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Bohdan Zelinka

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ON A PROBLEM OF R. HÄGGKVIST CONCERNING EDGE-COLOURINGS OF GRAPHS

BOHDAN ZELINKA, Liberec

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At the 5th Hungarian Colloquium on Combinatorics in Keszthely in 1976 R. HÄGGKVIST has proposed the following problem [1]:

Let $Q(n, G)$ be the set of n-line-colourings of G. Let $q \in Q(n, G)$. Define $L(q)$ $(l(q))$ as the maximal (minimal) length of a cycle with edges from two of q's line-colour classes. Put

$$L(n, G) = \min_{q \in Q(n,G)} L(q), \quad l(n, G) = \max_{q \in Q(n,Q)} l(q).$$

Give bounds on $L(n, G)$ and $l(n, G)$ for reasonable defined graphs G. Especially: Is $L(n, K_{n,n}) = 2n$?

In this paper we shall study $L(n, K_{n,n})$ for n which is a power of 2. Instead of "line" we shall say "edge".

Theorem. *Let $n = 2^m$, where m is a positive integer. Then $L(n, K_{n,n}) = 4$.*

Proof. For each positive integer m denote $G(m) = K_{n,n}$, where $n = 2^m$. Denote $N = \{1, 2, \ldots, n\}$, $P = \{n + 1, n + 2, \ldots, 2n\}$. The vertices of $G(m)$ are $u_1, \ldots, u_n$, $v_1, \ldots, v_n$, the edges are u_iv_j for each i and j from N. For each $G(m)$ we shall introduce an edge-colouring $q(m)$ by n colours such that no vertex of $G(m)$ is incident with any two edges of the same colour. We define it recurrently. In the graph $G(1)$ we colour the edges u_1v_1, u_2v_2 by the colour 1, the edges u_1v_2, u_2v_1 by the colour 2. Now let the colouring $q(m)$ of $G(m)$ by the colours from N be given for some m; we shall construct the colouring $Q(m + 1)$ of the edges of $G(m + 1)$. Consider four graphs H_1, H_2, H_3, H_4 which are all isomorphic to $G(m)$. The vertices of H_1 are denoted in the same way as in $G(m)$; we may consider $G(m)$ and H_1 as the same graph. The vertices of H_2 are $u_{n+1}, \ldots, u_{2n}, v_{n+1}, \ldots, v_{2n}$ and the edges are u_iv_j for all i and j from P. The vertices of H_3 are $u_1, \ldots, u_n, v_{n+1}, \ldots, v_{2n}$ and the edges are u_iv_j for each i from N and each j from P. The vertices of H_4 are $u_{n+1}, \ldots, u_{2n}$, $v_1, \ldots, v_n$ and the edges are u_iv_j for each i from P and each j from N. Now we shall colour the edges of the graphs H_1, H_2, H_3, H_4. The graph H_2 is considered the same as $G(m)$, therefore its edges will be coloured by the colours from N in the same way as the edges of $G(m)$. Also the edges of H_2 will be coloured by the colours from N;

the edge $u_i v_j$ is coloured by the same colour as the edge $u_{i-n} v_{j-n}$ of $G(m)$. The edges of H_3 will be coloured by the colours from P; the edge $u_i v_j$ is coloured by the colour $c + n$, where c is the colour of the edge $u_i v_{j-n}$ in $G(m)$. The edges of H_4 will be coloured also by the colours from P; the edge $u_i v_j$ is coloured by the colour $c + n$, where c is the colour of the edge $u_{i-n} v_j$ in $G(m)$. Now we shall take the graphs H_1, H_2, H_3, H_4 and identify all pairs of vertices which are denoted by the same symbol in two of these graphs; thus we obtain the graph $G(m + 1)$. We preserve the colours of edges; evidently the colouring thus obtained is a colouring of $G(m + 1)$ by $2n$ colours and no vertex of $G(m + 1)$ is incident with two edges of the same colour.

Now consider the cycles in $G(m + 1)$ whose edges are coloured only by two colours. In $G(1)$ we have only one cycle and it has the length 4. We shall proceed by induction; suppose that in $G(m)$ each cycle whose edges are coloured by two colours in $q(m)$ has the length 4 and consider the graph $G(m + 1)$ with the above constructed colouring $q(m + 1)$. Let c_1, c_2 be two of the colours $1, \ldots, 2n$. If $c_1 \in N$, $c_2 \in N$, such a cycle is either wholly in H_1, or wholly in H_2. As these graphs were coloured in the same way as $G(m)$, this cycle must have the length 4. Analogously if $n + 1 \leqq c_1 \leqq 2n$, $n + 1 \leqq c_2 \leqq 2n$, such a cycle is either wholly in H_3, or wholly in H_4 and it must have the length 4. Let $1 \leqq c_1 \leqq n$, $n + 1 \leqq c_2 \leqq 2n$. Let C be a cycle whose edges are coloured only by the colours c_1 and c_2. Without loss of generality we may suppose that C contains an edge $u_i v_j$ of H_1; it is coloured by c_1. Now let k be such a number that $v_j u_k$ is an edge of C coloured by c_2; this means that the edge $v_j u_{k-n}$ of $G(m)$ is coloured by $c_2 - n$. Let l be such a number that $u_k v_l$ is an edge of C coloured by c_1; then $u_{k-n} v_{l-n}$ in $G(m)$ is coloured also by c_1. The edge $v_l u_i$ is in H_3 and is coloured by the colour $c + n$, where c is the colour of the edge $u_i v_{l-n}$ in $G(m)$; but $c = c_2 - n$, because in $G(m)$ there is a cycle with the length 4 with the vertices $u_i, v_j, u_{k-n}, v_{l-n}$ whose edges are coloured by the colours c_1 and $c_2 - n$. (The only exception is $k = i + n$, $l = j + n$, but also in this case we have evidently a cycle of the length 4.) Thus $v_l u_i$ is coloured also by c_2 and C has the length 4. Analogously if $n + 1 \leqq c_1 \leqq 2n$, $1 \leqq c_2 \leqq n$. Thus for every positive integer m we have $L(q(m)) = 4$ and $L(n, G(m)) \leqq 4$. As in a bipartite graph without multiple edges no cycle has a length smaller than 4, we have $L(n, G(m)) = 4$ and this means $L(n, K_{n,n}) = 4$ for each $n = 2^m$, where m is a positive integer.

This is also the negative answer to the question at the end of the problem. If $n = 2^m$, where $m \geqq 2$ is an integer, then $L(n, K_{n,n}) \neq 2n$.

Reference

[1] Proceedings of the Fifth Hungarian Colloquium on Combinatorics held in Keszthely, June 28—July 3, 1976 (to appear).

Author's address: 460 01 Liberec, Komenského 2 (Vysoká škola strojní a textilní).