})$, $\varphi = 0$ on $\Omega - S_h$ $|\varphi_{x_N}| \leqq c/h$ on S_h and $\varphi(x', x_N) = 0$ if $\varphi(x', a(x')) = 0$. Let us denote

$$M = \{(x', x_N);\ (x', a(x')) \in \operatorname{supp} \varphi,\ 0 < x_N < a(x')\}.$$

Then $|M| < \delta b$ and with respect to $\alpha_N = 0$ on L we obtain

$$\int_L \varphi \nu_N \, d\beta = \int_\Omega u \varphi_{x_N} \, dx + \int_\Omega \varphi \, d\alpha_N .$$

Due to (62) and (63) the following estimate holds:

$$\left| \int_L \varphi \nu_N \, d\beta \right| \leqq \frac{c}{h} \int_M |u| \, dx + \int_{S_h} d|\alpha_N| \leqq c\, \varepsilon .$$

Since $\nu_N \geqq c > 0$ on L (see (9)),

$$\left| \int_L \varphi \, d\beta \right| \leqq \frac{1}{\nu_N} \left| \int_L \varphi \nu_N \, d\beta \right| \leqq c\, \varepsilon$$

holds. Thus (61) is proved.

Let $M \subset L$ be a Borel set with $\int_M dS = 0$. There exists an open in L set $G \supset M$, $G \subset L$ such that $\int_G dS < \delta$. Then we obtain with respect to (61)

$$|\beta| (M) \leqq |\beta| (G) = \sup \left\{ \left| \int_L \varphi \, d\beta \right| ;\ \varphi \in C^1(L),\ \|\varphi\|_{C(L)} \leqq 1,\ \operatorname{supp} \varphi \subset \bar{G} \right\} \leqq \varepsilon .$$

7. Restrictions and extensions of functions from W_μ^1

Suppose $\Omega' \subset \Omega$ is a domain of the class C^1 with $\bar{\Omega}' \subset \Omega$ and $(u, \alpha_i, \beta) \in W_\mu^1(\Omega)$. The restriction of this function may be defined in many ways. It depends on the part of the side on $\partial\Omega$ which we add to the restricted function.

Let us denote

(64) $$u' = u_{|\Omega'} , \quad \alpha_i' = \alpha_{i|\bar{\Omega}'} .$$

Theorem 11. *Under the above assumptions, the function (u', α_i') from (64) belongs to the space $W_\mu^1(\bar{\Omega}')$ and its trace $\beta' \in L_\mu(\partial\Omega')$ is absolutely continuous with respect to the measure dS on $\partial\Omega'$.*

Proof. With regard to Theorem 8 we can suppose $\alpha_i = 0$ on $\partial\Omega$ and hence $\beta = \beta^0 \in L_1(\partial\Omega)$ is the trace of (u, α_i). From Theorem 4 we conclude that there exist $u_n \in W_1^1(\Omega)$ such that $u_n \rightharpoonup (u, \alpha_i)$ in $W_\mu^1(\bar{\Omega})$, $\|u_n\|_{W_1^1(\Omega)} \leqq c\|(u, \alpha_i)\|_{W_\mu^1(\bar{\Omega})}$. Let us denote $\Omega'' = \Omega - \bar{\Omega}'$, $u'' = u|\Omega''$, $\alpha_i'' = \alpha_i$ in Ω'' and $\alpha_i'' = 0$ on $\partial\Omega''$. $\{u_{n|\Omega''}\}_{n=1}^\infty$ is a bounded sequence and there exists its subsequence such that $u_{n_k|\Omega''} \rightharpoonup (\bar{u}, \bar{\alpha}_i)$ in

$W_\mu^1(\bar{\Omega}'')$. Evidently $\bar{u} = u$ in Ω'' and $\bar{\alpha}_i = \alpha_i$ inside of Ω''. Due to Theorem 8 $(u'', \alpha_i'') \in W_\mu^1(\bar{\Omega}'')$. Let ν_i' be the i-th component of the exterior normal to Ω'. The function (u'', α_i'') possesses the trace β^0 on $\partial\Omega$. Let us denote its trace on $\partial\Omega'$ by β'. β' is at the same time inner trace of (u'', α_i'') and hence $\beta' \in L_1(\partial\Omega')$ according to Theorem 10. Now we prove that $(u', \alpha_i', \beta') \in W_\mu^1(\bar{\Omega}')$.

Green's theorem holds for (u, α_i) and (u'', α_i'') with a function $\varphi \in C^1(\bar{\Omega})$:

$$\int_{\partial\Omega} \varphi \nu_i \, d\beta^0 = \int_\Omega u \varphi_{x_i} \, dx + \int_\Omega \varphi \, d\alpha_i ,$$

$$\int_{\partial\Omega} \varphi \nu_i \, d\beta^0 - \int_{\partial\Omega'} \varphi \nu_i' \, d\beta' = \int_{\Omega''} u \varphi_{x_i} \, dx + \int_{\Omega''} \varphi \, d\alpha_i .$$

Subtracting these formulas we obtain Green's theorem for the function (u', α_i', β').

By the same method we can prove $(u_{|\Omega''}, \alpha_{i|\bar{\Omega}''}) \in W_\mu^1(\bar{\Omega}'')$ and at the same time the trace of this function is the inner trace of the function $(u', \alpha_i') \in W_\mu^1(\bar{\Omega}')$.

If two functions from W_μ^1 possess the same trace on the common boundary, then it is possible to join them together. Suppose $\Omega^* \supset \bar{\Omega}$ is a domain of the class C^1, $\Omega' = \Omega^* - \bar{\Omega}$.

Theorem 12. *Let $(u, \alpha_i, \beta) \in W_\mu^1(\bar{\Omega})$, $(u', \alpha_i') \in W_\mu^1(\bar{\Omega}')$ and suppose that (u', α_i') possesses the trace β^* on $\partial\Omega^*$ and β on $\partial\Omega$, i.e. the same trace as (u, α_i, β) on $\partial\Omega$. Let us set*

$$(65) \qquad u^* = u \text{ on } \Omega , \quad u^* = u' \text{ on } \Omega' ,$$
$$\alpha_i^* = \alpha_i \text{ on } \Omega , \quad \alpha_i^* = \alpha_i + \alpha_i' \text{ on } \partial\Omega , \quad \alpha_i^* = \alpha_i \text{ on } \bar{\Omega}^* - \bar{\Omega} .$$

Then $(u^*, \alpha_i^*, \beta^*) \in W_\mu^1(\bar{\Omega}^*)$.

Proof. It suffices to consider Green's theorem for (u, α_i) and for (u', α_i'). By adding them we obtain Green's theorem for the function (u^*, α_i^*).

8. Regularizations of functions from W_μ^1

In order that the mollified functions u_h of u satisfy $u_h \overset{w^\circ}{\rightharpoonup} u$, we must take into account the side of the function (u, α_i). We proceed in the following way:

First we extend (u, α_i) on a larger domain, so that the side of the extended function on $\partial\Omega$ is twice the side of (u, α_i). Then we mollify the extended function. This method we can use only for functions possessing the traces from $L_1(\partial\Omega)$. A similar result is obtained for the functions possessing the trace from $L_\mu(\partial\Omega)$ by the diagonal method and Lemma 1.

Theorem 13. *Suppose $(u, \alpha_i, \beta) \in W_\mu^1(\bar{\Omega})$ and let β be absolutely continuous with respect to the measure dS on $\partial\Omega$. Suppose $\Omega^* \supset \bar{\Omega}$ is a bounded domain of the* *class* c^1. *Then there exists* $(u^*, \alpha_i^*, \beta^*) \in W_\mu^1(\bar{\Omega}^*)$ *such that*

$$u^* = u \text{ on } \Omega, \quad \alpha_i^* = \alpha_i \text{ on } \Omega, \quad \alpha_i^* = 2\alpha_i \text{ on } \partial\Omega. \tag{66}$$

For small $h > 0$ *let us set*

$$u_h(x) = \int_{\Omega^*} K^h(x - y)\, u^*(y)\, dy, \quad x \in \Omega. \tag{67}$$

Then $u_h \in W_\mu^1(\Omega)$ *and*

$$u_h \to (u, \alpha_i) \text{ in } W_\mu^1(\bar{\Omega}), \quad \|u_h\|_{W_1^1(\Omega)} \to \|(u, \alpha_i)\|_{W_\mu^1(\bar{\Omega})}. \tag{68}$$

Proof. Suppose α_i' are the measures defined by (50). With regard to Theorem 8, $(0, \alpha_i') \in W_\mu^1(\bar{\Omega})$ and the trace of this function is equal to the side of the function (u, α_i), i.e. $(0, \alpha_i', \alpha_\nu) \in W_\mu^1(\bar{\Omega})$ on account of Theorem 9. From this fact it follows $(u, \alpha_i + \alpha_i', \beta + \alpha_\nu) = (u, \alpha_i, \beta) + (0, \alpha_i', \alpha_\nu) \in W_\mu^1(\bar{\Omega})$. β being absolutely continuous implies that $\alpha_\nu = \beta - \beta^0$ is absolutely continuous (see Theorem 10), i.e. $\beta + \alpha_\nu$ is absolutely continuous. With respect to [3] there exists $u' \in W_1^1(\Omega^* - \bar{\Omega})$ with the trace $\beta + \alpha_\nu$ on $\partial\Omega$ (see our agreement). Now, we join together the function $u' \in W_1^1(\Omega^* - \bar{\Omega})$ and the function $(u, \alpha_i + \alpha_i', \beta + \alpha_\nu) \in W_\mu^1(\bar{\Omega})$ and thus, by means of Theorem 12, we define the function $(u^*, \alpha_i^*) \in W_\mu^1(\bar{\Omega}^*)$. The first part of the theorem is proved. Evidently $u_h \to u$ in $L_1(\Omega)$. Suppose $\varphi \in C(\bar{\Omega}^*)$. Let us denote

$$S_h = \{x \in \Omega;\ \operatorname{dist}(x, \partial\Omega) < h\}, \quad S_h^* = \{x \in \Omega^* - \bar{\Omega};\ \operatorname{dist}(x, \partial\Omega) < h\}.$$

For $y \in \Omega^*$ we set

$$\varphi_h(y) = \int_\Omega K^h(x - y)\, \varphi(x)\, dx, \quad \bar{\varphi}_h(y) = \int_{\Omega^*} K^h(x - y)\, \varphi(x)\, dx.$$

With regard to (57) $\varphi_h \to \frac{1}{2}\varphi$ holds in $C(\partial\Omega)$. Easily we find that $\bar{\varphi}_h \to \varphi$ in $C(\bar{\Omega})$, $\varphi_h = \bar{\varphi}_h$ on $\Omega - S_h$ and

$$u_{hx_i}(x) = \int_{\Omega^*} K^h(x - y)\, d\alpha_i^*(y), \quad x \in \Omega. \tag{69}$$

Then

$$\int_\Omega u_{hx_i}\varphi\, dx = \iint_{\substack{x\in\Omega\\ y\in\Omega^*}} K^h(x - y)\, \varphi(x)\, dx\, d\alpha_i^*(y) =$$

$$= \iint_{\substack{x\in\Omega\\ y\in\Omega}} \ldots + \iint_{\substack{x\in\Omega\\ y\in\partial\Omega}} \ldots + \iint_{\substack{x\in\Omega\\ y\in S_h^*}} \ldots =$$

$$= \int_\Omega \bar{\varphi}_h\, d\alpha_i + \int_{S_h} (\varphi_h - \bar{\varphi}_h)\, d\alpha_i + \int_{\partial\Omega} 2\varphi_h\, d\alpha_i + \int_{S_h^*} \varphi_h\, d\alpha_i^*.$$

For $h \to 0$, $\int_\Omega \bar{\varphi}_h \, d\alpha_i \to \int_\Omega \varphi \, d\alpha_i$ holds

$$\left| \int_{S_h^*} (\varphi_h - \bar{\varphi}_h) \, d\alpha_i \right| \leqq 2\|\varphi\|_{C(\bar{\Omega}^*)}, \quad \int_{S_h} d(\alpha_i) \to 0 .$$

With respect to (57), $\int_{\partial\Omega} 2\varphi_h \, d\alpha_i \to \int_{\partial\Omega} \varphi \, d\alpha_i$,

$$\left| \int_{S_h^*} \varphi_h \, d\alpha_i^* \right| \leqq \|\varphi\|_{C(\bar{\Omega}^*)}, \quad \int_{S_h^*} d|\alpha_i^*| \to 0 .$$

This implies $u_h \rightharpoonup (u, \alpha_i)$ in $W_\mu^1(\bar{\Omega})$ and thus

$$\|(u, \alpha_i)\|_{W_\mu^1(\bar{\Omega})} \leqq \varliminf_{h \to 0} \|u_h\|_{W_1^1(\Omega)} .$$

Now we prove the converse inequality. $\int_\Omega |u_h| \, dx \to \int_\Omega |u| \, dx$ holds on account of the convergence $u_h \to u$ in $L_1(\Omega)$. From (69) we deduce

$$\int_\Omega |u_{hx_i}| \, dx \leqq \iint_{\substack{x \in \Omega \\ y \in \Omega^*}} K^h(x - y) \, d|\alpha_i^*(y)| \, dx =$$

$$= \iint_{\substack{x \in \Omega \\ y \in \Omega}} \ldots + \iint_{\substack{x \in \Omega \\ y \in \partial\Omega}} \ldots + \iint_{\substack{x \in \Omega \\ y \in S_h^*}} \ldots \leqq$$

$$\leqq \int_\Omega d|\alpha_i| + \iint_{\substack{x \in \Omega \\ y \in \partial\Omega}} 2K^h(x - y) \, dx \, d|\alpha_i| \, (y) + \int_{S_h^*} d|\alpha_i^*| .$$

The third right-hand side term converges to zero for $h \to 0$. From (56) we conclude $\int_\Omega K^h(x - y) \, dx \to \frac{1}{2}$ uniformly for $y \in \partial\Omega$ and thus the second term converges to $\int_{\partial\Omega} d|\alpha_i|$.

Theorem 14. *Suppose $(u, \alpha_i, \beta) \in W_\mu^1(\bar{\Omega})$. Then there exists a sequence $u_n \in W_1^1(\Omega)$ such that*

$$u_n \rightharpoonup (u, \alpha_i) \quad \textit{in} \quad W_\mu^1(\bar{\Omega}) \quad \textit{and} \quad \|u_n\|_{W_1^1}(\Omega) \to \|(u, \alpha_i)\|_{W_\mu^1(\bar{\Omega})} .$$

Proof. As in (50) we set

$$\alpha_i' = \alpha_i \quad \text{on} \quad \partial\Omega , \qquad \alpha_i' = 0 \quad \text{on} \quad \Omega , \qquad \bar{\alpha}_i = \alpha_i - \alpha_i' \quad \text{on} \quad \bar{\Omega} .$$

Then $(u, \bar{\alpha}_i, \beta^0)$, $(0, \alpha_i', \alpha_\nu) \in W_\mu^1(\bar{\Omega})$ where β^0 is the inner trace of the function (u, α_i) and α_ν is its side.

Similarly as in Lemma 1 we set

$$u_h'(x) = \int_{\partial\Omega} R^h(x - y) \, d\alpha_\nu(y) , \quad x \in \partial\Omega .$$

Making use of Lemma 1 we obtain

$$u'_h \rightharpoonup \alpha_\nu \quad \text{in} \quad L_\mu(\partial\Omega), \qquad \|u'_h\|_{L_1(\partial\Omega)} \to \|\alpha_\nu\|_{L_\mu(\partial\Omega)}.$$

From Theorem 9 we conclude $\alpha'_i = \nu_i \alpha_\nu$ on $\partial\Omega$. Let us define $\alpha'_{hi} \in L_\mu(\bar{\Omega})$ by

$$(70) \qquad \int_{\partial\Omega} \varphi \, \mathrm{d}\alpha'_{h_i} = \int_{\partial\Omega} \varphi \nu_i u'_h \, \mathrm{d}S, \quad \varphi \in C(\partial\Omega); \qquad \alpha'_{hi} = 0 \quad \text{in} \quad \Omega.$$

We find easily that $(0, \alpha'_{hi}, u'_h) \in W^1_\mu(\bar{\Omega})$ and

$$(71) \qquad (0, \alpha'_{hi}) \rightharpoonup (0, \alpha'_i) \quad \text{in} \quad W^1_\mu(\bar{\Omega}).$$

We prove that

$$(72) \qquad \|(0, \alpha'_{hi})\|_{W_\mu{}^1} \to \|(0, \alpha'_i)\|_{W_\mu{}^1}.$$

$$\int_{\partial\Omega} \mathrm{d}|\alpha'_{hi}| = \int_{\partial\Omega} |\nu_i| \, . \, |u'_h| \, \mathrm{d}S \leqq \iint\limits_{\substack{x\in\partial\Omega\\ y\in\partial\Omega}} |\nu_i(x)| \, R^h(x-y) \, \mathrm{d}|\alpha_\nu| \, (y) \, \mathrm{d}S(x) \leqq$$

$$\leqq \int_{\partial\Omega} (|\nu_i|)_h \, (y) \, \mathrm{d}|\alpha_\nu| \, (y) \to \int_{\partial\Omega} |\nu_i| \, \mathrm{d}|\alpha_\nu| = \|\alpha_i\|_{L_\mu(\partial\Omega)}.$$

We used

$$(73) \qquad |\psi\alpha| = |\psi| \, . \, |\alpha|, \quad \psi \in C(\partial\Omega), \quad \alpha \in L_\mu(\partial\Omega)$$

in our reasoning.

The converse inequality can be obtained from (71). Let us set

$$(74) \qquad (u, \alpha_{hi}, \beta_h) = (u, \bar{\alpha}_i + \alpha'_{hi}, \beta^0 + u'_h) \in W^1_\mu(\bar{\Omega})$$

(see our agreement), where β_h is absolutely continuous with respect to $\mathrm{d}S$. Theorem 13 implies the existence of $u_{hk} \in W^1_1(\Omega)$, $k > 0$ such that

$$(75) \qquad u_{hk} \underset{k\to 0}{\rightharpoonup} (u, \alpha_{hi}) \quad \text{in} \quad W^1_\mu, \qquad \|u_{hk}\|_{W_1{}^1} \underset{k\to 0}{\to} \|(u, \alpha_{hi})\|_{W_\mu{}^1}.$$

From (71), (72) and (74) we conclude

$$(76) \qquad (u, \alpha_{hi}) \rightharpoonup (u, \alpha_i) \quad \text{in} \quad W^1_\mu, \qquad \|(u, \alpha_{hi})\|_{W_\mu{}^1} \to \|(u, \alpha_i)\|_{W_\mu{}^1}.$$

Suppose $\{\varphi^j\}_{j=1}^\infty$ is a dense subset in the space $[C(\bar{\Omega})]^{N+1}$. For $\varphi = (\varphi_0, \ldots, \varphi_N) \in C^{N+1}$, $(u, \alpha_i) \in W^1_\mu(\bar{\Omega})$ we define

$$(77) \qquad \langle (u, \alpha_i), \varphi \rangle = \int_\Omega u\varphi_0 \, \mathrm{d}x + \sum_{i=1}^N \int_{\bar{\Omega}} \varphi_i \, \mathrm{d}\alpha_i.$$

For each positive integer n, there exists $h_n > 0$ such that

$$|\langle (u, \alpha_{h_n i}), \varphi^j \rangle - \langle (u, \alpha_i), \varphi^j \rangle| < \frac{1}{n} \quad \text{for} \quad j = 1, \ldots, n$$

$$\big| \|(u, \alpha_{h_n i})\|_{W_\mu{}^1} - \|(u, \alpha_i)\|_{W_\mu{}^1} \big| < \frac{1}{n}.$$

To this h_n there exists according to (75) such $k_n \geqq 0$ that

$$|\langle u_{h_n}k_n, \varphi^j \rangle - \langle (u, \alpha_{h_n i}), \varphi^j \rangle| < \frac{1}{n} \quad \text{for} \quad j = 1, \ldots, n,$$

$$\big| \|u_{h_n k_n}\|_{W_1{}^1} - \|(u, \alpha_{h_n i})\|_{W_\mu{}^1} \big| < \frac{1}{n}.$$

Hence we deduce $\|u_{h_n k_n}\|_{W_1{}^1} \underset{n \to \infty}{\to} \|(u, \alpha_i)\|_{W_\mu{}^1}$,

$$\langle u_{h_n k_n}, \varphi^j \rangle \underset{n \to \infty}{\to} \langle (u, \alpha_i); \varphi^j \rangle \quad \text{for} \quad j = 1, 2, \ldots$$

Thus, we obtain $u_{h_n k_n} \underset{n \to \infty}{\to} (u, \alpha_i)$ in $W_\mu^1(\bar{\Omega})$. The theorem is proved.

§. 2. SPACE W_μ^k

9. Definition and fundamental properties of W_μ^k

Let us denote $e_m = (0, \ldots, 0, 1, 0, \ldots, 0)$ the N-dimensional vector with the unit in the m-th place. Let $\varkappa$ be the number of multiindices i with $|i| = k$.

Definition 5. $W_\mu^k(\bar{\Omega})$ is the space of all $(\varkappa + 1)$-tuples $(u, \alpha_i)_{|i|=k}$ such that $u \in W_1^{k-1}(\Omega)$, $\alpha_i \in L_\mu(\bar{\Omega})$ and

(78) $$(D^i u, \alpha_{i+e_1}, \ldots, \alpha_{i+e_N}) \in W_\mu^1(\bar{\Omega}) \quad \text{for all} \quad |i| = k - 1.$$

The norm is defined by

$$\|(u, \alpha_i)\|_{W_\mu{}^k(\bar{\Omega})} = \|u\|_{W_1{}^{k-1}(\Omega)} + \sum_{|i|=k} \|\alpha_i\|_{L_\mu(\bar{\Omega})}.$$

The w^*-convergence in the space W_μ^k is defined as the w^*-convergence in the space W_1^{k-1} for the first component and as the w^*-convergence in $L_\mu(\bar{\Omega})$ for the other components. The space $W_1^k(\Omega)$ can be canonically imbedded into the space $W_\mu^k(\bar{\Omega})$ by the rule (in the sense of our agreement)

$$u \in W_1^k(\Omega) \to (u, D^i u)_{|i|=k} \in W_\mu^k(\bar{\Omega}).$$

The space $\mathring{W}_\mu^k(\bar{\Omega})$ is defined in the following way: $(u, \alpha_i) \in \mathring{W}_\mu^k(\bar{\Omega})$ iff $u \in \mathring{W}_1^{k-1}(\Omega)$ and for all $|i| = k - 1$ the function from (78) belongs to the space $\mathring{W}_\mu^1(\bar{\Omega})$.

Suppose $(u, \alpha_i) \in W_\mu^k(\bar{\Omega})$. The same decomposition as in Theorem 8 can be realized. For $|i| = k$ let us set

(79) $$\alpha_i' = \alpha_i \text{ on } \partial\Omega\,, \quad \alpha_i' = 0 \text{ on } \Omega\,, \quad \bar{\alpha}_i = \alpha_i - \alpha_i' \text{ on } \bar{\Omega}\,.$$

Regarding Theorem 8 we find that $(u, \bar{\alpha}_i)$, $(0, \alpha_i') \in W_\mu^k(\bar{\Omega})$.

Definition 6. The measure $\alpha_\nu \in L_\mu(\partial\Omega)$ defined by

(80) $$\alpha_\nu = \sum_{i_1,\ldots,i_k=1}^{N} \nu_{i_1} \ldots \nu_{i_k} \alpha_{e_{i_1}+\ldots+e_{i_k}} \quad \text{on} \quad \partial\Omega$$

is called the side of the function $(u, \alpha_i) \in W_\mu^k(\bar{\Omega})$.

The formula for α_ν corresponds to the formula for the k-th derivative with respect to the normal ν for the functions from W_1^k. An analogical theorem to Theorem 9 is valid.

Theorem 15. *If* $(u, \alpha_i) \in W_\mu^k(\bar{\Omega})$, *then*

$$\alpha_i = \nu_1^{i_1} \ldots \nu_N^{i_N} \alpha_\nu\,, \quad i = (i_1, \ldots, i_N) \quad on \quad \partial\Omega$$

for all $|i| = k$.

Proof. Let us rewrite the assertion of the theorem into a more suitable form

$$\alpha_{e_{i_1}+\ldots+e_{i_k}} = \nu_{i_1} \ldots \nu_{i_k} \alpha_\nu\,, \quad i_1, \ldots, i_k = 1, \ldots, N\,.$$

For $k = 1$, the assertion is proved in Theorem 9. With respect to the Definition 5 we have

$$(D^{e_{i_2}+\ldots+e_{i_k}} u, \alpha_{e_{i_1}+\ldots+e_{i_k}})_{i_1=1}^N \in W_\mu^1(\bar{\Omega})\,.$$

The side of this function is equal to $\sum_{i_1=1}^N \nu_{i_1} \alpha_{e_{i_1}+\ldots+e_{i_k}}$.

From Theorem 9 we obtain

(81) $$\alpha_{e_{i_1}+\ldots+e_{i_k}} = \nu_{i_1} \sum_{j_1=1}^N \nu_{j_1} \alpha_{e_{j_1}+e_{i_2}+\ldots+e_{i_k}}\,, \quad i_1 = 1, \ldots, N\,.$$

The same assertion is valid for the index i_2

(82) $$\alpha_{e_{i_1}+\ldots+e_{i_k}} = \nu_{i_2} \sum_{j_2=1}^N \nu_{j_2}\, \alpha_{e_{i_1}+e_{j_2}+e_{i_3}+\ldots+e_{i_k}}\,, \quad i_2 = 1, \ldots, N\,.$$

If we substitute (82) into (81), then

$$\alpha_{e_{i_1}+\ldots+e_{i_k}} = \nu_{i_1} \nu_{i_2} \sum_{j_1,j_2=1}^N \nu_{j_1} \nu_{j_2} \alpha_{e_{j_1}+e_{j_2}+e_{i_3}+\ldots+e_{i_k}}\,.$$

After k steps we obtain

$$\alpha_{e_{i_1}+\ldots+e_{i_k}} = v_{i_1} \ldots v_{i_k} \sum_{j_1,\ldots,j_k=1}^{N} v_{j_1} \ldots v_{j_k} \alpha_{e_{j_1}+\ldots+e_{j_k}} = v_{i_1} \ldots v_{i_k} \alpha_v .$$

Analogous theorems on imbedding are valid for $W_\mu^k(\bar{\Omega})$ as in the case of W_μ^1.

The imbedding $W_\mu^k(\bar{\Omega}) \to W_q^{k-1}(\Omega)$, $1/q = 1 - 1/N$ is continuous and the imbedding $W_\mu^k(\bar{\Omega}) \to W_{q^*}^{k-1}(\Omega)$, $q^* < q$ is compact. This imbeddings are defined by the rule $(u, \alpha_i) \to u$.

Proof. The morms $\|D^i u\|_{L_q}$, $|i| = k - 1$ can be estimated from Theorem 5 and the norms $\|D^i u\|_{L_q}$, $|i| \leqq k - 2$ can be estimated by means of the imbedding $W_1^{k-1} \to \to W_q^{k-2}$.

Compactness can be proved similarly.

Theorems on imbedding of $W_\mu^k(\bar{\Omega})$ into $W_p^e(\Omega)$, $C^{e,\alpha}(\bar{\Omega})$, $e \leqq k - 2$, are valid in the same form as for the space $W_1^k(\Omega)$. This is a consequence of the transitivity of imbeddings, which makes it possible to obtain them from the imbedding $W_\mu^k \to W_q^{k-1}$.

$\sum_{|i|=k} \|\alpha_i\|_{L_\mu(\bar{\Omega})}$ is an equivalent norm in the space $\mathring{W}_\mu^k(\bar{\Omega})$.

Proof. Using Theorem 7 we can estimate the norms $\|D^i u\|_{L_1}$, $|i| = k - 1$ and then we apply the theorem on equivalent norms in the space $\mathring{W}_1^{k-1}$.

If $(u, \alpha_i) \in W_\mu^k(\bar{\Omega})$, then $(u_{|\Omega'}, \alpha_{i|\bar{\Omega}'}) \in W_\mu^k(\bar{\Omega}')$, where $\Omega' \subset \Omega$. This assertion follows immediately from Theorem 11 and from the definition of the space W_μ^k.

Let us denote $[L_\mu(\bar{\Omega})]^{\kappa'} = \{\{\alpha_i\};\ |i| \leqq k,\ \alpha_i \in L_\mu(\bar{\Omega})\}$. The space $W_\mu^k(\bar{\Omega})$ can be canonically imbedded into $[L_\mu(\bar{\Omega})]^{\kappa'}$ by: the rule $(u, \alpha_i) \to \{\alpha_i\}_{|i| \leqq k}$, where α_i are the same for $|i| = k$ and $\alpha_i = D^i u$ (in sense of our agreement) for $|i| \leqq k - 1$. The next theorem is valuable in applications.

Theorem 16. *The space $W_\mu^k(\bar{\Omega})$ is closed with respect to the w*-topology as a subspace of $[L_\mu(\bar{\Omega})]^{\kappa'}$.*

The ball in the space $W_\mu^k(\bar{\Omega})$ is compact with respect to the w-topology.*

The same assertion is true for $\mathring{W}_\mu^k(\bar{\Omega})$.

Proof. Let $(u_n, \alpha_{n_i}) \rightharpoonup \{\alpha_i\}_{|i| \leqq k}$ (w^*-convergence) in $[L_\mu(\bar{\Omega})]^{\kappa'}$. By the same method as in the proof of Theorem 6 we find from the theorems on imbedding that $u = = \alpha_0 \in W_1^{k-1}(\Omega)$ and that $\alpha_i \in L_1(\Omega)$, $|i| \leqq k - 1$ (in the sense of our agreement). Analogously as in the proof of Theorem 1 we can prove that $\alpha_i = D^i u$, $|i| \leqq k - 1$ in the sense of distributions. It remains to prove that $(D^i u, \alpha_{i+e_m})_{m=1}^N \in W_\mu^1$ for $|i| = = k - 1$. However, this is a consequence of the Theorem 6 and of the fact that

$$(D^i u_n, \alpha_{n,i+e_m})_{m=1}^N \rightharpoonup (D^i u, \alpha_{i+e_m})_{m=1}^N \quad \text{in} \quad W_\mu^1(\bar{\Omega}) .$$

The rest of the proof is the same as that of Theorem 6.

Now we prove that $\mathring{W}_\mu^k$ is closed in $[L_\mu(\bar{\Omega})]^{\kappa'}$ with respect to the w^*-convergence. Suppose $(u_n, \alpha_{ni}) \rightharpoonup \{\alpha_i\}_{|i| \leq k}$ and $(u_n, \alpha_{ni}) \in \mathring{W}_\mu^k$. The first part of the proof implies $\{\alpha_i\}_{|i| \leq k} = (u, \alpha_i) \in W_\mu^k$ in the sense of canonical imbedding. Owing to the theorem on imbedding, there exists a subsequence $\{u_{n_k}\}$ converging to u in the norm of the space W_1^{k-1} and hence $u \in \mathring{W}_1^{k-1}$. For $|i| = k - 1$ we have

$$(D^i u_n, \alpha_{n,i+e_m}) \rightharpoonup (D^i u, \alpha_{i+e_m}) \quad \text{in} \quad W_\mu^1 .$$

From the fact $(D^i u_\mu, \alpha_{n,i+e_m}) \in \mathring{W}_\mu^1$ and from Theorem 2 we conclude $(D^i u, \alpha_{i+e_m}) \in \in \mathring{W}_\mu^1$.

The rest of the proof is the same as that of Theorem 6.

10. Regularisation of functions from W_μ^k

We use the same method as that in Section 8. In order to prove the existence of the extension similar to the extension (u^*, α_i^*) from Theorem 13, we prove first of all two lemmas. We shall assume that the boundary $\partial\Omega$ is sufficiently smooth, so that we were able to transform suitably pieces of the boundary in the proofs of lemmas 3 and 4. It suffices to assume that $\partial\Omega$ is of the class C^{k+1}.

Lemma 3. *Suppose $(u, \alpha_i) \in W_\mu^k(\bar{\Omega})$. If $\alpha_{i|\partial\Omega} = 0$, $|i| = k$ then there exists a domain $\Omega^* \supset \bar{\Omega}$ and a function $(u^*, \alpha_i^*) \in W_\mu^k(\bar{\Omega}^*)$ such that*

$$u^* = u \text{ on } \Omega , \quad \alpha_i^* = \alpha_i \text{ on } \bar{\Omega} .$$

Proof. First we prove the assertion in the case of the cube. Let us denote

$$K = \{x;\ 0 < x_i < b,\ i = 1, \dots, N-1,\ -b < x_N < 0\} ,$$
$$K_1 = \{x;\ 0 < x_i < b,\ i = 1, \dots, N-1,\quad 0 < x_N < b\} ,$$
$$L = \{x;\ 0 < x_i < b,\ i = 1, \dots, N-1,\quad x_N = 0\} .$$

Let us assume that the support of $(u, \alpha_i) \in W_\mu^k(\bar{\Omega})$ is a subset of $K \cup L$. We extend the function (u, α_i) by zero on $\{x; x_N \leq 0\}$. We use the method of Nikolsky – see [2]. Let $\lambda_1 \dots \lambda_k$ be real numbers such that

$$\sum_{m=1}^{k} \lambda_m (-m)^j = 1 \quad \text{for} \quad j = 0, \dots, N-1 . \tag{83}$$

Let us define the function $\bar{u} \in W_1^{k-1}(K_1)$ by the rule

$$\bar{u}(x', x_N) = \sum_{m=1}^{k} \lambda_m u(x', -m x_N) . \tag{84}$$

Then for $i = (i_1 \dots i_N)$ $|i| \leq k - 1$ we obtain

(85) $$D^i \bar{u}(x', x_N) = \sum_{m=1}^{k} \lambda_m (-m)^{i_N} D^i u(x', -m x_N) .$$

Let us define the measures $\bar{\alpha}_i \in L_\mu(\bar{K}_1)$ $|i| = k$ by formula

(86) $$\int_{\bar{K}_1} \varphi(x', x_N) \, d\bar{\alpha}_i(x', x_N) = \sum_{m=1}^{k} \lambda_m (-m)^{i_N} \frac{1}{m} \int_{\bar{K}} \varphi\left(x', -\frac{x_N}{m}\right) d\alpha_i(x', x_N) .$$

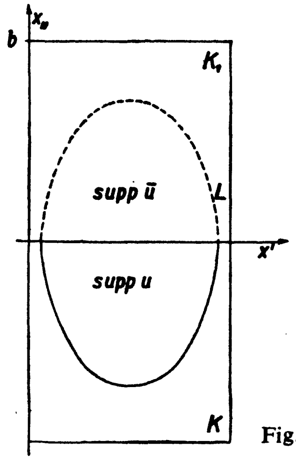

Fig. 2

If $|i| = k - 1$ then there exists $u_n \in W_1^1(K)$ such that

$$u_n \rightharpoonup (D^i u \ \alpha_{i+e_j})_{j=1}^N \quad \text{in} \quad W_\mu^1(\bar{K}) .$$

Now, let us define $\bar{u}_n \in W_1^1(K_1)$ by formula

(87) $$\bar{u}_n(x', x_N) = \sum_{m=1}^{k} \lambda_m (-m)^{i_N} u_n(x', -m x_N) , \quad (x', x_N) \in K_1 .$$

By direct computation we find that

$$\bar{u}_n \rightharpoonup (D^i \bar{u}, \bar{\alpha}_{i+e_j})_{j=1}^N \quad \text{in} \quad W_\mu^1(\bar{K}_1) .$$

From (84) we deduce that $\bar{u} \in W_1^{k-1}(K_1)$ and hence $(\bar{u}, \bar{\alpha}_i) \in W_\mu^k(\bar{K}_1)$. According to (83) $\bar{u}_n(x', 0) = u_n(x', 0)$ is valid on L in the sense of traces. Owing to Theorem 2 the functions $(D^i u, \alpha_{i+e_j})$ and $(D^i \bar{u}, \bar{\alpha}_{i+e_j})$ possess the same trace on L. Thus Theorem 12 enables us to fasten these functions together. Let us set $u^* = u$ on K, $u^* = \bar{u}$ on K_1, $\alpha_i^* = \alpha_i$ on K, $\alpha_i^* = \bar{\alpha}_i$ on K_1, $\alpha_i^* = \alpha_i + \bar{\alpha}_i$ on L, $|i| = K$.

We obtain $u^* \in W_1^{k-1}(K \cup L \cup K_1)$ from (83). Thus we conclude $(u^*, \alpha_i^*) \in W_\mu^k(\overline{K \cup K_1})$. From formula (86) it can be seen easily that $\bar{\alpha}_i = 0$ on L and hence

$\alpha_i^* = 0$ on L, $|i| = K$. The support of the function (u^*, α_i^*) is in $K \cup L \cup K_1$ and thus (u^*, α_i^*) can be extended by zero on any larger domain. It can be seen from the proof that the following estimate is true:

$$\|(u^*, \alpha_i^*)\|_{W_\mu{}^k(\overline{K \cup K_1})} \leqq c\|(u, \alpha_i)\|_{W_\mu{}^k(\bar{K})} \,.$$

Now, let us assume $(u, \alpha_i) \in W_\mu^k(\bar{\Omega})$ with $\alpha_i = 0$ on $\partial\Omega$, $|i| = k$. Let the cubes K_r cover $\partial\Omega$ similarly as in Section 2 and let $\gamma_r \in C^{k+1}(\bar{\Omega})$, $r = 0, \ldots, R$ be the corresponding decomposition of the unit. We extend smoothly each function γ_r, $r = 1, \ldots, R$ on E_N so that its support is in K_r and $\gamma_r = C_0^{k+1}(E_N)$. Let us denote $u_r = u \,.\, \gamma_r$ on Ω, $\alpha_{ri} = D^i u_r$ in Ω, $|i| = k$ in the sense of distributions and $\alpha_{ri} = 0$ on $\partial\Omega$. Owing to Theorem 3 and 8 we find easily that $(u_r, \alpha_{ri}) \in W_\mu^k(\bar{\Omega})$. Then we carry out a corresponding linear orthogonal transformation of coordinates, after which there will be $K_r = \{x; 0 < x_i < b\}$ and $\partial\Omega \cap K_r$ will be described by formula $x_N = a(x')$ where a possesses the corresponding smoothness. At last we use the transformation of coordinates

$$A : (x', x_N) \to \left(x', \frac{b}{a(x')}\, x_N\right)$$

A transforms the domain $\Omega \cap K_r$ onto K_r. We extend the function (u_r, α_{ri}) on (u_r^*, α_{ri}^*) as at the beginning of the proof, then we pass to the original coordinates and finally we put together the functions (u_r^*, α_{ri}^*).

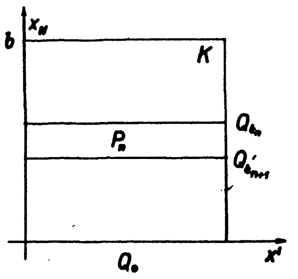

Fig. 3

Lemma 4. *If $u' \in L_1(\partial\Omega)$, then there exists a function $u \in W_1^k(\Omega)$ satisfying*

$$\|u\|_{W_1{}^k(\Omega)} \leqq c\|u'\|_{L_1(\partial\Omega)}$$

and

$$u = \frac{\partial u}{\partial v} = \ldots = \frac{\partial^{k-2} u}{\partial v^{k-2}} = 0\,, \quad \frac{\partial^{k-1} u}{\partial v^{k-1}} = u' \quad \text{on} \quad \partial\Omega\,,$$

where $\partial/\partial v$ is the derivative with respect to the exterior normal on $\partial\Omega$.

Proof. The proof is completely analogous to that of Theorem in [3]. First we prove the theorem in the case of the cube.

For $0 \leqq t \leqq b$ let us denote

$$Q_t = \{(x', t);\ 0 < x_i < b,\ i = 1, \dots, N - 1\}\,.$$

There exist functions $u'_n \in C_0^\infty(Q_0)$ satisfying

$$\sum_{n=1}^{\infty} \int_{Q_0} |u'_{n+1} - u'_n|\, dx' \leqq c\|u'\|_{L_1(Q_0)} \quad (\text{see } [3])\,. \tag{88}$$

Suppose that $t_1, t_2, \dots$ is a decreasing sequence of positive numbers, $t_n \to 0$. Let us denote

$$P_n = \{(x', x_N);\ 0 < x_i < b,\ i = 1, \dots, N - 1,\ t_{n+1} < x_N < t_n\}\,.$$

Let us define the function $\bar{u}$ in the following way:

$$\bar{u} = u'_n \text{ on } Q_{t_n}\,, \quad \text{i.e.} \quad \bar{u}(x', t_n) = u'_n(x')\,,$$

$$\bar{u}(x', x_N) = \frac{t_n - x_N}{t_n - t_{n+1}}\,\bar{u}(x', t_n) + \frac{x_N - t_{n+1}}{t_n - t_{n+1}}\,\bar{u}(x', t_{n+1}) \quad \text{for } (x', x_N) \in P_n \tag{89}$$

and $\bar{u}(x', x_N) = 0$ for $x_N \geqq t_1$.

Let us estimate $\partial\bar{u}/\partial x_N$ in $L_1(K)$. For $(x', x_N) \in P_n$ we obtain

$$\frac{\partial \bar{u}}{\partial x_N}(x', x_N) = \frac{u'_{n+1}(x') - u'_n(x')}{t_n - t_{n+1}}$$

and hence

$$\int_{P_n} \left|\frac{\partial \bar{u}}{\partial x_N}\right| dx'\, dx_N = \int_{Q_0} dx' \int_{t_{n+1}}^{t_n} \left|\frac{\partial \bar{u}}{\partial x_N}\right| dx_N = \int_{Q_0} |u'_{n+1}(x') - u'_n(x')|\, dx'\,.$$

With respect to (88) we obtain the estimate

$$\int_K \left|\frac{\partial \bar{u}}{\partial x_N}\right| dx_N \leqq c\|u'\|_{L_1(Q_0)}\,. \tag{90}$$

This estimate is independent of the sequence $t_1, t_2, \dots$. Now, let us estimate $\int_K |D^i\bar{u}|\, dx$, $|i| \leqq k$, where D^i is the tangent derivative (i.e. $i_N = 0$). Owing to (89) we obtain

$$|D^i\bar{u}(x', x_N)|\, dx \leqq |D^i u'_n(x')| + |D^i u'_{n+1}(x')|\,,$$

for $(x', x_N) \in P_n$. Let us denote

$$a_n = \|D^i u'_n\|_{L_1(Q_0)} + \|D^i u'_{n+1}\|_{L_1(Q_0)}\,.$$

Then

$$\int_K |D^i\bar{u}|\,\mathrm{d}x \leqq \sum_{n=1}^{\infty}(t_n - t_{n+1})\,a_n \leqq \sum_{n=1}^{\infty} t_{n+1}(a_n + a_{n+1}) + t_1 a_1\,.$$

The sequence $\{t_n\}$ can be chosen so that the following inequality be valid

$$\int_K |D^i\bar{u}|\,\mathrm{d}x = \|u'\|_{L_1(Q_0)} \quad \text{for} \quad |i| \leqq k\,, \quad i_N = 0\,. \tag{91}$$

Let us set

$$u_{k-1} = \bar{u}\,,$$
$$u_{k-2}(x', x_N) = \int_0^{x_N} u_{k-1}(x', \xi)\,\mathrm{d}\xi\,,$$
$$u_{k-3}(x', x_N) = \int_0^{x_N} u_{k-2}(x', \xi)\,\mathrm{d}\xi\,,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$u(x', x_N) = u_0(x', x_N) = \int_0^{x_N} u_1(x', \xi)\,\mathrm{d}\xi\,.$$

We shall estimate $\int_K |D^i u|\,\mathrm{d}x$ for $|i| \leq k$. If $i = (0, \dots, 0, k)$, then $D^i u = \partial\bar{u}/\partial x_N$ and thus (90) implies the required estimate. If $i = (i_1, \dots, i_N)$, $i_N \leqq k - 1$ then $D^i u = D^{(i_1, \dots, i_{N-1}, 0)} u_{i_N}$. Thus, it suffices to estimate the tangent derivative for the functions $u_0, \dots, u_{k-2}$ in $L_1(K)$. Let D^i denote the tangent derivative. Then, with respect to (91), we obtain

$$\int_K |D^i u_{k-2}|\,\mathrm{d}x \leqq \int_0^b \mathrm{d}x_N \int_{Q_0} \mathrm{d}x' \int_0^{x_N} |D^i\bar{u}(x', \xi)|\,\mathrm{d}\xi \leqq$$
$$\leqq \int_0^b \mathrm{d}x_N \int_K |D^i\bar{u}(x', \xi)|\,\mathrm{d}\xi\,\mathrm{d}x' \leqq b\|u'\|_{L_1(Q_0)}\,.$$

Similar estimates for the functions $u_0, \dots, u_{k-3}$ can be deduced recurrently. Altogether we obtain the estimate

$$\|u\|_{W_1{}^k(K)} \leqq c\|u'\|_{L_1(Q_0)}\,.$$

We find easily that

$$u = \frac{\partial u}{\partial x_N} = \dots = \frac{\partial^{k-2} u}{\partial x_N^{k-2}} = 0\,, \quad \frac{\partial^{k-1} u}{\partial x_N^{k-1}} = \bar{u} = u'$$

in the sense of traces on Q_0.

The assertion follows in the usual way, by means of the decomposition of the unit and by a transformation of the boundary.

Lemma 5. *Let $(u, \alpha_i) \in W^k_\mu(\bar{\Omega})$ be such a function that its side α_ν is absolutely continuous with respect to the measure* d*S on $\partial\Omega$. Then there exists a bounded domain $\Omega^* \supset \bar{\Omega}$ and a function $(u^*, \alpha_i^*) \in W^k_\mu(\bar{\Omega}^*)$ with compact support in Ω^* which satisfies*

$$u^* = u \text{ on } \Omega\,, \quad \alpha_i^* = \alpha_i \text{ on } \Omega\,, \quad \alpha_i^* = 2\alpha_i \text{ on } \partial\Omega\,.$$

Proof. Let us decompose the measure $\alpha_i = \bar{\alpha}_i + \alpha'_i$ as in (79). There exists a function $u' \in L_1(\partial\Omega)$, $u' = \alpha_\nu$ in the sense of the agreement. We choose such Ω^* that it contains the cubes K_r from the decompositions of the unit used in the proofs of Lemmas 3 and 4. Owing to Theorem 15 it holds $\alpha'_{i|\partial\Omega} = \nu_1^{i_1} \dots \nu_N^{i_N} u'$ (see the agreement), $i = (i_1, \dots, i_N)$. According to Lemma 4, there exists a function $u'_1 \in W^k_1(\Omega^* - \bar{\Omega})$ satisfying

(92)
$$u'_1 = \frac{\partial u'_1}{\partial \nu'} = \dots = \frac{\partial^{k-2} u'_1}{\partial \nu'^{k-2}} = 0\,, \quad \frac{\partial^{k-1} u'_1}{\partial \nu'^{k-1}} = (-1)^{k-1}\, u' \quad \text{on} \quad \partial\Omega\,, \quad \nu' = -\nu$$

and moreover $u'_1 = 0$ on the boundary $\partial\Omega^*$. Let us set

$$u_1 = u'_1 \quad \text{on} \quad \Omega^* - \bar{\Omega}\,, \qquad u_1 = 0 \quad \text{on} \quad \bar{\Omega}$$
$$\alpha_{1i} = D^i u'_1 \quad \text{on} \quad \Omega^* - \bar{\Omega}\,, \qquad \alpha_{1i} = \alpha'_i \quad \text{on} \quad \bar{\Omega}\,, \qquad |i| = k\,.$$

We prove that $(u_1, \alpha_{1i}) \in W^k_\mu(\bar{\Omega}^*)$. It is easily to find that $u_1 \in W^{k-1}_1(\Omega^*)$. Let us consider $|i| = k - 1$. We prove that the function $(D^i u_1, \alpha_{1,i+e_m})_{m=1}^N$ can be obtained by the fastening together the functions $(D^i u'_1, D^{i+e_m} u'_1) \in W'_\mu(\bar{\Omega}^* - \Omega)$ and $(0, \alpha'_{i+e_m}) \in W^1_\mu(\bar{\Omega})$. Hence it belongs to $W^1_\mu(\bar{\Omega}^*)$. The function $(0, \alpha'_{i+e_m})$ possesses the trace (Theorem 9)

$$\sum_{m=1}^N \nu_m \alpha'_{i+e_m} = \sum_{m=1}^N \nu_m \nu_1^{i_1} \dots \nu_N^{i_N} \nu_m u' = \nu_1^{i_1} \dots \nu_N^{i_N} u'\,.$$

We can see from (92) that for $|i| = k - 1$

$$D^i u'_1 = \nu_1'^{i_1} \dots \nu_N'^{i_N} (-1)^{k-1}\, u' = \nu_1^{i_1} \dots \nu_N^{i_N} u'\,,$$

holds on $\partial\Omega$ in the sense of traces. From Theorem 12 we conclude $(D^i u_1, \alpha_{1,i+e_m}) \in W^1_\mu(\bar{\Omega}^*)$ and $(u_1, \alpha_{1i}) \in W^k_\mu(\bar{\Omega}^*)$. According to Lemma 3 there exists a function $(u_2, \alpha_{2i}) \in W^k_\mu(\bar{\Omega}^*)$ such that $u_2 = u$ on Ω, $\alpha_{2i} = \alpha_i$ on $\bar{\Omega}$, $|i| = k$. It suffices to set $(u^*, \alpha_i^*) = 2(u_1, \alpha_{1i}) + (u_2, \alpha_{2i})$.

Now it is possible to prove theorems analogous to Theorem 13 and 14.

Theorem 17. *Let us consider a function $(u, \alpha_i) \in W^k_\mu(\bar{\Omega})$ with the side α_ν absolutely continuous with respect to* d*S on $\partial\Omega$. Suppose that $(u^*, \alpha_i^*) \in W^k_\mu(\bar{\Omega}^*)$ is the function* *from Lemma 5, i.e.* $u^* = u$ *on* Ω, $\alpha_i^* = \alpha_i$ *on* Ω, $\alpha_i^* = 2\alpha_i$ *on* $\partial\Omega$. *For small* $h > 0$ *let us set*

$$(93) \qquad u_h(x) = \int_{\Omega^*} K^h(x - y)\, u^*(y)\, dy\,, \quad x \in \Omega\,.$$

Then there holds

$$u_h \rightharpoonup (u, \alpha_i) \quad in \quad W_\mu^k(\bar{\Omega})\,, \qquad \|u_h\|_{W_1^k(\Omega)} \to \|(u, \alpha_i)\|_{W_\mu^k(\Omega)}\,.$$

Proof. For small $h > 0$ and $|i| = k - 1$ we obtain

$$D^i u_h(x) = \int_{\Omega^*} K^h(x - y)\, D^i u^*(y)\, dy\,, \quad x \in \Omega\,.$$

Since $(D^i u^*, \alpha^*_{i+e_m}) \in W_\mu^1(\bar{\Omega}^*)$ we deduce

$$D^{i+e_m} u_h(x) = \int_{\bar{\Omega}^*} K^h(x - y)\, d\alpha^*_{i+e_m}(y)\,, \quad x \in \Omega\,, \quad m = 1, \ldots, N\,,$$

where $|i| = k - 1$.

Thus we obtain

$$(94) \qquad D^i u_h(x) = \int_{\bar{\Omega}^*} K^h(x - y)\, d\alpha_i^*(y)\,, \quad x \in \Omega\,,$$

for $|i| = k$. Evidently

$$u_h \to u \quad \text{in} \quad W_1^{k-1}(\Omega)\,, \qquad \|u_h\|_{W_1^{k-1}(\Omega)} \to \|u\|_{W_1^{k-1}(\Omega)}\,.$$

Following step by step the proof of Theorem 13 we prove $D^i u_h \rightharpoonup \alpha_i$ in $L_\mu(\bar{\Omega})$, $|i| = k$ and

$$\|D^i u_h\|_{L_1(\Omega)} \to \|\alpha_i\|_{L_\mu(\bar{\Omega})}\,, \quad |i| = k\,.$$

Theorem 18. *For each function* $(u, \alpha_i) \in W_\mu^k(\bar{\Omega})$ *there exists* $u_n \in W_1^k(\Omega)$ *such that*

$$u_n \rightharpoonup (u, \alpha_i) \quad in \quad W_\mu^k(\bar{\Omega})\,, \qquad \|u_n\|_{W_1^k(\Omega)} \to \|(u, \alpha_i)\|_{W_\mu^k(\bar{\Omega})}\,.$$

Proof. Let us decompose the function $(u, \alpha_i) = (u, \bar{\alpha}_i) + (0, \alpha_i')$ as in formula (79). As a consequence of Theorem 15 it is $\alpha_{i|\partial\Omega} = v^i \alpha_v$, where α_v is the side of the function (u, α_i) as well as the side of the function $(0, \alpha_i')$, $v^i = v_1^{i_1} \ldots v_N^{i_N}$. Similarly as in Lemma 1 let us set

$$u_h'(x) = \int_{\partial\Omega} R^h(x - y)\, d\alpha_v(y)\,.$$

Let us define the measures $\alpha'_{hi} \in L_\mu(\bar{\Omega})$ by the rule $\alpha'_{hi} = v^i u_h'$ on $\partial\Omega$, $\alpha'_{hi} = 0$ on Ω.

It is easy to see that $(0, \alpha'_{hi}) \in W^k_\mu(\bar\Omega)$ and the function $(0, \alpha'_{hi})$ possesses the side u'_h. Because of Lemma 1 and Theorem 15 we obtain

$$\int_{\bar\Omega} \varphi \, d\alpha'_{hi} = \int_{\partial\Omega} \varphi v^i u'_h \, dS \to \int_{\partial\Omega} \varphi v^i \, d\alpha_v = \int_{\bar\Omega} \varphi \, d\alpha'_i$$

for all $\varphi \in C(\bar\Omega)$ and hence $(0, \alpha'_{hi}) \rightharpoonup (0, \alpha'_i)$ in $W^k_\mu(\bar\Omega)$. By the same argument as in the proof of Theorem 14 we deduce

$$\|\alpha'_{hi|\partial\Omega}\|_{L_\mu(\partial\Omega)} \to \|\alpha'_{i|\partial\Omega}\|_{L_\mu(\partial\Omega)}$$

and hence $\|\alpha'_{hi}\|_{L_\mu(\bar\Omega)} \to \|\alpha'_i\|_{L_\mu(\bar\Omega)}$. Let us set $(u, \alpha_{hi}) = (u, \bar\alpha_i + \alpha'_{hi}) \in W^k_\mu(\bar\Omega)$. It can be seen easily that

$$\|(u, \alpha_{hi})\|_{W_\mu{}^k} = \|(u, \bar\alpha_i)\|_{W_\mu{}^k} + \|(0, \alpha'_{hi})\|_{W_\mu{}^k} \to$$
$$\to \|(u, \bar\alpha_i)\|_{W_\mu{}^k} + \|(0, \alpha'_i)\|_{W_\mu{}^k} = \|(u, \alpha_i)\|_{W_\mu{}^k} .$$

At the same time the side of the function (u, α_{hi}) is absolutely continuous.

The rest of the proof is the same as that of Theorem 14. The duality is defined for $\varphi = \{\varphi_i\}_{|i| \leq k}$, $\varphi_i \in C(\bar\Omega)$ and for $(u, \alpha_i) \in W^k_\mu(\bar\Omega)$ by the formula

$$\langle (u, \alpha_i), \varphi \rangle = \sum_{|i| \leq k-1} \int_\Omega D^i u \varphi_i \, dx + \sum_{|i|=k} \int_{\bar\Omega} \varphi_i \, d\alpha_i .$$

The same theorem on equivalent norms is valid in the space $W^k_\mu(\bar\Omega)$ as in the space $W^k_1(\Omega)$.

Theorem 19. *The formula* $\|u\|_{L_1(\Omega)} + \sum_{|i|=k} \|\alpha_i\|_{L_\mu(\Omega)}$ *is an equivalent norm in the space* $W^k_\mu(\bar\Omega)$.

Proof. Let us suppose that the functions $u_n \in W^k_1$ are those from Theorem 18. $u_n \rightharpoonup (u, \alpha_i)$ in W^k_μ implies

$$\|D^i u\|_{L_1} \leq \lim \|D^i u_n\|_{L_1} \quad \text{for} \quad |i| \leq k-1 \quad \text{and}$$
$$\|\alpha_i\|_{L_\mu} \leq \lim \|D^i u_n\|_{L_1} \quad \text{for} \quad |i| = k .$$

The convergence $\|u_n\|_{W_1{}^k} \to \|(u, \alpha_i)\|_{W_\mu{}^k}$ implies

(95)
$$\|D^i u_n\|_{L_1} \to \|D^i u\|_{L_1} , \quad |i| \leq k-1$$
$$\|D^i u_n\|_{L_1} \to \|\alpha_i\|_{L_\mu} , \quad |i| = k .$$

However the expression $\|u\|_{L_1} + \sum_{|i|=k} \|D^i u\|_{L_1}$ is an equivalent norm in the space W^k_1.

Using (95) we conclude

$$\|(u, \alpha_i)\|_{W_\mu^k} = \lim \|u_n\|_{W_1^k} \leqq c \lim \left[\|u_n\|_{L_1} + \sum_{|i|=k} \|D^i u_n\|_{L_1}\right] =$$

$$= c\left[\|u\|_{L_1} + \sum_{|i|=k} \|\alpha_i\|_{L_\mu}\right].$$

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