Časopis pro pěstování matematiky

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Časopis pro pěstování matematiky, Vol. 108 (1983), No. 4, 409--411

Persistent URL: `http://dml.cz/dmlcz/118185`

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A NOTE ON THE INVARIANCE OF BAIRE SPACES UNDER MAPPINGS

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(Received January 21, 1983)

In 1961 Z. Frolík in his paper [3] proved that if f is an almost continuous and feebly open mapping of a Baire space X onto a space Y, then Y is a Baire space. In 1977 T. Neubrunn in his paper [4] proved that if f is a one-to-one feebly continuous and feebly open mapping of X onto Y, then X is a Baire space if and only if Y is a Baire space.

In the present paper we shall give a generalization of there results, assuming more generally that f is a feebly continuous mapping such that for every nowhere dense set $E \subset Y$ the set $f^{-1}(E)$ is nowhere dense.

For the basic properties of Baire spaces see [1], Chapter 9, and [2].

Definition 1. A space X is said to be *a Baire space* if every nonempty open subset of X is of the second category.

Definition 2. A mapping f from X onto Y is said to be *almost continuous* if

$$f^{-1}(G) \subset \mathrm{Cl}(\mathrm{Int}(f^{-1}(G)))$$

for any open set $G \subset Y$.

Definition 3. A mapping f from X onto Y is said to be *feebly continuous* (*feebly open*) if for any nonempty open set $V \subset Y$ ($U \subset X$), the set $\mathrm{Int}(f^{-1}(V))$ ($\mathrm{Int}(f(U))$) is nonempty.

Remark. A space is a Baire space if and only if the intersection of every countable family of open dense sets is a dense set (see [2]).

We shall prove the following

Theorem. *Let us suppose that f is a feebly continuous mapping of a space X onto a space Y such that for each $E \subset Y$,*

(1) $\qquad$ *E is nowhere dense in $Y \Rightarrow f^{-1}(E)$ is nowhere dense in X.*

If X is a Baire space then Y is a Baire space.

Proof. Suppose that X is a Baire space. Let U_n, $n = 1, 2, \ldots$, be open dense subsets of Y. We shall prove that

$$\bigcap_{n=1}^{\infty} U_n \quad \text{is dense in } Y.$$

Put

$$Z_n = \mathrm{Int}(f^{-1}(U_n)) \ (n = 1, 2, \ldots).$$

Since the sets $Y - U_n$ are nowhere dense in Y, by (1) we obtain that $f^{-1}(Y - U_n)$ are nowhere dense in X. Hence

$$Z_n = X - \mathrm{Cl}(f^{-1}(Y - U_n))$$

are dense in X. Since X is a Baire space, $\bigcap_{n=1}^{\infty} Z_n$ is dense in X. By the feeble continuity of f the set $f(\bigcap_{n=1}^{\infty} Z_n)$ is dense in Y. Hence by

$$f(\bigcap_{n=1}^{\infty} Z_n) \subset \bigcap_{n=1}^{\infty} U_n,$$

the set $\bigcap_{n=1}^{\infty} U_n$ is dense in Y. The proof is complete.

Corollary 1. (See [3; Theorem 1].) *Let us suppose that f is an almost continuous and feebly open mapping of a space X onto a space Y. If X is a Baire space then Y is a Baire space.*

Proof. We shall prove that f satisfies (1). Let E be a nowhere dense subset of Y. Hence $Y - \mathrm{Cl}(E)$ is dense in Y. Since f is feebly open the set $f^{-1}(Y - \mathrm{Cl}(E))$ is dense in X. By almost continuity of f we have

$$f^{-1}(Y - \mathrm{Cl}(E)) \subset \mathrm{Cl}(\mathrm{Int}(f^{-1}(Y - \mathrm{Cl}(E)))) \subset \mathrm{Cl}(X - \mathrm{Cl}(f^{-1}(E))).$$

Thus the set $X - \mathrm{Cl}(f^{-1}(E))$ is dense in X, i.e. the set $f^{-1}(E)$ is nowhere dense in X. The proof is complete.

Corollary 2. (See [4; Theorem].) *If f is a one-to-one feebly continuous and feebly open mapping of X onto Y, then X is a Baire space if and only if Y is a Baire space.*

Proof. First suppose that X is a Baire space. We shall prove that f satisfies (1). Let E be a nowhere dense subset of Y. Let U be a nonempty open subset of X. Put

$$V = \mathrm{Int}(U - f^{-1}(E)).$$

Evidently V is an open subset of U and $V \cap f^{-1}(E)$ is empty. We shall prove that V is nonempty. The set $Y - \mathrm{Cl}(E)$ is dense in Y. Since f is feebly open, $\mathrm{Int}(f(U))$ is nonempty. Then the set

$$(Y - \mathrm{Cl}(E)) \cap \mathrm{Int}(f(U))$$

is nonempty. Since f is feebly continuous and one-to-one we obtain

$$\emptyset \neq \operatorname{Int}(f^{-1}((Y - \operatorname{Cl}(E)) \cap \operatorname{Int}(f(U)))) \subset \operatorname{Int}(f^{-1}(Y - \operatorname{Cl}(E))) \cap$$
$$\cap f^{-1}(f(U)) \subset \operatorname{Int}(X - f^{-1}(\operatorname{Cl}(E))) \cap U \subset V.$$

Then V is nonempty. Thus the set $f^{-1}(E)$ is nowhere dense in X.

The "only if" part follows from the fact that the inverse mapping f^{-1} is also feebly continuous and feebly open. The proof is complete.

In the conclusion we show that the assumption "one-to-one" in Corollary 2 cannot be omitted.

Example. Put $X = (-\infty, \infty)$. Let Y be a dense countable subset of the interval $(0, 1)$. Let g be a mapping of the set of all integer numbers onto Y. Denote by $[x]$ the integer part of x. Put

$$T = \{x \in X;\ x - [x] \in Y\}.$$

Define a mapping $f : X \to Y$ as follows:

$$f(x) = \begin{cases} x - [x] & \text{if } x \in T, \\ g([x]) & \text{otherwise}. \end{cases}$$

Then the mapping f is feebly continuous and feebly open but X is a Baire space while Y is not.

First we shall prove that f is feebly continuous. Let $P \subset Y$, $\operatorname{Int}(P) \neq \emptyset$. Put

$$U = g^{-1}(P) + \operatorname{Int}(P).$$

Then U is a nonempty open subset of X and $U \subset f^{-1}(P)$. Hence the set $\operatorname{Int}(f^{-1}(P))$ is nonempty.

Now we shall prove that f is feebly open. Let $S \subset X$, $\operatorname{Int}(S) \neq \emptyset$. Since T is dense in X, there exist $u, v \in T$, $u < v$, such that $(u, v) \subset S$, $[u] = [v]$. Put

$$V = (u, v) \cap Y.$$

Then V is a nonempty open set such that $V \subset f(S)$. Hence $\operatorname{Int}(f(S))$ is nonempty.

Evidently, X is a Baire space but Y is not a Baire space.

References

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