ON 1-FACTORS IN THE CUBE OF A GRAPH

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Let G be a graph (in the sense of [1] or [3]) with the vertex set $V(G)$ and the edge set $E(G)$. The number $|V(G)|$ is referred to as the order of G. A spanning subgraph F of G is called a 1-factor of G if F is a regular graph of degree one. By the cube G^3 of G we mean the graph with the properties that $V(G^3) = V(G)$ and that two vertices u and v are adjacent in G^3 if and only if the distance between u and v in G does not exceed three (the square G^2 of G is defined analogously). A set $M \subseteq E(G)$ is called a matching in G if no two edges in M are incident with the same vertex. Obviously, M is a matching in G if and only if either $M = \emptyset$ or M is the edge set of a 1-factor of a subgraph of G. We shall say that a matching M in G is *cohesive* if either $M = \emptyset$ or there exist a connected subgraph H of G and a 1-factor F of H such that $M = E(F)$.

The following theorem is the main result of this note:

Theorem. *Let G be a connected graph of an even order $p \geqq 4$, and let M be a cohesive matching in G. Then $G^3 - M$ contains a 1-factor.*

We first prove one lemma:

Lemma. *Let T be a tree of an even order $p \geqq 4$, and let M be a cohesive matching in T. Then $T^3 - M$ contains a 1-factor.*

Proof. We denote by Z the set of vertices in T which are incident with an edge in M.

If $p = 4$, then T^3 is complete, and the statement of the lemma holds.

Let $p = 6$. If T contains a pair of vertices $u, v \notin Z$ such that $T - u - v$ is a tree and $1 \leqq d(u, v) \leqq 3$, then $T^3 - M - u - v$ contains a 1-factor, and thus $T^3 - M$

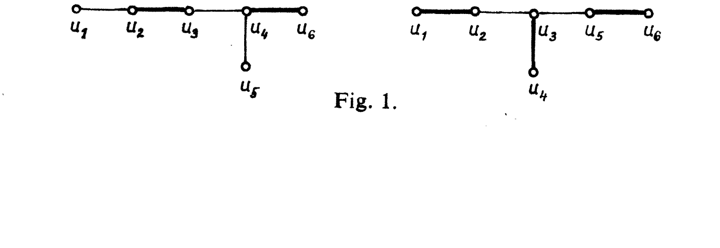

Fig. 1.

contains a 1-factor ($d(u, v)$ denotes the distance between u and v in T). Otherwise, T and M belong to one of the cases given in Fig. 1 (the edges of M are marked by thick lines); obviously, the edges u_1u_4, u_2u_5, and u_3u_6 induce a 1-factor in $T^3 - M$.

Let $p \geqq 8$. Assume that for every tree T' of an even order p', where $4 \leqq p' \leqq p - 2$, and for every cohesive matching M' in T', it is proved that $(T')^3 - M'$ contains a 1-factor. We distinguish two cases:

I. Assume that T and M fulfil at least one of the following conditions:

(1) there exist $v_1, w_1 \in V(T) - Z$ such that $v_1w_1 \in E(T) - M$, $\deg v_1 = 1$, and $\deg w_1 = 2$ ($\deg u$ denotes the degree of a vertex u in T);

(2) there exist v_2, $w_2 \in V(T) - Z$ such that $\deg v_2 = \deg w_2 = 1$, and $2 \leqq d(v_2, w_2) \leqq 3$;

(3) there exist $v_3, w_3, w_3' \in V(T)$ such that $v_3 \notin Z$, $v_3w_3 \in E(T)$, $w_3w_3' \in M$, $\deg v_3 = 1$, and $\deg w_3 = \deg w_3' = 2$;

(4) there exist $v_4, w_4, w_4' \in V(T)$ such that $v_4 \notin Z$, $w_4w_4' \in M$, $\deg v_4 = \deg w_4 = 1$, $\deg w_4' = 2$, and $d(v_4, w_4) = 3$;

(5) there exist $v_5, v_5', w_5, w_5' \in V(T)$ such that v_5v_5', $w_5w_5' \in M$, $v_5'w_5 \in E(T)$, $\deg v_5 = 1$, and $\deg v_5' = \deg w_5 = \deg w_5' = 2$;

(6) there exist $v_6, v_6', w_6, w_6' \in V(T)$ such that v_6v_6', $w_6w_6' \in M$, $\deg v_6 = \deg w_6 = 1$, $\deg v_6' = \deg w_6' = 2$, and $d(v_6, w_6) = 4$.

Denote

$$T_1 = T - v_1 - w_1\,, \quad M_1 = M\,, \quad A_1 = \{v_1w_1\}\,;$$
$$T_2 = T - v_2 - w_2\,, \quad M_2 = M\,, \quad A_2 = \{v_2w_2\}\,;$$
$$T_3 = T - v_3 - w_3\,, \quad M_3 = M - \{w_3w_3'\}\,, \quad A_3 = \{v_3w_3\}\,;$$
$$T_4 = T - v_4 - w_4\,, \quad M_4 = M - \{w_4w_4'\}\,, \quad A_4 = \{v_4w_4\}\,;$$
$$T_5 = T - v_5 - v_5' - w_5 - w_5'\,, \quad M_5 = M - \{v_5v_5', w_5w_5'\}\,,$$
$$A_5 = \{v_5w_5, v_5'w_5'\}\,;$$
$$T_6 = T - v_6 - v_6' - w_6 - w_6'\,, \quad M_6 = M - \{v_6v_6', w_6w_6'\}\,,$$
$$A_6 = \{v_6w_6', v_6'w_6\}\,.$$

There exists $i \in \{1, \ldots, 6\}$ such that T and M fulfil the condition (i). It is clear that T_i is a tree of an even order $\geqq 4$, and that M_i is a cohesive matching in T_i. According to the induction assumption, $(T_i)^3 - M_i$ contains a 1-factor, say F_i. Hence, the subgraph of T induced by the set of edges $E(F_i) \cup A_i$ is a 1-factor of T.

II. Assume that T and M fulfil none of the conditions (1)–(6). Then T is not a path.

By a terminal path in T we mean such a path P of a length $n \geqq 1$ in T that one of the end-vertices of P has degree one in T, the other end-vertex of P has a degree at least three in T, and $n - 1$ vertices of P have degree two in T. Let P be a terminal path in T, s the end-vertex of P such that $\deg s \geqq 3$, and let t be the vertex adjacent, to s in P; we shall say that P is strong or weak if $st \in M$ or $st \notin M$, respectively.

Since T and M fulfil none of the conditions (1)–(6), we have that every strong terminal path in T has a length at most three; and every weak terminal path in T has a length one or two. Moreover, no two weak terminal paths have the same end-vertex. This implies that T contains at least two vertices of a degree $\geqq 3$. It is easy to see that T contains a vertex r of degree three which is the end-vertex of exactly two terminal paths, say P_1 and P_2. Clearly, exactly one of the terminal paths P_1 and P_2 is strong. Since T contains a vertex of a degree $\geqq 3$ different from r, we have that $T - (V(P_1) \cup V(P_2))$ is a tree of an order $\geqq 3$. Obviously, the subtree of T induced by $V(P_1) \cup V(P_2)$ belongs to exactly one of the cases (7)–(11) given in Fig. 2 (the edges belonging to M are marked by thick lines).

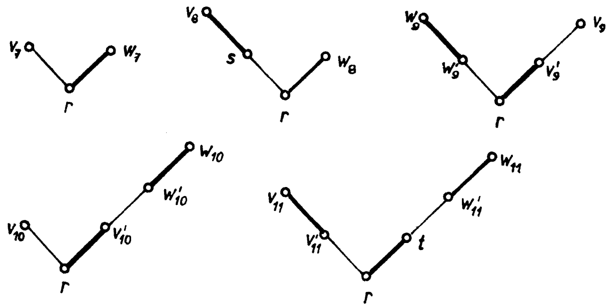

Fig. 2.

Denote

$$\begin{aligned}
T_7 &= T - v_7 - w_7\,, \quad M_7 = M - \{rw_7\}\,, \quad A_7 = \{v_7w_7\}\,;\\
T_8 &= T - r - s - v_8 - w_8\,, \quad M_8 = M - \{rw_8, sv_8\}\,,\\
&\qquad A_8 = \{rs, v_8w_8\}\,;\\
T_9 &= T - v_9 - v_9' - w_9 - w_9'\,, \quad M_9 = M - \{rv_9', w_9w_9'\}\,,\\
&\qquad A_9 = \{v_9w_9', v_9'w_9\}\,;\\
T_{10} &= T - v_{10} - v_{10}' - w_{10} - w_{10}'\,, \quad M_{10} = M - \{rv_{10}', w_{10}w_{10}'\}\,,\\
&\qquad A_{10} = \{v_{10}w_{10}', v_{10}'w_{10}\}\,;\\
T_{11} &= T - r - t - v_{11} - v_{11}' - w_{11} - w_{11}'\,,\\
&\qquad M_{11} = M - \{rt, v_{11}v_{11}', w_{11}w_{11}'\}\,, \quad A_{11} = \{rw_{11}, tv_{11}, v_{11}'w_{11}'\}\,.
\end{aligned}$$

There exists $i \in \{7, \ldots, 11\}$ such that T and M fulfil (i). Since T_i is a tree of an even order $\geqq 4$ and M_i is a cohesive matching in T_i, we have that $(T_i)^3 - M_i$ contains a 1-factor, say F_i. The subgraph of T induced by the set of edges $E(F_i) \cup A_i$ is a 1-factor of T, which completes the proof of the lemma.

Proof of Theorem. First, let $M = \emptyset$. Since G is connected, there exists a spanning tree T of G. According to the lemma, T^3, and therefore G^3, contains a 1-factor.

Next, let $M \neq \emptyset$. Then there exist a connected subgraph H of G and a 1-factor F of H such that $M = E(F)$. It is clear that there exists a spanning tree T_0 of H such that F is a 1-factor of T_0. Moreover, there exists a spanning tree T of G such that T_0 is a subtree of T. This means that M is a cohesive matching in T. According to the lemma, $T^3 - M$, and therefore $G^3 - M$, contains a 1-factor, which completes the proof of the theorem.

Corollary. *Let G be a connected grahh of an order $\geqq 4$, and let G contain a 1-factor F. Then there exists a 1-factor F' of G^3 such that $E(F) \cap E(F') = \emptyset$.*

Remark 1. In our theorem the word "cohesive" cannot be omitted. Consider the tree $T_{(1)}$ given in Fig. 3 and the matching $M_{(1)}$ formed by the strong edges. Obviously, $M_{(1)}$ is not cohesive. It is clear that $(T_{(1)})^3 - M_{(1)}$ contains no 1-factor.

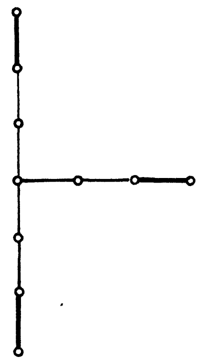

Fig. 3.

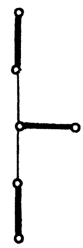

Fig. 4.

Remark 2. Chartrand, Polimeni and Stewart [2] and Sumner [6] have proved that if G is a connected graph of an even order, then G^2 contains a 1-factor. But in our theorem the power cannot be decreased. Consider the tree $T_{(2)}$ given in Fig. 4 and the cohesive matching $M_{(2)}$ formed by the strong edges. Then $(T_{(2)})^2 - M_{(2)}$ contains no 1-factor.

Remark 3. Sekanina [5] has proved that if G is a connected graph, then G^3 is hamiltonian-connected. This implies that if G is a connected graph of an even order $p \geqq 4$, then G^3 contains a 1-factor which is a subgraph of a hamiltonian cycle in G^3. Consider the tree $T_{(3)}$ in Fig. 5. It is easy to see that $T_{(3)}$ contains exactly one 1-factor, say F, and that F is a subgraph of none of the hamiltonian cycles in $(T_{(3)})^3$.

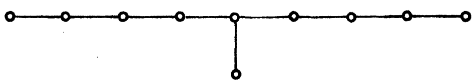

Fig. 5.

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