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ON THE RELATION OF SOLUTIONS AND COEFFICIENTS OF LINEAR DIFFERENTIAL EQUATION

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0. Let be given k functions

$$x_1(t), x_2(t), \ldots, x_k(t) \tag{1}$$

with continuous derivatives up to the n-th order in the interval $I = (a, b)$, $k < n$. Denote

$$H(t) = \begin{pmatrix} x_1(t), & \ldots, & x_k(t) \\ x_1'(t), & \ldots, & x_k'(t) \\ \ldots & \ldots & \ldots \\ x_1^{(n-1)}(t), & \ldots, & x_k^{(n-1)}(t) \end{pmatrix} \tag{2}$$

for $t \in I$ and assume

$$\operatorname{rank} H(t) = k \tag{3}$$

for all $t \in I$. The following theorem is proved in [1], p. 210 and in [2] by different methods: There is a differential equation

$$y^{(n)} + p_1(t)\, y^{(n-1)} + \ldots + p_{n-1}(t)\, y' + p_n(t)\, y = 0 \tag{4}$$

with coefficients $p_1(t), \ldots, p_n(t)$ continuous in I such that functions (1) are its solutions. It can be easily verified step by step that the method presented in [2] (i.e., completing the family of functions (1) to a system of n functions whose wronskian is different from zero for all $t \in I$) yields all such equations. On the other hand, Ascoli in [1] constructs just one equation. It is the aim of this paper to show that by an obvious generalization of Ascoli's method it is possible to obtain again all the equations of the required properties provided there is no $t \in I$ such that $x_i^{(n)}(t) = 0$ holds simultaneously for $i = 1, 2, \ldots, k$.

1. Theorem. *Let functions* (1) *be defined and have continuous n-th derivative in the interval* $I = (a, b)$. *For every* $t \in I$ *let there exist* $i \in (1, 2, \ldots, k)$ *such that* $x_i^{(n)}(t) \neq$

$\neq 0$. *Let* (3) *hold. Then functions* (1) *are solutions of equation* (4) *if and only if there exists a positive definite* $(n \times n)$*-matrix* $A(t)$ *defined and continuous for all* $t \in I$ *so that*

$$p = -AH(H^*AH)^{-1}\, x^{(n)} \tag{5}$$

where $p = p(t) = (p_n(t), \ldots, p_1(t))$, $x^{(n)} = x^{(n)}(t) = (x_1^{(n)}(t), \ldots, x_k^{(n)}(t))$ *and the asterisk denotes transposition of the matrix.* (All vectors occuring in the paper are to be considered column vectors.)

Remark 1. Matrix H^*AH is obviously regular, even positive definite:

$$(H^*AHu, u) = (AHu, Hu) = (Av, v) > 0\,, \quad v = Hu\,.$$

Remark 2. Let us mention the relation of Theorem to the result of Ascoli [1]. If φ is the solution of $H^*H\varphi = -x^{(n)}$ (which is unique), then putting $p = H\varphi$ we obtain an equation (4) with solutions (1). (This is Ascoli's result.) Taking the solution of $H^*AH\varphi = -x^{(n)}$ instead, then p is expressed by (5).

Proof. If (5) holds, then $H^*p = -x^{(n)}$, i.e.

$$p_n x_j + p_{n-1}x_j' + \ldots + p_1 x_j^{(n-1)} = -x_j^{(n)}$$

for $j = 1, 2, \ldots, k$. Hence functions (1) are solutions of (4).

To prove that (5) is a necessary condition, let us establish

2. Lemma. *If* u, v *are* n*-vectors,* $(u, v) > 0$, *then there is a symmetrical positive definite matrix* A *such that* $v = Au$.

Proof. Consider first $n = 2$, $u = (u_1, 0)$, $v = (v_1, v_2)$. By an elementary calculation we obtain from $v = Au$ that

$$A = \begin{pmatrix} \dfrac{v_1}{u_1}, & \dfrac{v_2}{u_1} \\ \dfrac{v_2}{u_1}, & c \end{pmatrix} \tag{6}$$

with an arbitrary c. Since in this case $(u, v) = u_1 v_1 > 0$, $\det A = v_1 c/u_1 - (v_2/u_1)^2$, it is sufficient to choose $c > v_2^2/(u_1 v_1) > 0$ in order that the matrix A be positive definite.

If now u, v are general n-vectors, then a rotation maps them onto vectors $p = (p_1, 0, \ldots, 0)$, $q = (q_1, q_2, 0, \ldots, 0)$. The matrix B of the rotation is orthonormal i.e. $B^{-1} = B^*$ and hence $(p, q) = (Bu, Bv) = (u, B^*Bv) = (u, v) > 0$.

According to the first part of the proof there is a symmetrical positive definite (2×2)-matrix C, $\tilde{q} = C\tilde{p}$ where $\tilde{p} = (p_1, 0)$, $\tilde{q} = (q_1, q_2)$. Put

$$M = \begin{bmatrix} C & \begin{matrix} 0, & \dots, & 0 \\ 0, & \dots, & 0 \end{matrix} \\ \begin{matrix} 0, & 0, \\ \dots & \\ 0, & 0, \end{matrix} & D \end{bmatrix}$$

where D is an arbitrary symmetrical positive definite matrix of order $n - 2$. Then obviously $q = Mp$. As $B^{-1} = B^*$, the matrix $A = B^{-1}MB = B^*MB$ is symmetrical positive definite and $v = Au$ which completes the proof.

3. Proof of Theorem (continuation). Now let $p(t) = (p_n(t), \dots, p_1(t))$ be the vector of coefficients of equation (4) which is satisfied by functions (1). We shall show that there is a matrix A with the required properties such that the equation

$$H\varphi = A^{-1}p \tag{7}$$

has a solution and hence (5) holds.

Denote by $\mathscr{H}$ the space of all n-vectors $H(t)\,\varphi(t)$ where $\varphi(t)$ is an arbitrary k-vector. Obviously $H^*(t)\,p(t) = -x^{(n)}(t)$ (cf. the preceding part of the proof). Hence $p(t)$ is not orthogonal to $\mathscr{H}$ since

$$(p, H\varphi) = (H^*p, \varphi) = (-x^{(n)}, \varphi)$$

and $x^{(n)} = x^{(n)}(t) \neq 0$. We can write $p = \xi + \eta$ where η is orthogonal to $\mathscr{H}$ and $0 \neq \xi \in \mathscr{H}$ is the orthogonal projection of p to $\mathscr{H}$. Consequently

$$(p, \xi) = (\xi, \xi) + (\eta, \xi) = (\xi, \xi) > 0$$

and according to Lemma there exists a positive definite matrix $A = A(t)$ such that $p = A\xi$. Hence $A^{-1}p \in \mathscr{H}$ which means that (7) has a solution $\varphi = \varphi(t)$. It holds $H^*AH\varphi = H^*p = -x^{(n)}$, $p = AH\varphi = -AH(H^*AH)^{-1}\,x^{(n)}$ and (5) is established.

It remains to prove that the matrix $A(t)$ (which is not uniquely determined) may be chosen so that it is continuous in I. Since the matrix $H(t)$ is continuous in I, there exists a continuous basis of the space $\mathscr{H}$. It is evident that the continuity of $p(t)$ then implies the continuity of the projection $\xi(t)$.

From the proof of Lemma it is easy to see that the matrix $A(t)$ mapping $\xi(t)$ onto $p(t)$ may now be chosen as a continuous function. In fact, this is obvious for the matrices B (representing a rotation) and D (which need not depend on t at all). The continuity of the matrix C follows from (6) and from the fact that $u_1(t) \neq 0$. Hence the proof is complete.

Remark 3. If $x_i^{(n)}(\tau) = 0$ for some $\tau \in I$ and for all $i = 1, 2, \ldots, k$, then only those vectors $p(t)$ satisfying $p_j(\tau) = 0, j = 1, 2, \ldots, n$ can be written in the form (5) (with a positive definite matrix A). Particularly, if $x_i^{(n)}(t) = 0$ in I, $i = 1, 2, \ldots, k$, then the single equation $y^{(n)} = 0$ is obtained independently of the choice of the matrix A.

References

[1] *Ascoli, G.:* Sulla decomposizione degli operatori differenziali lineari. Revista (Univ. Nac. Tucuman), Ser. A, *1* (1940), pp. 189—215.

[2] *Jarník, J.:* A note to the construction of a linear differential equation with given solutions. Čas. pěst. mat. *95* (1970), pp. 269—277.

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