Jozef Nagy; Eva Nováková

Local determinacy of symmetric pseudoprocesses

Časopis pro pěstování matematiky, Vol. 104 (1979), No. 3, 223--242

Persistent URL: `http://dml.cz/dmlcz/118018`

ČASOPIS PRO PĚSTOVÁNÍ MATEMATIKY
Vydává Matematický ústav ČSAV, Praha
SVAZEK 104 * PRAHA 20. 8 1979 * ČÍSLO 3

LOCAL DETERMINACY OF SYMMETRIC PSEUDOPROCESSES

JOZEF NAGY, EVA NOVÁKOVÁ, Praha

(Received January 20, 1976)

A number of various physical systems can be described by means of relations in the cartesian product $P \times T$, where P is the set of all possible states of the system concerned and T is a set of time instants. This approach to the study of the behaviour of systems was used in [1] to [5], where a wide class of such relations is investigated in detail. The present paper is a direct continuation of [5] so that as far as the notation and the terminology is concerned, the reader is refered to [5]. To make the text of the paper as self-contained as possible, the basic notions and notation from [5] will be recalled in the first point of the next section.

1. SYMMETRIC PSEUDOPROCESSES

1.1. Notation. In what follows, P denotes an arbitrary set, R the set of all reals, $R^* = R \cup \{+\infty, -\infty\}$ the extended real line with the ordering extended from R to R^* in the natural way, T a subset of R.

If X, Y are sets, then any subset of the cartesian product $X \times Y$ (in this order) is called a relation between X and Y. If $X = Y$, then a relation $\mathsf{r} \subset X \times X$ is called a relation in X. The relation inverse to a relation r is denoted by r^{-1}. The identity relation in X is denoted by 1_X. If $\mathsf{r} \subset X \times Y$, $\mathsf{s} \subset Y \times Z$, then the composition of the relations r and s (in this order) is denoted by $\mathsf{r} \circ \mathsf{s}$. If a pair $(x, y) \in X \times Y$ belongs to a relation $\mathsf{r} \subset X \times Y$, then we write either $(x, y) \in \mathsf{r}$ or $x\mathsf{r}y$. Given $\mathsf{r} \subset X \times Y$, we set

(1.1.1) $$D_{\mathsf{r}} = \{y \in Y \mid x\mathsf{r}y \text{ for some } x \in X\},$$

(1.1.2) $$I_{\mathsf{r}} = \{x \in X \mid x\mathsf{r}x\} \text{ if } X = Y,$$

(1.1.3) $$\mathsf{r}y = \{x \in X \mid (x, y) \in D_{\mathsf{r}}\},$$

(1.1.4) $$\mathsf{r}A = \{x \in X \mid (x, y) \in D_{\mathsf{r}} \text{ for some } y \in A\},$$

$$(1.1.5)\qquad x\mathsf{r} = \{y \in Y \mid (x, y) \in D_{\mathsf{r}}\},$$

$$(1.1.6)\qquad B\mathsf{r} = \{y \in Y \mid (x, y) \in D_{\mathsf{r}} \text{ for some } x \in B\},$$

$$(1.1.7)\qquad \mathsf{r}|_A = \mathsf{r} \cap (X \times A),$$

for each $y \in Y$, $A \subset Y$, $x \in X$, $B \subset X$.

In the present paper we shall be concerned mainly with relations p, q in $P \times T$, i.e. with subsets of $(P \times T) \times (P \times T)$. Each such relation $\mathsf{q} \subset (P \times T) \times (P \times T)$ can be uniquely described by the two-parametric system of relations ${}_v\mathsf{q}_u$ in P with $u, v \in T$ as follows:

$$(1.1.8)\qquad (y, v)\,\mathsf{q}(x, u) \quad \text{iff} \quad y\,{}_v\mathsf{q}_u x, \quad x, y \in P, \quad u, v \in T.$$

A relation p in $P \times T$ such that

$$(\mathrm{I})\qquad {}_u\mathsf{p}_u \subset 1_P \quad \text{for each} \quad u \in T$$

and

$$(\mathrm{R})\qquad {}_v\mathsf{p}_u \neq \emptyset \quad \text{implies} \quad u \leqq v \quad \text{for all} \quad u, v \in T$$

is called a *right pseudoprocess in P over T*. The set of all right pseudoprocesses in P over T i sdenoted by $\mathsf{Ps}(P, T)$. A right pseudoprocess $\mathsf{p} \in \mathsf{Ps}(P, T)$ is said to be a *compositive right pseudoprocess*, a *transitive right pseudoprocess* or a *right process in P over T* iff the condition

$$(\mathrm{RC})\qquad {}_v\mathsf{p}_u \subset {}_v\mathsf{p}_t \circ {}_t\mathsf{p}_u \quad \text{for all} \quad u \leqq t \leqq v \quad \text{in} \quad T,$$

$$(\mathrm{RT})\qquad {}_v\mathsf{p}_u \supset {}_v\mathsf{p}_t \circ {}_t\mathsf{p}_u \quad \text{for all} \quad u \leqq t \leqq v \quad \text{in} \quad T$$

or

$$(\mathrm{RP})\qquad {}_v\mathsf{p}_u = {}_v\mathsf{p}_t \circ {}_t\mathsf{p}_u \quad \text{for all} \quad u \leqq t \leqq v \quad \text{in} \quad T,$$

is satisfied, respectively. The set of all compositive right pseudoprocesses, transitive right pseudoprocesses and right processes in P over T will be denoted by $\mathsf{Psc}(P, T)$, $\mathsf{Pst}(P, T)$ and $\mathsf{P}(P, T)$. A more detailed explanation of the theory of right pseudoprocesses may be found in [5].

1.2. Definition. Let P be an arbitrary set, $T \subset R$, q a relation in $P \times T$. The relation q is called a *symmetric pseudoprocess in P over T* iff it satisfies the conditions

$$(\mathrm{I})\qquad {}_u\mathsf{q}_u \subset 1_P \quad \text{for all} \quad u \in T,$$

$$(\mathrm{S})\qquad {}_v\mathsf{q}_u = ({}_u\mathsf{q}_v)^{-1} \quad \text{for all} \quad u, v \in T.$$

The set of all symmetric pseudoprocesses in P over T will be denoted by $\mathsf{Ss}(P, T)$.

1.3. Remark. The property (S) in 1.2 may be reformulated as

$$(1.3.1) \qquad y_v \mathsf{q}_u x \quad \text{iff} \quad x_u \mathsf{q}_v y \quad \text{for all} \quad x, y \in P, \quad u, v \in T.$$

Hence, this property is equivalent with

$$(1.3.2) \qquad \mathsf{q} = \mathsf{q}^{-1}.$$

Thus

$$(1.3.3) \qquad {}_v\mathsf{q}_u = {}_u\mathsf{q}_v \quad \text{for all} \quad u, v \in T.$$

For a symmetric pseudoprocess q, the sets D_q and I_q from (1.1.1) and (1.1.2) may be characterized as follows:

$$D_\mathsf{q} = \{(x, u) \in P \times T \mid {}_v\mathsf{q}_u x \neq \emptyset \text{ for some } v \in T\}$$

and

$$I_\mathsf{q} = \{(x, u) \in D_\mathsf{q} \mid x_u \mathsf{q}_u x\}.$$

1.4. Construction. Let $\mathsf{p} \in \mathrm{Ps}(P, T)$. It is not difficult to verify that the relation $\mathsf{p} \cup \mathsf{p}^{-1}$ in $P \times T$ fulfils the conditions of Definition 1.2 so that it is a symmetric pseudoprocess in P over T. The symmetric pseudoprocess q in P over T defined by

$$(1.4.1) \qquad \mathsf{q} = \mathsf{p} \cup \mathsf{p}^{-1}$$

is said to be *induced by the right pseudoprocess* p in P over T.

Let us show that, given a symmetric pseudoprocess $\mathsf{q} \in \mathrm{Ss}(P, T)$, there exists a right pseudoprocess $\mathsf{p} \in \mathrm{Ps}(P, T)$ such that (1.4.1) holds.

1.5. Definition. Let $\mathsf{q} \in \mathrm{Ss}(P, T)$, $\mathsf{q}^+ \in \mathrm{Ps}(P, T)$. The right pseudoprocess q^+ is said to be *positively induced by the symmetric pseudoprocess* q iff it satisfies the condition

$$(1.5.1) \qquad {}_v\mathsf{q}^+{}_u = {}_v\mathsf{q}_u \quad \text{for all} \quad u \leqq v \quad \text{in} \quad T.$$

1.6. Remark. Since q^+ is a right pseudoprocess, it holds

$$(1.6.1) \qquad {}_v\mathsf{q}^+{}_u = \emptyset \quad \text{for all} \quad u > v \quad \text{in} \quad T$$

so that we obtain from 1.5 and (1.3.1) that

$$(1.6.2) \qquad D_{\mathsf{q}^+} \subset D_\mathsf{q}.$$

The inclusion in (1.6.2) cannot be in general replaced by the equality. However, if $\mathsf{q} \in \mathrm{Ss}(P, T)$ is such that for each $(x, u) \in D_\mathsf{q}$ there exists $t \in T$ fulfilling the conditions $t \geqq u$ and ${}_t\mathsf{q}_u x \neq \emptyset$, then equality

$$(1.6.3) \qquad D_{\mathsf{q}^+} = D_\mathsf{q}$$

holds.

1.7. Lemma. *Let* $\mathsf{q}, \tilde{\mathsf{q}} \in \mathsf{Ss}(P, T)$. *Then the following assertions hold*:

(i) $\mathsf{q} = \mathsf{q}^+ \cup (\mathsf{q}^+)^{-1}$.

(ii) $\mathsf{q} \subset \tilde{\mathsf{q}}$ iff $\mathsf{q}^+ \subset \tilde{\mathsf{q}}^+$.

1.8. Definition. Let $\mathsf{p} \in \mathsf{Ps}(P, T)$, $\mathsf{q} \in \mathsf{Ss}(P, T)$, $\mathsf{p}' \in \mathsf{Ps}(P, -T)$, $\mathsf{q}' \in \mathsf{Ss}(P, -T)$, where $-T = \{t \in R \mid -t \in T\}$.

The right pseudoprocess p' is said to be *orientation-change produced from the right pseudoprocess* p iff

$$(1.8.1) \qquad {}_v\mathsf{p}'_u = ({}_{-u}\mathsf{p}_{-v})^{-1} \quad \text{for all} \quad u \leqq v \quad \text{in} \quad -T\,.$$

The symmetric pseudoprocess q' is said to be *orientation-change produced from the symmetric pseudoprocess* q iff

$$(1.8.2) \qquad {}_v\mathsf{q}'_u = ({}_{-u}\mathsf{q}_{-v})^{-1} \quad \text{for all} \quad u, v \quad \text{in} \quad -T\,.$$

1.9. Definition. Let $\mathsf{q} \in \mathsf{Ss}(P, T)$, let $\mathsf{q}^+ \in \mathsf{Ps}(P, T)$ be positively induced by q, and let $\mathsf{q}^- \in \mathsf{Ps}(P, -T)$.

The right pseudoprocess q^- is said to be *negatively induced* by the symmetric pseudoprocess q iff q^- is orientation-change produced from q^+.

1.10. Remark. A right pseudoprocess q^- from the preceding definition can be described directly by the relations ${}_v\mathsf{q}_u$ as follows.

The equality

$$(1.10.1) \qquad {}_v\mathsf{q}^-{}_u = ({}_{-u}\mathsf{q}^+{}_{-v})^{-1} = ({}_{-u}\mathsf{q}_{-v})^{-1} = {}_{-v}\mathsf{q}_{-u}$$

holds for all $u \leqq v$ in $-T$, hence

$$(1.10.2) \qquad {}_v\mathsf{q}_u = {}_{-v}\mathsf{q}^-{}_{-u} \quad \text{for all} \quad v \leqq u \quad \text{in} \quad T\,.$$

Clearly

$$(1.10.3) \qquad D_{\mathsf{q}^-} \subset D_{\mathsf{q}'} = \{(x, u) \in P \times (-T) \mid (x, -u) \in D_{\mathsf{q}}\},$$

where the symmetric pseudoprocess q' is orientation-change produced from the symmetric pseudoprocess q. If for each $(x, u) \in D_{\mathsf{q}}$ there exists $t \leqq u$ in T such that ${}_t\mathsf{q}_u x \neq \emptyset$, then the inclusion in (1.10.3) can be replaced by the equality.

1.11. Lemma. *Let* $\mathsf{q}, \tilde{\mathsf{q}} \in \mathsf{Ss}(P, T)$ *and let* $\mathsf{q}' \in \mathsf{Ss}(P, -T)$ *be orientation-change produced from* q. *Then the following assertions hold*:

(i) $\mathsf{q}' = \mathsf{q}^- \cup (\mathsf{q}^-)^{-1}$.

(ii) $\mathsf{q} \subset \tilde{\mathsf{q}}$ *iff* $\mathsf{q}^- \subset \tilde{\mathsf{q}}^-$.

Proof. The assertions follow from (1.8.2), 1.2 (ii), (1.10.1), (1.10.2) and from Lemmas 1.2 and 1.3 in [5].

1.12. Corollary. *Let* q, $\tilde{}q \in Ss(P, T)$. *Then the following three inclusions are equivalent*:

(i) $q \subset \tilde{}q$;

(ii) $q^+ \subset \tilde{}q^+$;

(iii) $q^- \subset \tilde{}q^-$.

1.13. Definition. Let $q \in Ss(P, T)$, $s \subset P \times T$. The relation s is called a *solution* of the symmetric pseudoprocess q iff the following three conditions are satisfied:

(i) the domain D_s of s is an interval in T;

(ii) s is a map of D_s into P;

(iii) $s(v)_v q_u s(u)$ holds for all $u, v \in D_s$.

The set of all solutions of q will be denoted by S_q.

1.14. Theorem. *Let* q, $\tilde{}q \in Ss(P, T)$, *let* $q' \in Ss(P, -T)$ *be orientation-change produced from* q *and let* $q^+ \in Ps(P, T)$ *and* $q^- \in Ps(P, -T)$ *be the right pseudoprocesses positively and negatively induced by* q, *respectively. Let* $s : T \to P$ *and* $s' : -T \to P$ *be maps such that* D_s *is an interval in* T; $D_{s'} = \{t \mid -t \in D_s\}$ *and* $s'(-t) = s(t)$ *for all* $t \in D_s$. *Then the following assertions hold*:

(i) $s \in S_q$ *iff* $s \times s \subset q$.

(ii) $s \in S_q$ *iff* $s' \in S_{q'}$.

(iii) $S_q = S_{q^+}$.

(iv) $S_{q'} = S_{q^-}$.

(v) $S_{q \cap \tilde{}q} = S_q \cap S_{\tilde{}q}$.

1.16. Definition. Let $q \in Ss(P, T)$. The maps

$$e^+ : D_q \to R^\#, \quad e^- : D_q \to R^\# \tag{1.16.1}$$

defined by

$$e^+(x, u) = \sup \{t \in T \mid {}_t q_u x \neq \emptyset\}, \tag{1.16.2}$$

$$e^-(x, u) = \inf \{t \in T \mid {}_t q_u x \neq \emptyset\} \tag{1.16.3}$$

are called the *positive* and the *negative extent of existence* of q, respectively.

1.17. Remark. Let us recall that if q^+ or q^- is a right pseudoprocess positively or negatively induced by a symmetric pseudoprocess q, then the extents of existence e or e' of these pseudoprocesses are defined, according to Definition 2.3 in [5], by

$$e(x, u) = \sup \{t \in T \mid {}_t q^+{}_u x \neq \emptyset\}, \quad (x, u) \in D_{q^+} \tag{1.17.1}$$

or

(1.17.2) $$e'(x, u) = \sup \{t \in T \mid {}_t\mathsf{q}^-{}_u x \neq \emptyset\}, \quad (x, u) \in D_{\mathsf{q}^-}.$$

If $(x, u) \in D_{\mathsf{q}^+}$, then $(x, u) \in D_\mathsf{q}$ so that $e^+(x, u)$ as well as $e(x, u)$ are defined and it is evident that the equality

(1.17.3) $$e^+(x, u) = e(x, u), \quad (x, u) \in D_{\mathsf{q}^+}$$

holds. Analogously, if $(x, -u) \in D_{\mathsf{q}^-}$, then $(x, u) \in D_\mathsf{q}$ so that both $e'(x, -u)$ and $e^-(x, u)$ are defined and the equality

(1.17.4) $$e^-(x, u) = -e'(x, -u)$$

takes place. From (1.16.2) and (1.16.3) we obtain immediately the inequality

(1.17.5) $$e^-(x, u) \leqq e^+(x, u) \quad \text{for all} \quad (x, u) \in D_\mathsf{q}.$$

1.18. Definition. Let $\mathsf{q} \in \mathsf{Ss}(P, T)$.

The symmetric pseudoprocess q is said to have *positive (negative, bilateral) local existence at a point* $(x, u) \in D_\mathsf{q}$ iff $e^+(x, u) > u$ $(e^-(x, u) < u,\ e^-(x, u) < u < e^+(x, u))$.

The symmetric pseudoprocess q is said to have *positive (negative, bilateral) local existence* iff it has positive (negative, bilateral) local existence at each point $(x, u) \in D_\mathsf{q}$.

The symmetric pseudoprocess q is said to have *positive (negative, bilateral) global existence at a point* $(x, u) \in D_\mathsf{q}$ iff $e^+(x, u) = \sup T$ $(e^-(x, u) = \inf T$, $e^+(x, u) = \sup T$ and $e^-(x, u) = \inf T)$.

The symmetric pseudoprocess q is said to have *positive (negative, bilateral) global existence* iff it has positive (negative, bilateral) global existence at each point $(x, u) \in D_\mathsf{q}$.

If $\mathsf{q}^+ \in \mathsf{Ps}(P, T)$ is a right pseudoprocess positively induced by a symmetric pseudoprocess q, then q^+ is said to have some of the properties described above iff q has the property.

1.19. Definition. Let $\mathsf{q} \in \mathsf{Ss}(P, T)$. A point $(x, u) \in I_\mathsf{q}$ is called a *start point* or an *end point* of the symmetric pseudoprocess q iff ${}_t\mathsf{q}_u x = \emptyset$ holds for all $t \in T$ such that $t < u$ or $u < t$, respectively.

1.20. Remark. Let a (right or symmetric) pseudoprocess r in P over T be given. In accordance with the notation introduced in [5], item 5.1, the symbol $\mathscr{S}_\mathsf{r}$ or $\mathscr{E}_\mathsf{r}$ will stand for the set of all start or end points of the pseudoprocess r, respectively.

Let $\mathsf{q} \in \mathsf{Ss}(P, T)$, $(x, u) \in D_\mathsf{q}$. Then $(x, u) \in \mathscr{S}_\mathsf{q}$ or $(x, u) \in \mathscr{E}_\mathsf{q}$ iff $e^-(x, u) = u$ or $e^+(x, u) = u$, respectively. If $(x, u) \in \mathscr{S}_\mathsf{q}$ or $(x, u) \in \mathscr{E}_\mathsf{q}$ and $s \in S_\mathsf{q}$ is such that $s(u) = x$, then $u = \min D_s$ or $u = \max D_s$, respectively. The converse of this assertion is not valid.

1.21. Lemma. *Let* $q \in Ss(P, T)$. *Let* $q^+ \in Ps(P, T)$ *or* $q^- \in Ps(P, -T)$ *be positively or negatively induced by* q, *respectively. Let* $(x, u) \in I_q$. *Then the following assertions hold.*

(i) $\mathscr{S}_q = \mathscr{S}_{q^+}$, $\mathscr{E}_q = \mathscr{E}_{q^+}$.

(ii) $(x, u) \in \mathscr{S}_q$ *iff* $(x, -u) \in \mathscr{E}_{q^-}$, $(x, u) \in \mathscr{E}_q$ *iff* $(x, -u) \in \mathscr{S}_{q^-}$.

Proof follows from 1.2 (ii), (1.5.1) and (1.10.2).

1.22. Definition. Let $q \in Ss(P, T)$. The maps

(1.22.1) $$d^+ : D_q \to R^\# \quad \text{and} \quad d^- : D_q \to R^\#$$

defined by

(1.22.2) $$d^+(x, u) = \sup \{w \in R \mid \text{card}\,({}_t q_u x) \leqq 1 \text{ for all } t \in T \cap \langle u, w \rangle\},$$

(1.22.3) $$d^-(x, u) = \inf \{w \in R \mid \text{card}\,({}_t q_u x) \leqq 1 \text{ for all } t \in T \cap \langle w, u \rangle\}$$

are called the *positive* and the *negative extent of unicity* of the symmetric pseudoprocess q.

1.23. Remark. Notice that $+\infty$ may belong to the range of the function d^+ and that d^+ can assume this value also in the case of a bounded T. Similarly for d^- and $-\infty$. In general, it holds

(1.23.1) $$-\infty \leqq d^-(x, u) \leqq u \leqq d^+(x, u) \leqq +\infty .$$

Let q^+ and q^- be the right pseudoprocesses positively and negatively induced by a symmetric pseudoprocess q, respectively. Then, according to Definition 2.5 in [5] the extents of unicity d and d' of q^+ and q^- are defined by

(1.23.2) $$d(x, u) = \sup \{w \in R \mid \text{card}\,({}_t q^+{}_u x) \leqq 1 \text{ for all } t \in T \cap \langle u, w \rangle\} \quad \text{for all } (x, u) \in D_{q^+},$$

and

(1.23.3) $$d'(x, u) = \sup \{w \in R \mid \text{card}\,({}_t q^-{}_u x) \leqq 1 \text{ for all } t \in (-T) \cap \langle u, w \rangle\} \quad \text{for all } (x, u) \in D_{q^-},$$

respectively. Then for each $(x, u) \in D_{q^+} \subset D_q$ both $d(x, u)$ and $d^+(x, u)$ are defined and

(1.23.4) $$d^+(x, u) = d(x, u) \quad \text{for all } (x, u) \in D_{q^+} .$$

Analogously, for each $(x, -u) \in D_{q^-}$ it holds $(x, u) \in D_q$ so that $d'(x, -u)$ as well as $d^-(x, u)$ are defined and the equality

(1.23.5) $$d^-(x, u) = -d'(x, -u) \quad \text{for all } (x, -u) \in D_{q^-}$$

takes place.

1.24. Definition. Let $\mathsf{q} \in \mathrm{Ss}(P, T)$.

The symmetric pseudoprocess q is said to have *positive (negative, bilateral) local unicity at a point* $(x, u) \in D_{\mathsf{q}}$ iff $d^+(x, u) > u$ $(d'(x, u) < u, d'(x, u) < u < d^+(x, u))$.

The symmetric pseudoprocess q is said to have *positive (negative, bilateral) local unicity* iff it has positive (negative, bilateral) local unicity at each point $(x, u) \in D_{\mathsf{q}}$.

The symmetric pseudoprocess q is siad to have *positive (negative, bilateral) global untcity at a point* $(x, u) \in D_{\mathsf{q}}$ iff $d^+(x, u) = +\infty$ $(d^-(x, u) = -\infty, d^+(x, u) = = -d^-(x, u) = +\infty)$.

The symmetric pseudoprocess q is said to have *positive (negative, bilateral) global unicity* iff it has positive (negative, bilateral) global unicity at each point $(x, u) \in D_{\mathsf{q}}$.

If $\mathsf{q}^+ \in \mathrm{Ps}(P, T)$ is a right pseudoprocess positively induced by a symmetric pseudoprocess q, then q^+ is said to have some of the properties described above iff q has the property.

1.25. Lemma. *Let* $\mathsf{q} \in \mathrm{Ss}(P, T)$. *Then the following assertions hold*:

(i) *If* $(x, u) \in D_{\mathsf{q}}$ *is a start or an end point of* q, *then* $d^-(x, u) = -\infty$ *or* $d^+(x, u) = = +\infty$, *respectively.*

(ii) *If* $(x, u) \in D_{\mathsf{q}}$ *and* $u < d^+(x, u) < +\infty$ *or* $-\infty < d^-(x, u) < u$, *then* $d^+(x, u) < < e^+(x, u)$ *or* $e^-(x, u) < d^-(x, u)$, *respectively.*

1.26. Lemma. *Let* $\mathsf{q}, {}^{\sim}\mathsf{q} \in \mathrm{Ss}(P, T)$, $\mathsf{q} \subset {}^{\sim}\mathsf{q}$. *Let* e^+, e^-, d^+, d^- *and* $\tilde{e}^+, \tilde{e}^-, \tilde{d}^+, \tilde{d}^-$ *be the corresponding extents of existence and unicity of* q *and* ${}^{\sim}\mathsf{q}$, *respectively. Then the following assertions hold*:

(i) *If* $(x, u) \in D_{\mathsf{q}}$, *then*

$$e^+(x, u) \leqq \tilde{e}^+(x, u), \quad \tilde{e}^-(x, u) \leqq e^-(x, u), \tag{1.26.1}$$

$$\tilde{d}^+(x, u) \leqq d^+(x, u), \quad d^-(x, u) \leqq \tilde{d}^-(x, u). \tag{1.26.2}$$

(ii) *If* $(x, u) \in D_{\sim\mathsf{q}}$ *is a start point or an end point of* ${}^{\sim}\mathsf{q}$, *then it is a start point or an end point of* q, *respectively.*

(iii) *If* q *has positive, negative, bilateral local or global existence at a point* $(x, u) \in D_{\mathsf{q}}$, *then* ${}^{\sim}\mathsf{q}$ *has the same property.*

(iv) *If* ${}^{\sim}\mathsf{q}$ *has positive, negative, bilateral local or global unicity at a point* $(x, u) \in \in D_{\sim\mathsf{q}}$, *then* q *has the same property.*

1.27. Definition. Let $\mathsf{q} \in \mathrm{Ss}(P, T)$. The symmetric pseudoprocess q is said to be *solution complete* iff for each pair $((y, v), (x, u)) \in \mathsf{q}$ there exists $s \in S_{\mathsf{q}}$ such that $s(u) = x$, $s(v) = y$.

1.28. Theorem. *Let* $q \in Ss(P, T)$. *Then the symmetric pseudoprocess* q *is solution complete iff the right pseudoprocess* q^+ *positively induced by* q *is solution complete.*

Proof follows directly from 1.7 (i), 1.14 (iii) and 1.27.

2. SYMMETRIC PROCESSES

2.1. Definition. Let $q \in Ss(P, T)$ and let $q^+ \in Ps(P, T)$ be positively induced by q. The symmetric pseudoprocess q is said to be *compositive* or *transitive* iff the right pseudoprocess q^+ is compositive or transitive, respectively.

The symmetric pseudoprocess q is called a *symmetric process in* P *over* T iff it is compositive and transitive.

The sets of all compositive symmetric pseudoprocesses, of all transitive symmetric pseudoprocesses and of all symmetric processes in P over T will be denoted by $Ssc(P, T)$, $Sst(P, T)$ and $S(P, T)$, respectively.

2.2. Lemma. *Let* $q \in Ss(P, T)$, *let* $q^+ \in Ps(P, T)$ *be positively induced by* q *and let* $q^- \in Ps(P, -T)$ *be negatively induced by* q. *Then the following assertions are equivalent*:

(i) $q \in Ssc(P, T)$.

(ii) $q^+ \in Psc(P, T)$.

(iii) $q^- \in Psc(P, -T)$.

(iv) ${}_vq_u \subset {}_vq_t \circ {}_tq_u$ *for all* $u \leqq t \leqq v$ *in* T.

(v) ${}_vq_u \subset {}_vq_t \circ {}_tq_u$ *for all* $v \leqq t \leqq u$ *in* T.

(vi) ${}_vq_u \subset {}_vq_t \circ {}_tq_u$ *for all* $u, v, t \in T$, t *between* u, v.

Proof follows from 2.1, (1.10.2), (RC) in 1.1, (1.5.1) and (1.10.4).

2.3. Lemma. *Let* $q \in Ss(P, T)$, *let* $q^+ \in Ps(P, T)$ *be positively induced by* q *and let* $q^- \in Ps(P, -T)$ *be negatively induced by* q. *Then the following assertions are equivalent*:

(i) $q \in Sst(P, T)$.

(ii) $q^+ \in Pst(P, T)$.

(iii) $q^- \in Pst(P, -T)$.

(iv) ${}_vq_t \circ {}_tq_u \subset {}_vq_u$ *for all* $u \leqq t \leqq v$ *in* T.

(v) ${}_vq_t \circ {}_tq_u \subset {}_vq_u$ *for all* $v \leqq t \leqq u$ *in* T.

(vi) ${}_vq_t \circ {}_tq_u \subset {}_vq_u$ *for all* $u, v, t \in T$, t *between* u, v.

2.4. Lemma. *Let* $\mathsf{q} \in \mathsf{Ss}(P, T)$. *Then the following assertions are equivalent:*

(i) $\mathsf{q} \in \mathsf{S}(P, T)$.

(ii) ${}_v\mathsf{q}_u = {}_v\mathsf{q}_t \circ {}_t\mathsf{q}_u$ *for all* $u, v, t \in T$, t *between* u, v.

2.5. Lemma. *If* $\mathsf{q} \in \mathsf{Ssc}(P, T)$, *then*

$$D_{\mathsf{q}^+} = D_{\mathsf{q}} = I_{\mathsf{q}} ,$$
$$D_{\mathsf{q}^-} = \{(x, u) \in P \times (-T) \mid (x, -u) \in D_{\mathsf{q}}\} .$$

2.6. Lemma. *Let* $\mathsf{q} \in \mathsf{Ss}(P, T)$ *and let* I *be an arbitrary set. Then the following assertions hold:*

(i) *If* $s \in S_{\mathsf{q}}$, J *an interval in* T, *then* $s|_J \in S_{\mathsf{q}}$.

(ii) *If* $s_i \in S_{\mathsf{q}}$ *for* $i \in I$ *are such that* $D_{\cap s_i}$ *is an interval in* T, *then* $\cap s_i \in S_{\mathsf{q}}$.

(iii) *If* q *is transitive and if* $s_i \in S_{\mathsf{q}}$ *with* $i \in I$ *are such that* $D_{s_i} \cap D_{s_j} \neq \emptyset$ *and* $s_i \cup s_j$ *is a map for all* $i, j \in I$, *then* $s_i \in S_{\mathsf{q}}$.

2.7. Lemma. *Let* $\mathsf{q} \in \mathsf{Sst}(P, T)$, $(x, u) \in D_{\mathsf{q}}$. *Then the following assertions hold:*

(i) *If* $v, w \in T$, v *between* u, w, $z\,{}_w\mathsf{q}_v\,y$, $y\,{}_v\mathsf{q}_u\,x$, *then also* $z\,{}_w\mathsf{q}_u\,x$.

(ii) *Let* $(y, v) \in D_{\mathsf{q}}$ *be such that* $y\,{}_v\mathsf{q}_u\,x$. *If* $v = e^-(x, u)$ *or* $v = e^+(x, u)$, *then* (y, v) *is a start point or an end point of* q, *respectively.*

2.8. Lemma. *Let* $\mathsf{q} \in \mathsf{Ssc}(P, T)$, $(x, u) \in D_{\mathsf{q}}$, $u, v, w \in T \cap \langle d^-(x, u), d^+(x, u)\rangle$, v *between* u, w. *If* $y\,{}_v\mathsf{q}_u\,x$, $z\,{}_w\mathsf{q}_u\,x$, *then also* $z\,{}_w\mathsf{q}_v\,y$.

2.9. Theorem. *Let* $\mathsf{q} \in \mathsf{Ssc}(P, T)$ *have global unicity and let* $\mathsf{s} : T \to P$. *Then* $\mathsf{s} \in S_{\mathsf{q}}$ *iff the following two conditions are satisfied:*

(i) D_s *is an interval in* T;

(ii) *there exists* $u \in D_s$ *such that* $\mathsf{s}(v)\,{}_v\mathsf{q}_u\,\mathsf{s}(u)$ *holds for all* $v \in D_s$.

Proof follows easily from Definition 1.13 and Lemma 2.8.

2.10. Theorem. *Let* $\mathsf{q} \in \mathsf{Ss}(P, T)$. *If* q *is solution complete, then it is compositive.*

Proof. According to Theorem 1.28 the right pseudoprocess q^+ is solution complete so that it is compositive (see Theorem 3.8 in [5]). Now apply Definition 2.1.

2.11. Definition. Let $\mathsf{q} \in \mathsf{Ss}(P, T)$, $(x, u) \in D_{\mathsf{q}}$, $\mathsf{s} \subset P \times T$. The relation s is called a *characteristic solution of* q *through the point* (x, u) iff it satisfies the following two conditions:

(i) $D_s = \{v \in T \mid \text{card}\,({}_t\mathsf{q}_u x) = 1$ *for all* $t \in T$, t *between* $u, v\}$;

(ii) $\mathsf{s}(v)\,{}_v\mathsf{q}_u\,x$ *holds for all* $v \in D_s$.

2.12. Lemma. *Let* $\mathsf{q} \in \mathrm{Ssc}(P, T)$ *have bilateral local unicity at a point* $(x, u) \in D_{\mathsf{q}}$ *and let* s *be the characteristic solution of* q *through* (x, u). *Then* $s \in S_{\mathsf{q}}$ *with*

$$\sup D_s = \min\{e^+(x,u), d^+(x,u)\} = \begin{cases} e^+(x,u) & \text{if } d^+(x,u) = +\infty, \\ d^+(x,u) & \text{if } d^+(x,u) < +\infty, \end{cases}$$

$$\inf D_s = \max\{e^-(x,u), d^-(x,u)\} = \begin{cases} e^-(x,u) & \text{if } d^-(x,u) = -\infty, \\ d^-(x,u) & \text{if } d^-(x,u) > -\infty. \end{cases}$$

2.13. Remark. In Definition 3.10 in [5], each right pseudoprocess p in P over T is associated with the maximal compositive right pseudoprocess $^\wedge\mathsf{p}$ in P over T contained in p, the so called lower modification of p. The construction of the lower modification of a right pseudoprocess described in item 3.12 in [5] is applicable to symmetric pseudoprocesses as well. However, this is not necessary, because, as will be shown, the maximal compositive symmetric pseudoprocess contained in a given symmetric pseudoprocess q, which will be called again the lower modification of q, can be constructed directly from the lower modifications of the pertinent right pseudoprocesses q^+ and q^-.

2.14. Theorem. *Let* $\mathsf{q} \in \mathrm{Ss}(P, T)$, *let* $\mathsf{q}^+ \in \mathrm{Ps}(P, T)$ *be positively induced by* q, *let* $\mathsf{q}^- \in \mathrm{Ps}(P, T)$ *be negatively induced by* q, *let* $^\wedge\mathsf{q}^+$ *be the lower modification of* q^+ *and let* $^\wedge\mathsf{q}^-$ *be orientation-change produced from* $^\wedge\mathsf{q}^+$. *Then* $^\wedge\mathsf{q}^-$ *is the lower modification of* q^-.

Proof. According to Definition 1.8, $^\wedge\mathsf{q}^- \in \mathrm{Ps}(P, -T)$. First we shall prove that $^\wedge\mathsf{q}^-$ is compositive.

The right pseudoprocess $^\wedge\mathsf{q}^+$ being the lower modification of q^+ is compositive, i.e.

(2.14.1) $\quad {}_v{}^\wedge\mathsf{q}^+{}_u \supset {}_v{}^\wedge\mathsf{q}^+{}_t \circ {}_t{}^\wedge\mathsf{q}^+{}_u$ for all $u \leqq t \leqq v$ in T.

Hence

(2.14.2) $\quad ({}_v{}^\wedge\mathsf{q}^+{}_u)^{-1} \supset ({}_v{}^\wedge\mathsf{q}^+{}_t \circ {}_t{}^\wedge\mathsf{q}^+{}_u)^{-1}$ for all $u \leqq t \leqq v$ in T.

According to (1.10.1) it is

(2.14.3) $$({}_v{}^\wedge\mathsf{q}^+{}_u)^{-1} = {}_{-u}{}^\wedge\mathsf{q}^-{}_{-v}$$

so that

(2.14.4) $\quad ({}_v{}^\wedge\mathsf{q}^+{}_t \circ {}_t{}^\wedge\mathsf{q}^+{}_u)^{-1} = ({}_t{}^\wedge\mathsf{q}^+{}_u)^{-1} \circ ({}_v{}^\wedge\mathsf{q}^+{}_t)^{-1} = {}_{-u}{}^\wedge\mathsf{q}^-{}_{-t} \circ {}_{-t}{}^\wedge\mathsf{q}^-{}_{-v}$

holds for all $u \leqq t \leqq v$ in T. Substituting from (2.14.3) and (2.14.4) into (2.14.2) we obtain

$${}_{-u}{}^\wedge\mathsf{q}^-{}_{-v} \subset {}_{-u}{}^\wedge\mathsf{q}^-{}_{-t} \circ {}_{-t}{}^\wedge\mathsf{q}^-{}_{-v} \quad \text{for all} \quad -v \leqq -t \leqq -u \quad \text{in} \quad -T.$$

Thus $^\wedge\mathsf{q}^-$ is compositive.

Now we shall prove that $^\wedge q^-$ is the maximal compositive right pseudoprocess in P over $-T$ contained in q^-.

Let $p \in Psc(P, -T)$ be such that $^\wedge q^- \subset p \subset q^-$. Then

$$(2.14.5) \qquad {}_v{}^\wedge q^-{}_u \subset {}_v p_u \subset {}_v q^-{}_u \quad \text{for all} \quad u \leqq v \quad \text{in} \quad -T.$$

Denote by p' the right pseudoprocess in P over T which is orientation-change produced from p. From (2.14.5) and (1.10.1) one obtains

$$({}_{-u}{}^\wedge q^+{}_{-v})^{-1} \subset {}_{-u}p'{}_{-v} \subset {}_{-u}q^+{}_{-v} \quad \text{for all} \quad -v \leqq -u \quad \text{in} \quad T,$$

which can be written equivalently as

$$^\wedge q^+ \subset p' \subset q^+ .$$

Since $^\wedge q^+$ and p' are compositive, it is necessarily $^\wedge q^+ = p'$, hence $^\wedge q^- = p$.

We have proved that $^\wedge q^-$ is the lower modification of q^-.

2.15. Theorem. *Let* $q \in Ss(P, T)$, *let* $q^+ \in Ps(P, T)$ *be positively induced by* q *and let* $^\wedge q^+$ *be the lower modification of* q^+. *Then the symmetric pseudoprocess*

$$(2.15.1) \qquad ^\wedge q = {}^\wedge q^+ \cup (^\wedge q^+)^{-1}$$

is the maximal compositive symmetric pseudoprocess in P over T contained in q.

Proof. Since $^\wedge q^+$ is compositive, $^\wedge q$ is compositive as well.

Let $^\sim q \in Ssc(P, T)$ be such that $^\wedge q \subset {}^\sim q \subset q$. Then

$$(^\wedge q^+ \cup (^\wedge q^+)^{-1}) \subset (^\sim q^+ \cup (^\sim q^+)^{-1}) \subset (q^+ \cup (q^+)^{-1}) .$$

Hence, according to 1.12, one obtains $^\wedge q^+ \subset {}^\sim q^+ \subset q^+$. Since $^\wedge q^+$ is the lower modification of q^+ and $^\sim q^+$ is compositive, it is necessarily $^\wedge q^+ = {}^\sim q^+$, hence $^\wedge q = {}^\sim q$ follows by virtue of Lemma 1.12.

2.16. Definition. Let $q \in Ss(P, T)$. The symmetric compositive pseudoprocess $^\wedge q$ in P over T defined by

$$(2.16.1) \qquad ^\wedge q = {}^\wedge q^+ \cup (^\wedge q^+)^{-1}$$

is called the *lower modification* of the symmetric pseudoprocess q.

2.17. Remark. One may verify easily that

$$D_{^\wedge q} = D_{^\wedge q^+} \subset D_{q^+} \subset D_q .$$

If, in addition, $D_q = I_q$, then

$$D_{^\wedge q} = D_{^\wedge q^+} = D_{q^+} = D_q .$$

2.18. Theorem. *Let* $q_i \in Ss(P, T)$ *for* $i = 1, 2$ *and let* q_i^+ *be the right pseudoprocess positively induced by* q_i. *Then* $q_1 \cap q_2 \in Ss(P, T)$ *and its lower modification is the compositive symmetric pseudoprocess* $q_1 \wedge q_2$ *in* P *over* T *defined by*

$$q_1 \wedge q_2 = (q_1^+ \wedge q_2^+) \cup (q_1^+ \wedge q_2^+)^{-1}, \tag{2.18.1}$$

where $q_1^+ \wedge q_2^+$ *denotes the lower modification of the right pseudoprocess* $q_1^+ \cap q_2^+$.

Proof. According to the assertion (i) of Lemma 1.7 it is

$$q_1 = q_1^+ \cup (q_1^+)^{-1}, \quad q_2 = q_2^+ \cup (q_2^+)^{-1}$$

so that

$$q_1 \cap q_2 = (q_1^+ \cap q_2^+) \cup (q_1^+ \cap q_2^+)^{-1}.$$

Hence and from 1.7 (i) one obtains

$$(q_1 \cap q_2)^+ = q_1^+ \cap q_2^+. \tag{2.18.2}$$

Since the lower modification of $q_1^+ \cap q_2^+$ is $q_1^+ \wedge q_2^+$, the equality (2.18.1) follows now directly from (2.18.2) and (2.16.1).

2.19. Theorem. *Let* $q \in Ss(P, T)$, *let* $^\wedge q$ *be its lower modification. Then* $S_{^\wedge q} = S_q$.

Proof. Applying Theorem 6.14 to the equalities

$$q = q^+ \cup (q^+)^{-1}, \quad ^\wedge q = {}^\wedge q^+ \cup (^\wedge q^+)^{-1}$$

we obtain $S_q = S_{q^+}$, $S_{^\wedge q} = S_{^\wedge q^+}$. Theorem 3.14 in [5] yields $S_{^\wedge q^+} = S_{q^+}$. Thus $S_{^\wedge q} = S_q$.

2.20. Corollary. *Let* $q_1, q_2 \in Ss(P, T)$. *Then* $S_{q_1 \wedge q_2} = S_{q_1 \cap q_2}$.

3. LOCAL BEHAVIOUR OF SYMMETRIC PSEUDOPROCESSES

3.1. In Section 5 of the paper [5] we have investigated the local behaviour of right pseudoprocesses. Let us recall the basic notions and notation which will be used in the sequel.

Given a (right or symmetric) pseudoprocess r in P over T, the symbol L_r will denote the set

$$L_r = \{(s, u) \in S_r \times T \mid u \in D_s\}. \tag{3.1.1}$$

Let $p \in Ps(P, T)$, $(x, u) \in D_p$. Then p is said to have right (or left) local existence of solutions at the point (x, u) iff the following conditions are fulfilled:

(i) $(x, u) \notin \mathscr{E}_p$ (or $(x, u) \notin \mathscr{S}_p$);

(ii) There exist $\varepsilon > 0$ and $s \in S_p$ such that

$$\langle u, u + \varepsilon\rangle \cap T \subset D_s \quad (\text{or } \langle u - \varepsilon, u\rangle \cap T \subset D_s).$$

A pseudoprocess p is said to have bilateral local existence of solutions at the point (x, u) iff it has right local existence of solutions at the point (x, u) if $(x, u) \in D_p - \mathscr{E}_p$ and left local existence of solutions at the point (x, u) if $(x, u) \in$ $\in D_p - \mathscr{S}_p$. A pseudoprocess p is said to have right or left or bilateral local existence of solutions iff it has the property at each point $(x, u) \in D_p$.

Let p, $\tilde{}$p $\in$ Ps(P, T). Then $\tilde{}$p is said to determine the local behaviour of p (which is shortly written as $\tilde{}$p $\prec$ p) iff $\tilde{}$p $\subset$ p and there exists a map

$$k : L_p \to R \tag{3.1.2}$$

such that $k(s, u) > u$ for $u < \sup D_s$, $k(s, u) = u$ for $u = \max D_s$ and

$$s|_{\langle u, k(s,u) \rangle} \in S_{\sim p} . \tag{3.1.3}$$

A pseudoprocess $\tilde{}$p is said to determine the bilateral local behaviour of p (which is shortly written as $\tilde{}$p $\preccurlyeq$ p) iff $\tilde{}$p $\subset$ p and there exist maps

$$k_1, k_2 : L_p \to R \tag{3.1.4}$$

such that

$$k_1(s, u) < u \quad \text{for} \quad \inf D_s < u\,, \qquad k_1(s, u) = u \quad \text{for} \quad \min D_s = u\,,$$

$$k_2(s, u) > u \quad \text{for} \quad \sup D_s > u\,, \qquad k_2(s, u) = u \quad \text{for} \quad \max D_s = u$$

and

$$s|_{\langle k_1(s,u), k_2(s,u) \rangle} \in S_{\sim p} . \tag{3.1.5}$$

Now, let q $\in$ Ss(P, T) and $q^+ \in$ Ps(P, T) be positively induced by q. According to Theorem 1.14 (iii) it holds $S_q = S_{q^+}$. This enables us to define the corresponding notions related to the local existence of solutions and to the local behaviour of pseudoprocesses for symmetric pseudoprocesses in a natural way as follows.

3.2. Definition. Let q $\in$ Ss(P, T), let $q^+ \in$ Ps(P, T) be positively induced by q and let $(x, u) \dot{\in} D_q$. The symmetric pseudoprocess q is said to have *right* or *left* or *bilateral local existence of solutions at the point* (x, u) iff the right pseudoprocess q^+ has right or left or bilateral local existence of solutions at the point (x, u), respectively.

The symmetric pseudoprocess q is said to have *right* or *left* or *bilateral local existence of solutions* iff it has the property at each point $(x, u) \in D_q$.

3.3. Definition. Let q $\in$ Ss(P, T), p, $\tilde{}$p, $q^+ \in$ Ps(P, T), $\approx$p, $q^- \in$ Ps$(P, -T)$, where q^+ is positively and q^- negatively induced by q. The right pseudoprocess $\approx$p, $\tilde{}$p or p is said to *determine the negative local behaviour, the positive local behaviour* or *the local behaviour* of the symmetric pseudoprocess q (which is shortly written as $\approx$p $\leqq$ q, $\tilde{}$p $\prec$ q or p $\preccurlyeq$ q) iff $\approx$p $\prec q^-$, $\tilde{}$p $\prec q^+$, or p $\preccurlyeq q^+$.

Let $q, {}^{\sim}q \in Ss(P, T)$. The symmetric pseudoprocess ${}^{\sim}q$ is said to *determine the local behaviour* of the symmetric pseudoprocess q (shortly written as ${}^{\sim}q \preccurlyeq q$) iff ${}^{\sim}q^{+} \preccurlyeq q$, where ${}^{\sim}q^{+}$ is the right pseudoprocess positively induced by ${}^{\sim}q$.

3.4. Remark. If $q, {}^{\sim}q \in Ss(P, T)$, ${}^{\sim}q \preccurlyeq q$, then $D_{\sim q} \subset D_q$. If, in addition, $D_q = I_q$, then $D_{\sim q} = D_q$. This equality holds in particular if $q \in Ssc(P, T)$.

Since Definitions 3.2 and 3.3 are immediate generalizations of Definitions 5.2, 5.3, 5.5 and 5.6 from [5], it is natural that many results valid for right pseudoprocesses remain valid as well when formulated for symmetric pseudoprocesses. Some results of this kind are given in what follows.

Similarly as in [5] many assertions concerning the relations $\prec$ and $\preccurlyeq$ may be formulated simultaneously. It will be done using the symbol $<$. In these assertions the symbol $<$ has to be replaced either by $\prec$ or by $\preccurlyeq$.

3.5. Lemma. *Let* $q \in Ss(P, T)$, $p \in Ps(P, T)$, ${}^{\sim}p \in Ps(P, -T)$. *Then the following assertions hold*:

(i) *If* $p < q$, *then* $S_p \subset S_q$, $D_p \subset D_q$, $\mathscr{E}_p = \mathscr{E}_q \cap D_p$.

(ii) *If* ${}^{\sim}p \leqq q$, *then* $S_{\sim p} \subset S_{q^-}$.

(iii) *If* $p \leqq q$, *then* $\mathscr{E}_p = \mathscr{E}_q \cap D_p$, $\mathscr{S}_p = \mathscr{S}_q \cap D_p$.

(iv) *If* $p \prec q$ *and* p *has right or left local existence of solutions at each point, then* $D_p = D_q$, $\mathscr{E}_p = \mathscr{E}_q$; *if, in addition,* $p \preccurlyeq q$, *then also* $\mathscr{S}_p = \mathscr{S}_q$.

3.6. Lemma. *Let* $q, {}^{\sim}q, {}^{\approx}q \in Ss(P, T)$. *If* $q \;{}^{\approx}\!\!\preccurlyeq {}^{\sim}q$, ${}^{\sim}q \preccurlyeq q$, *then* ${}^{\approx}q \preccurlyeq q$.

3.7. Lemma. *Let* $p, {}^{\sim}p \in Ps(P, T)$, $q \in Ss(P, T)$, ${}^{\sim}p \subset p \subset q$. *If* ${}^{\sim}p < q$, *then* $p < q$.

3.8. Lemma. *Let* $p \in Ps(P, T)$, $q, {}^{\sim}q \in Ss(P, T)$, $p \subset {}^{\sim}q \subset q$. *If* $p < q$, *then* $p < {}^{\sim}q$.

3.9. Lemma. *Let* $p \in Ps(P, T)$, $q \in Ss(P, T)$. *Then* $p < q$ *iff* ${}^{\wedge}p < q$, *where* ${}^{\wedge}p$ *is the lower modification of* p.

3.10. Lemma. *Let* $p, {}^{\sim}p \in Ps(P, T)$, $q \in Ss(P, T)$. *Then the following assertions are equivalent.*

(i) $p < q$, ${}^{\sim}p < q$.

(ii) $p \cap {}^{\sim}p < q$.

(iii) $p \wedge {}^{\sim}p < q$.

3.11. Lemma. *Let* $q, {}^{\sim}q, {}^{\approx}q \in Ss(P, T)$. *If* ${}^{\sim}q \preccurlyeq q$, *then* ${}^{\sim}q \cap q^{\approx} \preccurlyeq q \cap {}^{\approx}q$. *Especially, if* ${}^{\sim}q \preccurlyeq q$, ${}^{\approx}q \preccurlyeq q$, *then* ${}^{\sim}q \cap {}^{\approx}q \preccurlyeq {}^{\sim}q$, ${}^{\sim}q \cap {}^{\approx}q \preccurlyeq {}^{\approx}q$.

3.12. Lemma. *Let* $q \in Ssc(P, T)$, $\tilde{q}, \approx q \in Ss(P, T)$, $q \preccurlyeq \tilde{q}$, $q \preccurlyeq {}^{\approx}q$. *Then* $q^+ \subset \subset \tilde{q}^+ \wedge {}^{\approx}q^+$, $q^- \subset \tilde{q}^- \wedge {}^{\approx}q^-$, $q \subset \tilde{q} \wedge {}^{\approx}q$.

3.13. Theorem. *Let* $q \in Ss(P, T)$, $p \in Ps(P, T)$ *and let* $p' \in Ps(P, -T)$ *be orientation change produced from* p. *Then the following assertions hold*:

(i) $p' \leqq q$ *iff* $p \subset q$ *and there exists a map* $h^- : L_q \to R$ *such that*

$$h^-(s, u) < u \quad for \quad \inf D_s < u\,, \quad h^-(s, u) = u \quad for \quad \min D_s = u$$

and

$$(3.13.1) \qquad s(t)\ {}_tp_v\ s(v) \quad for\ all \quad h^-(s, u) \leqq v \leqq t \leqq u \quad in \quad D_s\,.$$

(ii) $p \,\dot{\prec}\, q$ *iff* $p \subset q$ *and there exists a map* $h^+ : L_q \to R$ *such that*

$$h^+(s, u) > u \quad for \quad u < \sup D_s\,, \quad h^+(s, u) = u \quad for \quad u = \max D_s$$

and

$$(3.13.2) \qquad s(t)\ {}_tp_v\ s(v) \quad for\ all \quad u \leqq v \leqq t \leqq h^+(s, u) \quad in \quad D_s\,.$$

(iii) $p \preccurlyeq q$ *iff* $p \subset q$ *and there exist maps* $h^+, h^- : L_q \to R$ *such that*

$$h^+(s, u) > u \quad for \quad u < \sup D_s\,, \qquad h^+(s, u) = u \quad for \quad u = \max D_s\,,$$

$$h^-(s, u) < u \quad for \quad \inf D_s < u\,, \qquad h^-(s, u) = u \quad for \quad u = \min D_s$$

and

$$(3.13.3) \qquad s(t)\ {}_tp_v\ s(v) \quad for\ all \quad h^-(s, u) \leqq v \leqq t \leqq h^+(s, u) \quad in \quad D_s\,.$$

Proof. Before proving the assertion (i) let us recall that (1.8.1) and (1.10.1) yield $p' \subset q^-$ iff $p \subset q^+$ and according to 1.14 (ii), $s \in S_q = S_{q^+}$ iff there exists $s' \in S_{q^-}$ such that $s'(t) = s(-t)$ for all $-t \in D_s$.

Suppose $p' \leqq q$, i.e. $p' \prec q^-$ and prove that (3.13.1) is fulfilled. Take $(s, u) \in L_q$ arbitrary. According to the assumption there exists a real $h'(s', -u)$ such that

$$(3.13.4) \qquad s'(-v)\ {}_{-v}p'_{-t}\ s'(-t) \quad \text{for all} \quad -u \leqq -t \leqq -v \leqq h'(s', -u) \quad \text{in} \quad D_s\,.$$

Hence, setting $h^-(s, u) = -h'(s', -u)$ and using (1.8.1), one easily obtains (3.13.1).

Suppose now that the condition (3.13.1) is fulfilled and prove that $p' \prec q^-$. The condition (3.13.1) can be written in the form

$$(3.13.5) \qquad s(-t)\ {}_{-t}p_{-v}\ s(-v) \quad \text{for all} \quad h^-(s, u) \leqq -v \leqq -t \leqq u \quad \text{with} \quad v, t \in D_{s'}\,,$$

i.e.

$$s'(v)\ {}_vp'_t\ s'(t) \quad \text{for all} \quad -u \leqq t \leqq v \leqq -h^-(s, u) \quad \text{with} \quad v, t \in D_{s'}\,.$$

Setting $h'(s', u) = -h^-(s, -u)$ we conclude that for each $(s', u) \in L_{q^-}$ there exists a real $h'(s', u)$ such that

$$s'(t)\ {}_t\mathrm{p}_v\ s'(v) \quad \text{for all} \quad u \leqq v \leqq t \leqq h'(s', u) \quad \text{in} \quad D_{s'}.$$

Thus $\mathrm{p}' \prec \mathrm{q}^-$.

The assertions (ii) and (iii) follow immediately from Definition 3.3.

3.14. Theorem. *Let* $\mathrm{q}, \tilde{\mathrm{q}} \in \mathrm{Ss}(P, T)$. *Then the following three assertions are equivalent*:

(i) $\tilde{\mathrm{q}} \preccurlyeq \mathrm{q}$;

(ii) $\tilde{\mathrm{q}}^+ \preccurlyeq \mathrm{q}^+$;

(iii) $\tilde{\mathrm{q}}^- \preccurlyeq \mathrm{q}^-$.

Proof follows easily from Theorem 1.14 and Definition 3.3.

3.15. Theorem. *Let* $\mathrm{q} \in \mathrm{Ss}(P, T)$, $\tilde{\mathrm{q}} \in \mathrm{Sst}(P, T)$. *If* $\tilde{\mathrm{q}}^+ \prec \mathrm{q}$, $\tilde{\mathrm{q}}^- \leqq \mathrm{q}$, *then* $\tilde{\mathrm{q}} \preccurlyeq \mathrm{q}$.

Proof. According to 1.12 the inclusion $\tilde{\mathrm{q}} \subset \mathrm{q}$ is equivalent to any one of the inclusions $\tilde{\mathrm{q}}^+ \subset \mathrm{q}^+$ and $\tilde{\mathrm{q}}^- \subset \mathrm{q}^-$.

To each $(s, u) \in L_\mathrm{q}$ we can assign reals $h^+(s, u)$ and $h^-(s, u)$ as in Theorem 3.13 such that

$$s(t)\ {}_t\tilde{\mathrm{q}}_v\ s(v) \quad \text{for all} \quad u \leqq v \leqq t \leqq h^+(s, u) \quad \text{in} \quad T \tag{3.15.1}$$

and

$$s(t)\ {}_t\tilde{\mathrm{q}}_v\ s(v) \quad \text{for all} \quad h^-(s, u) \leqq v \leqq t \leqq u \quad \text{in} \quad T. \tag{3.15.2}$$

Especially,

$$s(t)\ {}_t\tilde{\mathrm{q}}_u\ s(u)\,, \quad s(u)\ {}_u\tilde{\mathrm{q}}_v\ s(v) \quad \text{for all} \quad h^-(s, u) \leqq v \leqq u \leqq t \leqq h^+(s, u) \quad \text{in} \quad T.$$

Since $\tilde{\mathrm{q}}$ is transitive, it holds also

$$s(t)\ {}_t\tilde{\mathrm{q}}_v\ s(v) \quad \text{for all} \quad h^-(s, u) \leqq v \leqq u \leqq t \leqq h^+(s, u) \quad \text{in} \quad T. \tag{3.15.3}$$

Finally, $({}_t\tilde{\mathrm{q}}_v)^{-1} = {}_v\tilde{\mathrm{q}}_t$ so that $s(v)\ {}_v\tilde{\mathrm{q}}_t\ s(t)$ holds iff $s(t)\ {}_t\tilde{\mathrm{q}}_v\ s(v)$. This together with (3.15.1), (3.15.2) and (3.15.3) yields

$$s(t)\ {}_t\tilde{\mathrm{q}}_v\ s(v) \quad \text{for all} \quad v, t \in D_s \cap \langle h^-(s, u), h^+(s, u)\rangle\,,$$

which was to be proved.

3.16. Definition. Let $q, {}^{\sim}q \in \mathrm{Ss}(P, T)$. The symmetric pseudoprocesses q and ${}^{\sim}q$ are said to be *negatively locally equivalent* or *positively locally equivalent* or *locally equivalent* (which is shortly written as $q \leqq\geqq {}^{\sim}q$ or $q \prec\succ {}^{\sim}q$ or $q \preccurlyeq\succcurlyeq {}^{\sim}q$) iff there exists $p \in \mathrm{Ps}(P, -T)$ or ${}^{\sim}p \in \mathrm{Ps}(P, T)$ or ${}^{\approx}p \in \mathrm{Ps}(P, T)$ such that $p \leqq q$ and $p \leqq {}^{\sim}q$ or ${}^{\sim}p \prec q$ and ${}^{\sim}p \prec {}^{\sim}q$ or ${}^{\approx}p \preccurlyeq q$ and ${}^{\approx}p \preccurlyeq {}^{\sim}q$, respectively.

3.17. Lemma. *Let* $q, {}^{\sim}q \in \mathrm{Ss}(P, T)$. *Then the following assertions hold*:

(i) $q \leqq\geqq {}^{\sim}q$ iff $q^- \prec\succ {}^{\sim}q^-$.

(ii) $q \prec\succ {}^{\sim}q$ iff $q^+ \prec\succ {}^{\sim}q^+$.

(iii) $q \preccurlyeq\succcurlyeq {}^{\sim}q$ iff $q^+ \preccurlyeq\succcurlyeq {}^{\sim}q^+$.

3.18. Lemma. *Let* $q, {}^{\sim}q \in \mathrm{Ss}(P, T)$. *Then the following three assertions are equivalent*:

(i) $q \leqq\geqq {}^{\sim}q$;

(ii) $q^- \cap {}^{\sim}q^- \prec q^-$, $q^- \cap {}^{\sim}q^- \prec {}^{\sim}q^-$;

(iii) $q^- \wedge {}^{\sim}q^- \prec q^-$, $q^- \wedge {}^{\sim}q^- \prec {}^{\sim}q^-$.

3.19. Lemma. *Let* $q, {}^{\sim}q \in \mathrm{Ss}(P, T)$. *Then the following three assertions are equivalent*:

(i) $q \prec\succ {}^{\sim}q$;

(ii) $q^+ \cap {}^{\sim}q^+ \prec q^+$, $q^+ \cap {}^{\sim}q^+ \prec {}^{\sim}q^+$;

(iii) $q^+ \wedge {}^{\sim}q^+ \prec q^+$, $q^+ \wedge {}^{\sim}q^+ \prec {}^{\sim}q^+$.

3.20. Lemma. *Let* $q, {}^{\sim}q \in \mathrm{Ss}(P, T)$. *Then the following three assertions are equivalent*:

(i) $q \preccurlyeq\succcurlyeq {}^{\sim}q$;

(ii) $q^+ \cap {}^{\sim}q^+ \preccurlyeq q^+$, $q^+ \cap {}^{\sim}q^+ \preccurlyeq {}^{\sim}q^+$;

(iii) $q^+ \wedge {}^{\sim}q^+ \preccurlyeq q^+$, $q^+ \wedge {}^{\sim}q^+ \preccurlyeq {}^{\sim}q^+$.

3.21. Theorem. *The positive local equivalence, the negative local equivalence and the local equivalence of symmetric pseudoprocesses in* P *over* T *are equivalence relations in the set* $\mathrm{Ss}(P, T)$.

Proof. See 3.17 and Theorem 5.15 in [5].

3.22. Theorem. *Let* $q, {}^{\sim}q \in \mathrm{Ss}(P, T)$ *have right local existence of solutions. Then* $q \prec\succ {}^{\sim}q$ *iff the following conditions are fulfilled*:

(i) $D_q = D_{\sim q}$, $\mathscr{E}_q = \mathscr{E}_{\sim q}$;

(ii) *there exists a map*

$$r^+ : L_q \to R$$

such that

$$r^+(s, u) > u \quad for \quad u < \sup D_s\,, \qquad r^+(s, u) = u \quad for \quad u = \max D_s$$

and

$$s|_{\langle u, r^+(s,u)\rangle} \in S_{\sim q}\,.$$

Proof follows from (1.6.3), 1.21, 1.14 (iii), 3.17 (ii) and Theorem 5.16 in [5].

3.23. Theorem. *Let* $q, {}^\sim q \in Ss(P, T)$ *have left local existence of solutions. Then* $q \leqq\geqq {}^\sim q$ *iff the following conditions are fulfilled:*

(i) $D_q = D_{\sim q}$, $\mathscr{S}_q = \mathscr{S}_{\sim q}$;

(ii) *there exists a map*

$$r^- : L_q \to R$$

such that

$$r^-(s, u) < u \quad for \quad \inf D_s < u\,, \qquad r^-(s, u) = u \quad for \quad \min D_s = u$$

and

$$s|_{\langle r^-(s,u), u\rangle} \in S_{\sim q}\,.$$

Proof is similar to that of Theorem 5.16 in [5].

3.24. Theorem. *Let* $q, {}^\sim q \in Ss(P, T)$ *have bilateral local existence of solutions. Then* $q \preccurlyeq\succcurlyeq {}^\sim q$ *iff the following conditions are fulfilled:*

(i) $D_q = D_{\sim q}$, $\mathscr{E}_q = \mathscr{E}_{\sim q}$, $\mathscr{S}_q = \mathscr{S}_{\sim q}$;

(ii) *there exist maps*

$$r^+, r^- : L_q \to R$$

such that

$$r^+(s, u) > u \quad for \quad \sup D_s > u\,, \qquad r^+(s, u) = u \quad \text{for} \quad \max D_s = u\,,$$

$$r^-(s, u) < u \quad for \quad \inf D_s < u\,, \qquad r^-(s, u) = u \quad for \quad \min D_s = u$$

and

$$s|_{\langle r^-(s,u), r^+(s,u)\rangle} \in S_{\sim q}\,.$$

Proof follows from (1.6.3), 1.21, 1.14, 3.17 and Theorem 5.21 in [5].

3.25. Theorem. *Let* T *be a closed subset of* R *and let* $q, {}^\sim q \in S(P, T)$ *be solution complete processes. Then* $q \preccurlyeq\succcurlyeq {}^\sim q$ *iff* $q = {}^\sim q$.

Proof follows from 3.17 (iii) and Theorem 5.21 in [5].

References

[1] *Hájek, O.:* Theory of processes I, Czech. Math. Journal *17* (*92*), (1967), 159—199.
[2] *Hájek, O.:* Theory of processes II, Czech. Math. Journal *17* (*92*), (1967), 372—398.
[3] *Nagy, J.:* Stability of sets with respect to abstract processes, Lecture Notes in Operations Research and Mathematical Economics, Vol. 12, Mathematical Systems Theory and Economics II, pp. 354—378, Springer Verlag, Berlin—Heidelberg—New York 1969.
[4] *Nováková, E.:* A contribution to axiomatization of differential equations. Thesis, Fac. of Electrical Engineering, Czech. Techn. Univ., Prague 1976. (Czech.)
[5] *Nagy, J., Nováková, E.:* Local determinacy of abstract right pseudoprocesses, Čas. pěst. mat. *104* (1979), 113—133.

Authors' address: 166 27 Praha 6, Suchbátarova 2 (Katedra matematiky elektrotechnické fakulty ČVUT).