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# ON GRAPHS WITH NON-ISOMORPHIC 2-NEIGHBOURHOODS

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## 1. INTRODUCTION

Let $G = (V(G), E(G))$ be a finite undirected graph with the vertex set $V(G)$ and the edge set $E(G)$. We assume that $G$ is a graph without loops and multiple edges. *The distance* $d_G(x, y)$ between vertices $x$ and $y$ in $G$ is the least number of edges in the path from $x$ to $y$. Let $L_j(x, G) = \{y \in V(G) : d_G(x, y) = j\}$ and $L_j^+(x, G) = \{y \in V(G) : : d_G(x, y) > j\}$, for $x \in V(G)$. The subgraph of $G$ induced by $L_j(x, G)$ is called the *j-neighbourhood* of $x$ in $G$ and denoted by $N_j(x, G)$. The subgraph of $G$ induced by $L_j^+(x, G)$ is called the *$j^+$-neighbourhood* of $x$ in $G$ and denoted by $N_j^+(x, G)$.

At the first Czechoslovak symposium on graph theory (Smolenice 1963) A. A. Zykov posed the problem: Given a graph $H$, does there exist a graph $G$ such that $H$ is isomorphic to $N_1(x, G)$ for all $x \in V(G)$? This problem, known as the Trahtenbrot-Zykov problem, has been investigated in many papers (see [1], [3], [5] and [6]). We have studied the generalization of the Trahtenbrot-Zykov problem to the j-neighbourhoods, for $j \geqq 1$, [2]. Another direction of research was proposed by J. Sedláček [7] in 1979. He studied the class $\mathscr{C}_1$ of connected graphs $G$ with the following property: If $x$ and $y$ are two vertices of $G$, then $N_1(x, G)$ and $N_1(y, G)$ are not isomorphic. He proved

**Theorem 1.1** [7]. *For every positive integer $m \geqq 6$ there exists a graph $G$ on $m$ vertices belonging to $\mathscr{C}_1$.*

In this paper we deal with the class $\mathscr{C}_2$ of graphs $G$ with the property: If $x$ and $y$ are two vertices of $G$, then $N_2(x, G)$ and $N_2(y, G)$ are not isomorphic. We derive a result similar to Theorem 1.1 for the class $\mathscr{C}_2$ and for every $m \geqq 7$. We also study relationships between the classes $\mathscr{C}_1$ and $\mathscr{C}_2$. In Section 2 we consider graphs $G$ belonging to $\mathscr{C}_1$ and/or $\mathscr{C}_2$, for which $L_2(x, G) \neq \emptyset$ for all $x \in V(G)$, and in Section 3 we omit this last condition. In our considerations we also use $\mathscr{C}_1^+$, the class of graphs with non-isomorphic $N_1^+(x, G)$ for all $x \in V(G)$.

Graph-theoretic terms not defined here can be found in [4].

## 2. MAIN RESULTS

In this section we study graphs $G$ in the class $\mathscr{C}_2$, and assume $L_2(x, G) \neq \emptyset$ for all vertices $x$ of $G$. Such graphs on 7, 8, 9, 10 and 11 vertices are presented in Figs. 2 and 3. We also study relationships between the classes $\mathscr{C}_1$ and $\mathscr{C}_2$. The results of this section are based on the following construction.

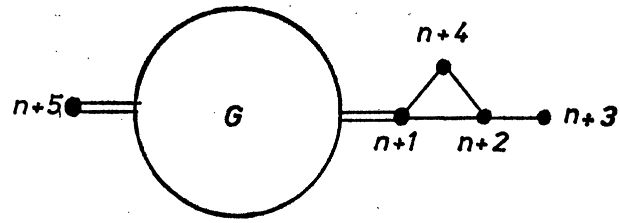



Fig. 1. The graph $G^*$.

Let $G$ be a graph with $n$ vertices. We consider the graph $G^*$ presented in Fig. 1, where a double line between two subgraphs indicates that every vertex of the first subgraph is adjacent to every vertex inthesecond one, while a single line between two vertices indicates that they are adjacent. Table 1 lists all 1-, 2- and $1^+$-neighbourhoods in the graph $G^*$.

We have the following observations:

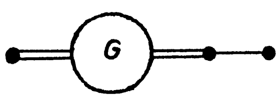
**Proposition 2.1.** *If $G$ is a graph with at least two vertices, then $G$ belongs to $\mathscr{C}_1$ if and only if $G^*$ belongs to $\mathscr{C}_1$.*

Proof follows directly from the second column of Tab. 1. □

Table 1

| vertex $x$ | $N_1(x, G^*)$ | $N_2(x, G^*)$ | $N_1^+(x, G^*)$ |
|---|---|---|---|
| $n+1$ | $G \cup K_2$ | $2K_1$ | $2K_1$ |
| $n+2$ | $K_1 \cup K_2$ | $G$ | $G + K_1$ |
| $n+3$ | $K_1$ | $K_2$ | $F$ |
| $n+4$ | $K_2$ | $G \cup K_1$ | $G + K_1 \cup K_1$ |
| $n+5$ | $G$ | $K_1$ | $S$ |
| $1 \leq i \leq n$ | $N_1(i, G) + 2K_1$ | $N_1^+(i, G) \cup K_2$ | $N_1^+(i, G) \cup P_3$ |

F:

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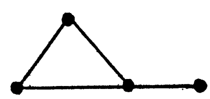

**Proposition 2.2.** *If $G$ is a graph with at least one vertex, then $G$ belongs to $\mathscr{C}_1^+$ if and only if $G^*$ belongs to $\mathscr{C}_1^+$.*

Proof follows directly from the fourth column of Tab. 1. □

**Proposition 2.3.** *Let $G$ be a connected graph with at least three vertices and $\Delta(G) < n - 1$, where $\Delta(G)$ is the maximum degree of $G$. If $G$ belongs to the intersection of $\mathscr{C}_2$ and $\mathscr{C}_1^+$, then $G^*$ belongs to the intersection of $\mathscr{C}_2$ and $\mathscr{C}_1^+$.*

Proof follows directly from the third and fourth columns of Tab. 1. □

To present further results we define the sequence of graphs $G_0^*, G_1^*, \ldots, G_i^*, \ldots$, as follows: $G_0^* = G$ and $G_{i+1}^* = (G_i^*)^*$, for a given graph $G$.

**Theorem 2.1.** *For every integer $m \geq 7$ there exists a graph on $m$ vertices belonging to $\mathscr{C}_1 \cap \mathscr{C}_2$.*

Proof. If $m$ is an integer greater than or equal to 7, then the graph $G_i^*$, where $i =$ entire $((m - 7)/5)$ and $G$ is isomorphic to the $(m - 6 - 5i)$th graph of Fig. 2, has $m$ vertices and by Propositions 2.1–2.3 it belongs to $\mathscr{C}_1$ and $\mathscr{C}_2$. □

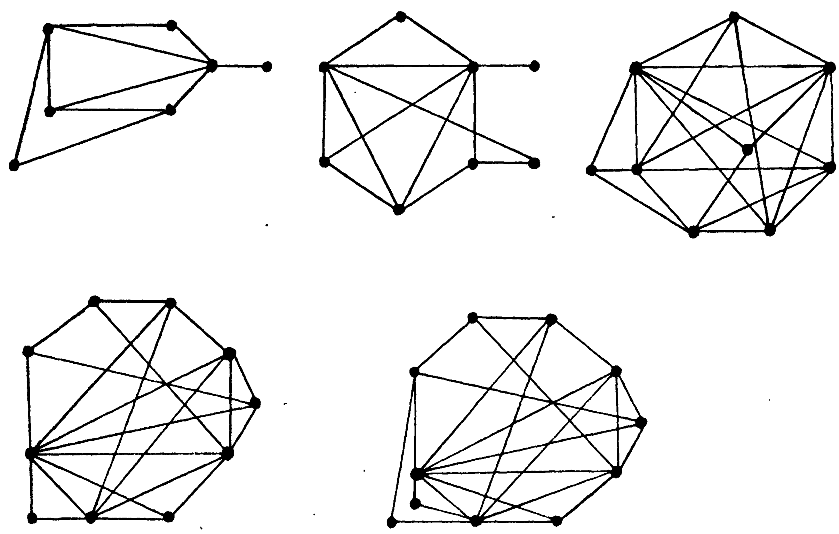

Fig. 2. Graphs in the classes $\mathscr{C}_1$, $\mathscr{C}_2$ and $\mathscr{C}_1^+$.

**Theorem 2.2.** *For every integer $m \geq 7$ there exists a graph on $m$ vertices belonging to $\mathscr{C}_2 - \mathscr{C}_1$.*

Proof. The proof of this theorem is similar to that of Theorem 2.1. For $G$ we take the $(m - 6 - 5i)$th graph of Fig. 3. □

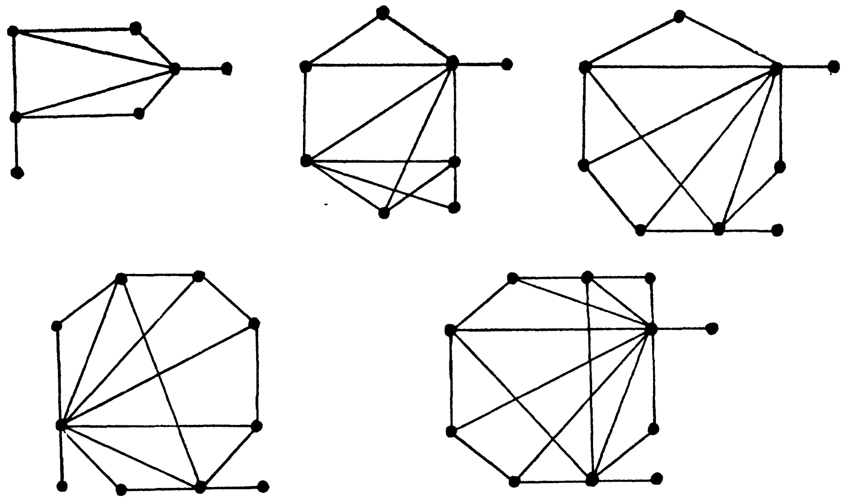

Fig. 3. Graphs in the classes $\mathscr{C}_2$, $\mathscr{C}_1^+$ but not in the class $\mathscr{C}_1$.

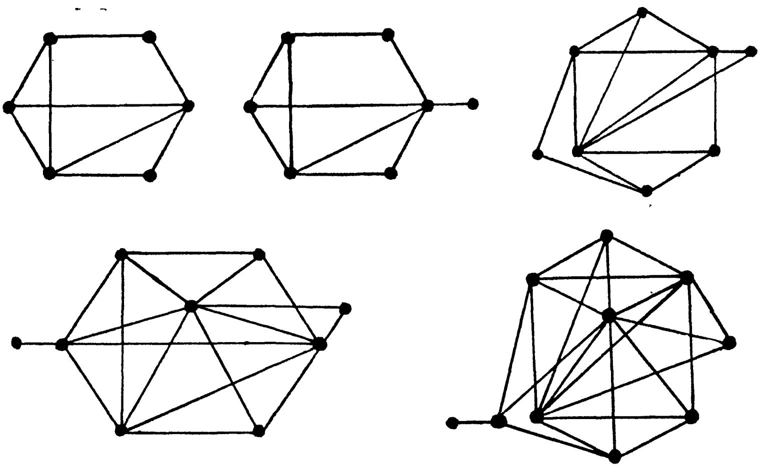

Fig. 4. Graphs in the class $\mathscr{C}_1$ but not in the classes $\mathscr{C}_1^+$ and $\mathscr{C}_2$.

**Theorem 2.3.** *For every integer $m \geqq 6$ there exists a graph on $m$ vertices belonging to $\mathscr{C}_1 - \mathscr{C}_2$.*

Proof. Let $m$ be an integer greater than or equal to 6 and assume $i$ = entire $((m - 6)/5)$. The graph $G_i^*$, where $G$ is isomorphic to the $(m - 5(i + 1))$th graph of Fig. 4, has $m$ vertices and belongs to $\mathscr{C}_1 - \mathscr{C}_2$. This follows from Propositions 2.1 and 2.2, and from the fact that if $G \notin \mathscr{C}_1^+$, then $G_i^* \notin \mathscr{C}_2$, for $i \geqq 1$. □

## 3. REMARKS

Let us now consider the graph $G^-$ presented in Fig. 5.

Note that $G^-$ has exactly one vertex $x$ for which $L_2(x, G^-) = \emptyset$, namely $x = n + 1$.

We use this construction to derive another subclass of $\mathscr{C}_1 \cap \mathscr{C}_2$ (and $\mathscr{C}_2 - \mathscr{C}_1$, $\mathscr{C}_1 - \mathscr{C}_2$ as well). To this end we define the sequence of graphs for a given graph $G$ with $n$ vertices:

$$G_0^-, G_1^-, \ldots, G_i^-, \ldots,$$

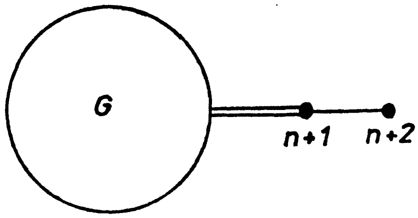


Fig. 5. The graph $G^-$.

where $G_0^- = G$ and $G_{i+1}^- = (G_i^-)^-$. One can easily see that starting with $G$ isomorphic to the first or the second graph in Fig. 2 (Fig. 3, Fig. 4) one obtains graphs with $m$ vertices in the class $\mathscr{C}_2 \cap \mathscr{C}_1$ ($\mathscr{C}_2 - \mathscr{C}_1$, $\mathscr{C}_1 - \mathscr{C}_2$, resp.), where $m \geqq 7$ ($m \geqq 7$, $m \geqq 6$). All 1-, 2- and $1^+$-neighbourhoods in $G^-$ are shown in Tab. 2.

Table 2

| vertex $x$ | $N_1(x, G^-)$ | $N_2(x, G^-)$ | $N_1^+(x, G^-)$ |
|---|---|---|---|
| $n+1$ | $G \cup K_1$ | $K_0$ | $K_0$ |
| $n+2$ | $K_1$ | $G$ | $G$ |
| $1 \leqq i \leqq n$ | $N_1(i, G) + K_1$ | $N_1^+(i, G) \cup K_1$ | $N_1^+(i, G) \cup K_1$ |

$K_0 = (\emptyset, \emptyset)$.

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