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ON CHROMATIC AND ACHROMATIC NUMBERS OF UNIFORM HYPERGRAPHS

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1. BASIC NOTIONS

(Cf. BERGE [1].) By a hypergraph H is meant a couple $(X, \mathscr{E})$, where X is a finite set of elements (called *vertices*) and $\mathscr{E} = \{E_1, \ldots, E_m\}$ is a finite system of subsets $E_i \neq \emptyset$ of X (called *edges*) such that $\bigcup_{i=1}^{m} E_i = X$ and for $i \neq j$ is $E_i \neq E_j$. A hypergraph is k-uniform, $k > 1$, if all edges have cardinality k. A 2-uniform hypergraph is called *graph*. A k-uniform hypergraph with $n \geqq k$ vertices is *complete* if its set of edges consists of all k-tuples formed from the n vertices.

The *complement* of a k-uniform hypergraph $H = (X, \mathscr{E})$ is the hypergraph $\bar{H} = (\bar{X}, \bar{\mathscr{E}})$ whose edges are all those k-tuples $\bar{E}_1, \bar{E}_2, \ldots$ formed from vertices of X which are not contained in $\mathscr{E}$ and whose vertex set $\bar{X}$ is the union of all these edges. (Notice that our uniform hypergraphs have no "isolated" vertices and that for the complement $\bar{H} = (\bar{X}, \bar{\mathscr{E}})$ of $H = (X, \mathscr{E})$ $|\bar{X}| \leqq |X|$ holds.) (By $|X|$ the cardinality of the set X is denoted.)

$H' = (X', \mathscr{E}')$ is a *partial hypergraph* of the hypergraph $H = (X, \mathscr{E})$, defined by the set of edges $\mathscr{E}' \subseteq \mathscr{E}$, if X' consists of all vertices belonging to edges from $\mathscr{E}'$.

A *coloring* of the hypergraph H is an assignment of colors to all vertices of H such that not all vertices of an edge of H are assigned the same color and every vertex is assigned one color. Two colors c_1, c_2 in a coloring of the hypergraph H are *adjacent* if there exists an edge of H containing two vertices colored by c_1 and c_2. A coloring of a hypergraph H is *complete* if all pairs of used colors are mutually adjacent. (Clearly, if a coloring uses the minimum possible number of colors then it is complete.) In the paper we deal with the *chromatic number* $\chi(H)$ or the *achromatic number* $\psi(H)$ of a hypergraph H which means the minimum or maximum number, respectively, of colors used in a complete coloring of H.

2. CHROMATIC NUMBERS

We want to estimate the number $\chi(H) + \chi(\bar{H})$ supposing H to be a k-uniform hypergraph. For $k = 2$, i.e. for a graph G, NORDHAUS and GADDUM [6] proved the inequality

$$\chi(G) + \chi(\bar{G}) \leqq n + 1,$$

where n is the number of vertices of the graph G.

Trying to generalize this result for $k > 2$ and wishing to give a relation between the numbers $\chi(H) + \chi(\bar{H})$, the uniformity k and the number of vertices n of H only, we obtain the following bound:

$$\chi(H) + \chi(\bar{H}) \leqq 2 \left] \frac{n}{k-1} \right[. \tag{1}$$

($]x[$ denotes the smallest integer $\geqq x$.)

Although hypergraphs can be constructed for which the equality in (1) is attained (e.g. the 3-uniform hypergraph with vertices 1, 2, 3, 4 and edges (1, 2, 3), (1, 2, 4)) there exist many hypergraphs for which (1) is too rough. We present here estimates depending on other invariants of hypergraphs. In course of the proof a well-known theorem by GALLAI will be generalized. First the new invariants must be introduced. (Cf. Berge [1]).

A set of vertices of the hypergraph H is called *stable* if it does not contain the vertices of a whole edge of H; the maximum cardinality of a stable set of H is called the *stability number* of H and denoted by $\alpha(H)$.

A set T of vertices of the hypergraph H is said to be *transversal* if every edge of H has a non-empty intersection with T; the *transversal-number* of H, $\tau(H)$, is the minimum cardinality of such a set.

A set N of edges of the hypergraph H is called *independent* if all the edges of N are pairwise disjoint; the maximum cardinality of an independent set, $\nu(H)$, is called the *independence number* of H.

A set of edges of the hypergraph H is called a *covering* set if its union is the whole vertex set of H; the minimum cardinality of a covering set of edges of H is called the *covering number* of H and denoted by $\varrho(H)$.

Theorem 1. *For a k-uniform hypergraph* $H = (X, \mathscr{E})$

$$\chi(H) + \chi(\bar{H}) \leqq \left] \frac{\tau(H)}{k-1} \right[+ \varrho(H) + 1 \tag{2a}$$

$$\chi(H) + \chi(\bar{H}) \leqq \left] \frac{|X| - \alpha(H)}{k-1} + 1 \right[+ \left] \frac{|X| - \nu(H)}{k-1} \right[\tag{2b}$$

$$\chi(H) + \chi(\bar{H}) \leqq \left] \frac{\tau(H)}{k-1} + 1 \right[+ \left] \frac{|X| - \nu(H)}{k-1} \right[\tag{2c}$$

holds.

To prove Theorem 1 we need some lemmas.

Lemma 1. *For a hypergraph* $H = (X, \mathscr{E})$,

$$\alpha(H) + \tau(H) = |X| . \tag{3}$$

Proof of Lemma 1. a) Let $S \subset X$ be a stable set of maximal cardinality. Then $T = X - S$ is a transversal set of the hypergraph H and we have

$$|T| = |X - S| = |X| - |S| = |X| - \alpha(H)$$

and $\tau(H) \leqq |T|$, so that

$$\tau(H) \leqq |X| - \alpha(H) . \tag{4}$$

b) Let $T \subset X$ be a transversal set of minimal cardinality. Then $S = X - T$ is a stable set of the hypergraph H because no edge of H has all its vertices in S. We have

$$|S| = |X - T| = |X| - |T| = |X| - \tau(H)$$

and $\alpha(H) \geqq |S|$, so that

$$\alpha(H) \geqq |X| - \tau(H) . \tag{5}$$

From (4) and (5), (3) follows.

Lemma 2. *For a k-uniform hypergraph* $H = (X, \mathscr{E})$ *the following inequalities hold*:

$$\varrho(H) + (k - 1) . \nu(H) \leqq |X| , \tag{6}$$

$$\nu(H) + (k - 1) . \varrho(H) \geqq |X| . \tag{7}$$

Proof of Lemma 2. a) First we prove the inequality (6). Let $N \subseteq \mathscr{E}$ be an independent set of edges of H of maximal cardinality. Denote by S the set of all vertices of H not lying in any edge of N. The maximality of N implies that S is a stable set. To every vertex v of S associate an edge incident to v and denote by M the set of these edges. Obviously $|M| \leqq |S|$. Consider the set of edges $K = N \cup M$ belonging either to N or to M. Since the sets of edges N, M are disjoint, we have

$$|K| = |N| + |M| \leqq |N| + |S| .$$

As $|S| = |X| - k . |N|$, we have

$$|K| \leqq |N| + |X| - k . |N| .$$

However, K is a covering set of edges of H, i.e., $\varrho(H) \leqq |K|$ which implies

$$\varrho(H) \leqq \nu(H) + |X| - k . \nu(H)$$

and (6) follows.

b) Let P be a covering set of edges of the hypergraph $H = (X, \mathscr{E})$ of minimal cardinality. For the partial hypergraph $H_0 = (X, P)$, it follows from its construction that

(8) $$\varrho(H_0) = \varrho(H) \quad \text{and} \quad \nu(H_0) \leqq \nu(H)\,.$$

We first prove that

(9) $$|X| - \nu(H_0) \leqq (k - 1)\,\varrho(H_0)$$

holds. Let Q be an independent set with maximal number of edges of the hypergraph H_0, i.e., $|Q| = \nu(H_0)$. Obviously $Q \subseteq P$. For the set of edges $B = P - Q$ we have

(10) $$|B| = |P - Q| = |P| - |Q| = \varrho(H_0) - \nu(H_0) = \omega\,.$$

Every edge from the set B has at least one vertex in an edge belonging to the set Q, because otherwise Q would not be a maximal independent set. Thus the number of vertices belonging to edges from B and not belonging to edges from Q is at most $(k - 1)\,.\,\omega$. The number of vertices belonging to edges from Q is $k\,.\,\nu(H_0) \leqq |X|$. From this the relation

$$|X| - k\,.\,\nu(H_0) \leqq (k - 1)\,.\,\omega$$

follows. Now using (10) we get

$$|X| - k\,.\,\nu(H_0) \leqq (k - 1)\,.\,(\varrho(H_0) - \nu(H_0))$$

and (9). Using further the relations (8) we get the assertion (7) of Lemma 2.

Remark 1. (3), (6), (7) are generalizations of Gallai's [2] relations for graphs. The proofs of (3), (6) are in fact Gallai's arguments. Clearly, the inequalities (6), (7) are for all $k \geqq 2$ the best possible.

Lemma 3. *For a k-uniform hypergraph H with n vertices,*

$$\chi(H) \leqq \left]\frac{n}{k - 1}\right[$$

holds.

Proof of Lemma 3 is quite simple and can be omitted.

Proof of Theorem 1. Let N be an independent set of edges of the hypergraph H with maximal cardinality, i.e., $|N| = \nu(H)$. The number of vertices belonging to edges from N is $k\,.\,\nu(H)$. The vertices of the hypergraph H not belonging to edges from the set N are colorable with $](|X| - k\,.\,\nu(H))/(k - 1)[$ colors (by Lemma 3). The edges of N do not belong to the hypergraph $\bar{H}$. Hence if we assign all vertices of an edge of N

the same color we obtain a coloring of the hypergraph $\bar{H}$. We have

$$\chi(\bar{H}) \leqq \left]\frac{|X| - \nu(H)}{k - 1}\right[.$$

Using the relation (7) of Lemma 2 we have

$$\chi(\bar{H}) \leqq \varrho(H) . \tag{11}$$

Let S be a stable set of vertices of the hypergraph H with maximal cardinality, i.e., $|S| = \alpha(H)$. Lemma 3 implies that the vertices of the set $(X - S)$ are colorable by $](X - \alpha(H))/(k - 1)[$ colors. Assigning the vertices of S the same color we get a coloring of the hypergraph H, i.e.,

$$\chi(H) \leqq \left]\frac{|X| - \alpha(H)}{k - 1}\right[+ 1 .$$

By Lemma 1 we have

$$\chi(H) \leqq \left]\frac{\tau(H)}{k - 1}\right[+ 1 . \tag{12}$$

Adding different expressions for $\chi(H)$ and $\chi(\bar{H})$ we get the assertion of Theorem 1.

Equality in (2a) – (2c) is attained for a k-uniform hypergraph H with n vertices such that $n - 1 \equiv 0 \pmod{k - 1}$ and there is a vertex in H which is the unique common vertex of every pair of edges of H. For such a hypergraph with a "large" number of edges (2) is much better than the above mentioned generalization (1) of the Nordhaus-Gaddum estimate. Many other hypergraphs can be constructed for which equality in (2a) – (2c) is attained.

3. ACHROMATIC NUMBERS

Harary and Hedetniemi [5] have given some bounds for the achromatic number of a graph G using a homomorphic mapping of G onto the complete graph $K_{\psi(G)}$. We do not see how this technique could be employed for treating the problem in case of k-uniform hypergraphs for $k > 2$. We give here some simple bounds for the achromatic number of a hypergraph H. They are strict; however, for many hypergraphs they give rather rough estimates.

Theorem 2. *For the achromatic number $\psi(H)$ of a k-uniform hypergrah H with h edges, the inequality*

$$\psi(H) \leqq \xi \tag{13}$$

holds where ξ is the positive solution of the equation

$$x^2 - x - h(k^2 - k) = 0 .$$

Proof. If H is completely colored by $\psi(H) = m$ colors then in H there are $\binom{m}{2}$ pairs of adjacent colors. In one edge of H at most $\binom{k}{2}$ different pairs of adjacent colors can occur. Then we have

$$h \geqq \frac{\binom{m}{2}}{\binom{k}{2}}.$$

From this our statement follows.

From the argument above it follows that in (13) equality is attained for a hypergraph H admitting such a coloring that each pair of colors is adjacent in exactly one edge of H; balanced incomplete block designs $(m, k; 1)$ (formed from m elements, each block having k elements, each pair of elements occurring in exactly one block) are such hypergraphs. E.g. the finite projective plane with m points (lines are edges of the hypergraph) has achromatic number m (cf. HALL [4]).

The *strong-stability number* of a hypergraph H, $\bar{\alpha}(H)$, is the maximum cardinality of a set of vertices of H no two of which belong to the same edge of H.

Let $E(v) = \{E^1, \ldots, E^n\}$ be the set of all edges of the hypergraph H such that the vertex v of H belongs to all of them. By the *degree* $m(v)$ of the vertex v we mean the minimum cardinality of a subset of $E(v)$ whose union of vertices is equal to the union of vertices of all edges from $E(v)$.

Theorem 3. *For a k-uniform hypergraph H with maximum degree of a vertex equal to m,*

$$\psi(H) \leqq m \,.\, \bar{\alpha}(H) \,.\, (k - 1) + 1 \,. \tag{14}$$

The proof is based on the following Lemma which is a generalization of a statement by Harary-Hedetniemi [5].

Lemma 4. *For a hypergraph H with p vertices,*

$$\psi(H) \leqq p - \bar{\alpha}(H) + 1 \,. \tag{15}$$

Proof of Lemma 4. Consider any complete coloring of the hypergraph $H = (X, \mathscr{E})$ and any strong-stable set S of vertices of H. If all vertices of S have the same color then the total number of colors used in the coloring is not greater than $p - |S| + 1$. If two vertices x and y from S are colored by different colors $k(x)$, $k(y)$ there must be in $X - S$ a vertex colored by $k(x)$ or $k(y)$ because the considered coloring is a complete one. Generally, at least $s - 1$ vertices of $X - S$ are assigned colors which occur in S, where s is the total number of colors appearing in S. From this (15) follows.

Equality in (15) is attained e.g. for the hypergraph with 6 vertices 1, ..., 6 and edges (1, 2, 3), (3, 4, 5), (5, 6, 1).

Proof of Theorem 3. Associate to every vertex v of a strong-stable set S of maximal cardinality the edges to which it belongs. Such a "star" contains at most $(k - 1) \,.\, d + 1$ vertices where d is the degree of v. The union of all "stars" associated to vertices of S contains all vertices of the hypergraph H, because otherwise S would not be a strong-stable set of maximal cardinality. This implies $v \leqq (k - 1) \,.\, m \,.\, \bar{\alpha}(H) + \bar{\alpha}(H)$, and using (15) we obtain the assertion of our theorem.

Equality in (14) is attained for complete k-uniform hypergraphs with p vertices if $(p - 1) \equiv 0 \pmod{k - 1}$. Evidently, for these and "similar" hypergraphs, (14) is a better bound than (13). Nevertheless, e.g. for the 3-uniform hypergraph consisting of seven disjoint edges equality in (13) holds while (14) is almost meaningless.

Remark 2. For a graph G with n vertices and its complement $\bar{G}$ the following relations are known:

$$\psi(G) + \psi(\bar{G}) \leqq \left]\frac{4}{3} n\right[\quad \text{(Gupta [3])},$$

$$\psi(G) + \chi(\bar{G}) \leqq n + 1 \quad \text{(Harary-Hedetniemi [5])}.$$

Trying to generalize these bounds to a k-uniform hypergraph H with $k > 2$ having n vertices we obtain very easily (using Lemma 3) the relations:

$$\psi(H) + \psi(\bar{H}) \leqq 2n,$$

$$\psi(H) + \chi(\bar{H}) \leqq n + \left]\frac{n}{k - 1}\right[.$$

Examples can be constructed showing that these bounds are sharp, too. The first estimate is sharp e.g. for the finite projective plane with 7 vertices. Equality in the second one is attained for the hypergraph $(X, \mathscr{E})$, $X = \{1, 2, \ldots, 6\}$, $\mathscr{E} = \{(1, 2, 3,),$ $(1, 2, 4), (1, 2, 5), (1, 2, 6), (1, 3, 4), (1, 3, 5), (1, 4, 5), (1, 4, 6), (1, 5, 6\}$. However, for many hypergraphs these bounds are rough, and it would be desirable to find better ones depending on different invariants of the hypergraph.

Added in proof (February 1974): The arguments employed in the proof of Theorem 1 yield also the following estimate for a k-uniform hypergraph H with n vertices

$$\chi(H) + \chi(\bar{H}) \leqq \left]\frac{n \,.\, (2k - 1)}{k\,(k - 1)}\right[+ 1.$$

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