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A FUNCTION WHICH PRESERVES CONNECTED SPACES

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1. INTRODUCTION

In [2] N. Levine introduced the notion of semi-open sets and semi-continuity in topological spaces. S. G. Crossley and S. K. Hildebrand [1] introduced the notion of semi-homeomorphisms and investigated topological properties which are preserved under such functions. In [1], among others, they showed that (1) connected spaces are preserved under semi-homeomorphisms, (2) the images of connected sets under a semi-homeomorphism are not necessarily connected and (3) the images of open connected sets under a semi-homeomorphism are connected. P. E. Long and D. A. Carnahan [3] showed that connected spaces are preserved under almost-continuous (in the sense of Singal [6]) surjections. Moreover, P. E. Long and L. L. Herrington [4] stated that the images of open connected sets are connected under open almost-continuous functions. The purpose of the present note is to introduce a weak form of continuity, called strongly semi-continuous, which is stronger than semi-continuity due to N. Levine and to show that the image of an open connected set under a strongly semi-continuous function is connected.

Throughout the present note, X and Y will denote topological spaces on which no separation axioms are assumed, and a function f of X into Y will be denoted by $f : X \to Y$. Let S be a subset of X. The closure (resp. interior) of S in X will be denoted by $\mathrm{Cl}_X(S)$ (resp. $\mathrm{Int}_X(S)$). A set S of X is said to be *semi-open* [2] (resp. an *α-set* [5]) if $S \subset \mathrm{Cl}_X(\mathrm{Int}_X(S))$ (resp. $S \subset \mathrm{Int}_X(\mathrm{Cl}_X(\mathrm{Int}_X(S)))$). The family of all semi-open sets (resp. α-sets) of X will be denoted by $SO(X)$ (resp. $\alpha(X)$).

2. STRONGLY SEMI-CONTINUOUS FUNCTIONS

Definition 2.1. A function $f : X \to Y$ is said to be *strongly semi-continuous* (resp. *semi-continuous* [2]) if $f^{-1}(V) \in \alpha(X)$ (resp. $f^{-1}(V) \in SO(X)$) for every open set V of Y.

Strongly semi-continuous functions will be denoted by s.s.c. functions.

Definition 2.2. A function $f : X \to Y$ is said to be *almost-continuous* [6] if for each $x \in X$ and each open set V containing $f(x)$, there exists an open set U containing x such that $f(U) \subset \mathrm{Int}_Y(\mathrm{Cl}_Y(V))$.

It is obvious that continuity implies strong semi-continuity and strong semi-continuity implies semi-continuity. However, the converses are not necessarily true as the following two examples show.

Example 2.3. Let $X = \{a, b, c\}$, $\Gamma = \{\emptyset, \{a\}, \{a, b\}, X\}$ and $\Gamma^* = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}$. Then, the identity function $f : (X, \Gamma) \to (X, \Gamma^*)$ is s.s.c. but not continuous

Example 2.4. Let $X = \{a, b, c\}$, $\Gamma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\Gamma^* = \{\emptyset, \{a\}, \{b, c\}, X\}$. Then, the identity function $f : (X, \Gamma) \to (X, \Gamma^*)$ is semi-continuous but not s.s.c.

Definition 2.5. A function $f : X \to Y$ is said to be a *semi-homeomorphism* [1] if (1) f is bijective, (2) $f^{-1}(V) \in SO(X)$ for every $V \in SO(Y)$ and (3) $f(U) \in SO(Y)$ for every $U \in SO(X)$.

In Example 2.3 f is a semi-homeomorphism because it is bijective and $SO(X, \Gamma) = SO(X, \Gamma^*)$. Therefore, a semi-homeomorphism need not imply continuity. Moreover, as the following example shows, a continuous function need not be a semi-homeomorphism and hence neither an almost-continuous function nor a s.s.c. function need be a semi-homeomorphism.

Example 2.6. Let $X = \{a, b, c, d\}$, $\Gamma = \{\emptyset, \{a\}, \{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}, X\}$ and $\Gamma^* = \{\emptyset, \{a\}, \{a, c\}, X\}$. Then, the identity function $f : (X, \Gamma) \to (X, \Gamma^*)$ is continuous but not a semi-homeomorphism.

3. CONNECTED SPACES

It is well known that connected spaces are preserved under continuous surjections. It is also known that connected spaces are preserved under semi-homeomorphisms [1, Theorem 2.12] or almost-continuous surjections [3, Theorem 4]. Similarly, we have

Theorem 3.1. *If X is connected and $f : X \to Y$ is a s.s.c. surjection, then Y is connected.*

Proof. Suppose that Y is not connected. Then, there exist nonempty open sets V_1 and V_2 of Y such that $V_1 \cup V_2 = Y$ and $V_1 \cap V_2 = \emptyset$; hence $f^{-1}(V_1) \cup f^{-1}(V_2) = X$ and $f^{-1}(V_1) \cap f^{-1}(V_2) = \emptyset$. Since f is s.s.c. and surjective, we have $\emptyset \neq f^{-1}(V_j) \subset \mathrm{Int}_X(\mathrm{Cl}_X(\mathrm{Int}_X(f^{-1}(V_j))))$ for $j = 1, 2$. Now, put $U_j = \mathrm{Int}_X(\mathrm{Cl}_X(\mathrm{Int}_X(f^{-1}(V_j))))$

for $j = 1, 2$, then U_j is a nonempty open set of X and $U_1 \cup U_2 = X$. Since $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are disjoint, we obtain $\mathrm{Int}_X(f^{-1}(V_1)) \cap \mathrm{Cl}_X(\mathrm{Int}_X(f^{-1}(V_2))) = \emptyset$ and hence $\mathrm{Int}_X(f^{-1}(V_1)) \cap U_2 = \emptyset$. Consequently, we obtain $U_1 \cap U_2 = \emptyset$ which implies that X is not connected. This is a contradiction.

Remark 3.2. In Example 2.4 (X, Γ) is connected and f is a semi-continuous surjection but (X, Γ^*) is not connected. Therefore, the condition "s.s.c." on f in Theorem 3.1 can not be replaced by "semi-continuous".

We recall that a function $f : X \to Y$ is said to be *connected* if the image $f(C)$ is connected for every connected set C of X. It is well known that every continuous function is connected but not conversely. It is shown that semi-homeomorphisms are not connected functions [1, Example 1.5]. However, it is known that the images of *open* connected sets are connected under semi-homeomorphisms [1, Theorem 2.14] or open almost-continuous surjections [4, Theorem 6]. We shall show that the images of open connected sets are connected under s.s.c. surjections.

Lemma 3.3. *If U is an open set of X and $A \in \alpha(X)$, then $U \cap A \in \alpha(X)$.*

Proof. We obtain $U \cap A \in \alpha(X)$ from

$$\mathrm{Int}_X(\mathrm{Cl}_X(\mathrm{Int}_X(U \cap A))) \supset \mathrm{Int}_X(U \cap \mathrm{Cl}_X(\mathrm{Int}_X(A))) \supset U \cap A.$$

Lemma 3.4. *Let U be an open set of X and A a subset of U. Then, $A \in \alpha(X)$ if and only if $A \in \alpha(U)$.*

Proof. Since U is open in X, $\mathrm{Int}_X(B) = \mathrm{Int}_U(B)$ for every subset B of U. Thus, we have $\mathrm{Int}_U(\mathrm{Cl}_U(\mathrm{Int}_U(A))) = \mathrm{Int}_X(\mathrm{Cl}_X(\mathrm{Int}_X(A))) \cap U$ which completes the proof.

Lemma 3.5. *If U is an open set of X and $f : X \to Y$ is s.s.c., then a function $f_U : U \to$ $\to f(U)$, defined by $f_U(x) = f(x)$ for every $x \in U$, is s.s.c.*

Proof. Let V_U be any open set of a subspace $f(U)$. Then, there exists an open set V of Y such that $V_U = V \cap f(U)$. Since f is s.s.c., $f^{-1}(V) \in \alpha(X)$ and hence, by Lemma 3.3 $f_U^{-1}(V_U) = f^{-1}(V) \cap U \in \alpha(X)$ because U is open in X. Therefore, by Lemma 3.4 we obtain $f_U^{-}\ (V_U) \in \alpha(U)$ which shows that f_U is s.s.c.

Theorem 3.6. *If $f : X \to Y$ is s.s.c., then $f(U)$ is connected for every open connected set U of X.*

Proof. Suppose that $f : X \to Y$ is s.s.c. and U is an open connected set of X. Then, by Lemma 3.5 $f_U : U \to f(U)$ is s.s.c. and hence, by Theorem 3.1 $f_U(U) = f(U)$ is connected.

Remark 3.7. In Example 2.3 f is s.s.c. and a subset $\{b, c\}$ is closed connected in

(X, Γ) but $f(\{b, c\})$ is not connected. Therefore the condition "open" on U in Theorem 3.6 can not be replaced by "closed". This shows that a s.s.c. function is not necessarily connected.

Remark 3.8. The inverse function $f^{-1} : (X, \Gamma^*) \to (X, \Gamma)$ of f in Example 2.4 is connected but not semi-continuous and hence not s.s.c. This shows that connected function is not necessarily s.s.c. Moreover, we observe that the converse to Theorem 3.6 is not true in general.

Remark 3.9. The condition "s.s.c." on f in Theorem 3.6 can not be replaced by "semi-continuous" as we have noted in Remark 3.2.

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