Časopis pro pěstování matematiky

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On formal theory of differential equations. I.

Časopis pro pěstování matematiky, Vol. 111 (1986), No. 4, 353--383

Persistent URL: http://dml.cz/dmlcz/118285

Časopis pro pěstování matematiky, roč. 111 (1986), Praha

ON FORMAL THEORY OF DIFFERENTIAL EQUATIONS I

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(Received October 26, 1983)

Expressively saying, the formal theory of differential equations is not especially interested in the existence proper of some actual solutions satisfying certain additional (boundary, initial, asymptotic, smoothness, etc.) conditions but deals with other topics more closely related to the equations themselves (as are the compatibility conditions, canonical forms, group symmetries, laws of conservation, presence of variational principle, geometry of characteristics, explicit integrability theory, to name a few). We cannot go into the history of the subject, it is however worth while to mention the classical results by Cartan, Darboux, Goursat, Lie, Riquier and Vessiot, the recent revivification due to Goldschmidt, Guillemin, Spencer and Sternberg, and the present development related to the theory of Lie-Bäcklund infinitesimal transforms (Ibragimov), group symmetries (Ovsjannikov), equivalence problem and formal integrability (Pommaret), variational complexes (Tulczyjew), continuous cohomologies (Tzujishita, Gelfand) and other actual areas of analysis and geometry.

In spite of abundance of various deep and rather complicated results, very little is known about the most fundamental problem of the theory which may be indicated as follows: Starting with a system

$$f_k(x^1, x^2, \ldots, y^1, y^2, \ldots, \partial y^1/\partial x^1, \ldots) \equiv 0\,,$$

various substitutions can be performed. For instance, one can take $\xi^1 = x^1\, \partial y^1/\partial x^2$, $\xi^2 = x^1 + \sin y^2, \ldots$ in the role of the new independent variables and, say, $\eta^1 = x^1$, $\eta^2 = x^2\, \partial y^1/\partial x^1, \ldots$ as the unknown functions. After some unpleasant calculations, one can derive certain relations

$$\varphi_\kappa(\xi^1, \xi^2, \ldots, \eta^1, \eta^2, \ldots, \partial\eta^1/\partial\xi^1, \ldots) \equiv 0$$

among the new variables. In favourable cases (which we have in mind) there exists an inverse substitution carrying the new system $\varphi_\kappa \equiv 0$ back to the original one $f_k \equiv 0$. It follows that both these systems may be considered equivalent in a certain obvious sense and the classes of equivalent systems (*diffieties*) may serve as rather natural objects of investigations.

Following this way, we obtain a remarkable category in which the notions of dependent and independent variables (and hence of partial derivatives, jets, usual differential operators, and others) do not make any invariant sense. The arising difficulties may be compared with analogous circumstances appearing in algebraic geometry (cf. the theory of birational correspondences). But we shall see that this rather vague similarity is deeper than one can expect at a glance. (At this place, it is of interest to mention the surprising but unsuccessful and forgotten paper [1] and remark that our setting of the problem has very little in common with the familiar differential algebra [2] which deals with rather special questions of purely algebraic character.) At least from the technical point of view, the general theory cannot be restricted only to finite sets of differential equations (identities, operators, etc.) quite analogously as the algebraic geometry needs some kinds of infinite dimensional spaces and necessarily handles polynomials of arbitrarily high degrees. Explicitly saying, we shall deal with the infinite prolongation of the general system of partial differential equations from the intrinsical point of view. Fortunately, this object of our investigations can be completely described by four simple axioms. (We also refer to the papers [3], [4] where the above mentioned concept of a diffiety was introduced for the first time.)

The differential equations considered are invariantly rewritten as a Pfaffian system and we use the common category of C^∞-smooth real manifolds and mappings. Our first aim is to demonstrate the advantages of the approach presented as transparently as possible by deriving several concrete and nontrivial results. Links to the common approach will be discussed later on. (In the meantime, we refer to the already mentioned papers [3], [4] and to a forthcomming book of the same author.)

INTRODUCTION AND BASIC STRUCTURES

1. The fundamental space. Our considerations will be carried out in a space J that is representable by an inverse limit of the type

$$(1) \qquad J = \lim \operatorname{inv} J^l: \ldots \to J^{l+1} \to J^l \to \ldots \to J^1 \to J^0$$

where J^l $(l = 0, 1, \ldots)$ are certain n^l-dimensional manifolds and the mappings $j_l^{l+1}: J^{l+1} \to J^l$ $(l = 0, 1, \ldots)$ are surjective submersions. The manner in which the space J is expressed is, however, not a consistent part of the structure under consideration. The same space J may be also represented by an analogous limit

$$J = \lim \operatorname{inv} I^l: \ldots \to I^{l+1} \to I^l \to \ldots \to I^1 \to I^0$$

with quite other manifolds I^l and mappings $i_l^{l+1}: I^{l+1} \to I^l$ (temporary notation) provided that for every $l, k \geqq 0$ there exist integers $K(l) \geqq l$, $L(k) \geqq k$ and mappings $p^l: I^{K(l)} \to J^l$, $q^k: J^{L(k)} \to I^k$ satisfying

$$p^l \circ q^{K(l)} = j_l^{l+1} \circ \ldots \circ j_{L(K(l))-1}^{L(K(l))}, \quad q^k \circ p^{L(k)} = i_k^{k+1} \circ \ldots \circ i_{K(L(k))-1}^{K(L(k))}.$$

We refer to [5] for similar reasonings. All the following concepts should not depend on the manner in which J is expressed, of course.

Denote (for a moment) by $\Psi_s^{[l]}$ ($l, s = 0, 1, \ldots$) the space of all differential s-forms on the manifold J^l. Owing to the injections $j_l^{l+1*}: \Psi_s^{[l]} \to \Psi_s^{[l+1]}$, the space $\Psi_s^{[l]}$ may be considered as a subspace of $\Psi_s^{[l+1]}$ and hence of all spaces $\Psi_s^{[l+c]}$ ($c = 0, 1, \ldots$) so that the direct limit

$$\Psi_s = \lim \operatorname{dir} \Psi_s^{[l]} = \Psi_s^{[0]} \cup \Psi_s^{[1]} \cup \ldots$$

makes a good sense. It is called *the space of exterior s-forms on the space J*. A form $\psi \in \Psi_s$ may be identified either with an array of the type

$$\vartheta, j_l^{l+1*}\vartheta, (j_l^{l+1} \circ j_{l+1}^{l+2})^* \vartheta, \ldots \quad (\vartheta \in \Psi_s^{[l]})$$

or, even simpler, with any form of this array. Every space Ψ_s is a module over the ring $\Psi_0 = C^\infty(J)$ of functions on J and the direct sum $\Psi = \Psi_0 \oplus \Psi_1 \oplus \ldots$ is an exterior graded differential algebra with the common exterior differential $\mathrm{d}: \Psi_s \to \Psi_{s+1}$.

Vector fields on J may be introduced as the derivations of the $\mathbb{R}$-algebra Ψ_0. That means, we postulate the familiar rules

$$Xf \in \Psi_0\,, \quad X(f + g) = Xf + Xg\,, \quad Xc \equiv 0\,, \quad X(fg) = gXf + fXg$$
$$(f, g \in \Psi_0,\ c \in \mathbb{R})\,,$$

for every vector field $X \in TJ$. Alternatively, a vector field X may be identified with certain Ψ_0-homomorphisms of the module Ψ_1 into the algebra Ψ_0. Denoting this homomorphism by $X \rfloor$, we postulate the rules

$$X \rfloor \varphi \in \Psi_0\,, \quad X \rfloor (\varphi + \psi) = X \rfloor \varphi + X \rfloor \psi\,, \quad X \rfloor (f\varphi) = fX \rfloor \varphi$$
$$(\varphi, \psi \in \Psi_1,\ f \in \Psi_0)\,.$$

The equivalent approaches are related by $Xf = X \rfloor \mathrm{d}f$, of course. Then, following the common way, the homomorphism $X \rfloor$ can be extended to the familiar derivative $X \rfloor : \Psi \to \Psi$ of the Ψ_0-algebra Ψ, we may introduce the Lie derivative $\mathscr{L}_X = X \rfloor \mathrm{d} + \mathrm{d}X \rfloor$ of the $\mathbb{R}$-algebra Ψ and finally, the space TJ of all vector fields on J is a Ψ_0-module and a Lie algebra with the well-known Lie bracket $[X, Y] = XY - YX$ ($X, Y \in TJ$). The common rules of calculations are valid without any changes.

Note also that algebraic methods will be often applied at a fixed point of the space J. In this case the values of various objects at this point will be denoted by subscripts. For example, if $z \in J$, then we have the objects $f_z = f(z)$, $\psi_z \in \Psi_z$, $X_z \in T_zJ$, and so on. Occasionally these subscripts will be omitted for typhographic reasons.

2. Diffieties. This rather suitable new term denotes the main object of our forthcoming considerations. In more detail, a Ψ_0-submodule Ω of the Ψ_0-module Ψ_1 is called a *diffiety* if it possesses certain properties $\mathscr{L}oc$, $\mathscr{D}im$, $\mathscr{C}los$ and $\mathscr{F}in$ specified as follows:

The property $\mathscr{L}oc$ means that a form $\psi \in \Psi_1$ lies in Ω if it belongs to Ω locally. In other words, let $\psi \in \Psi_1$ and for every $z \in J$ let there exist a function $f \in \Psi_0$ such that $f(z) \neq 0, f\psi \in \Omega$. Then we postulate the inclusion $\psi \in \Omega$. (Note that the property $\mathscr{L}oc$ is not an essential one, being of a technical character.)

The property $\mathscr{D}im$ means that the codimension of Ω in the Ψ_0-module Ψ_1 is locally finite. More in detail, we suppose that for every $z \in J$ there exist $f \in \Psi_0$ with $f(z) \neq 0$ and $\xi_1, \ldots, \xi_n \in \Psi_1$ such that we have a decomposition of the type

$$f\psi = f_1\xi_1 + \ldots + f_n\xi_n + \omega \quad (f_1, \ldots, f_n \in \Psi_0,\ \omega \in \Omega)$$

for every form $\psi \in \Psi_1$ and this decomposition is unique near the point z. Still in other words, the factor-module Ψ_1/Ω is locally free of a certain finite dimension n over the ground ring Ψ_0. (The $\mathscr{D}im$ property can be slightly weakened for the needs of Cartan's theory of pseudogroups.)

Before passing to the remaining properties, denote by $\mathscr{H}$ (= *horizontal*) the Ψ_0-submodule of TJ consisting of all vector fields $X \in TJ$ such that $X \rfloor \omega \equiv 0$ for all $\omega \in \Omega$. Owing to $\mathscr{L}oc$, every form $\psi \in \Psi_1$ satisfying $X \rfloor \psi \equiv 0$ $(X \in \mathscr{H})$ necessarily lies in Ω. Owing to $\mathscr{D}im$, the space $\mathscr{H}_z$ (consisting of all vectors X_z where $X \in \mathscr{H}$) is an n-dimensional $\mathbb{R}$-linear space. We also mention the obvious formula

$$\mathscr{L}_X\omega \equiv X \rfloor \mathrm{d}\omega \quad (X \in \mathscr{H},\ \omega \in \Omega)$$

which will be important in the sequel.

After this small digression, the property $\mathscr{C}los$ may be expressed by the inclusion $[X, Y] \in \mathscr{H}$ $(X, Y \in \mathscr{H})$, that means, $\mathscr{H}$ is a Lie subalgebra of the Lie algebra TJ. One can verify that the mentioned property is equivalent both to the identity $X \rfloor Y \rfloor \mathrm{d}\omega \equiv 0$ and to the identically satisfied inclusion $\mathscr{L}_X\omega \in \Omega$ $(X, Y \in \mathscr{H}, \omega \in \Omega)$. (The property $\mathscr{C}los$ is the most important one. It exactly corresponds to the compatibility conditions for the classical approach and can be neither omitted nor weakened.)

The property $\mathscr{F}in$ is satisfied if the Ψ_0-module Ω can be generated from a finite number of forms $\omega_1, \ldots, \omega_c \in \Omega$ by repeated use of the operators $\mathscr{L}_X$ $(X \in \mathscr{H})$. In more detail, we assume that there exist forms $\omega_1, \ldots, \omega_c \in \Omega$ such that every form $\omega \in \Omega$ can be expressed by a finite sum of terms of the type

$$f\mathscr{L}_{X_1} \ldots \mathscr{L}_{X_s}\omega_j \quad (f \in \Psi_0;\ X_1, \ldots, X_s \in \mathscr{H};\ s = 0, 1, \ldots;\ j = 1, \ldots, c)\,. \tag{2}$$

(The last condition ensures that we deal with differential equations involving a finite number of unknown functions. Note that the (obvious) local variant of the condition $\mathscr{F}in$ does not bring any essential generalization but leads to some technical difficulties.)

3. A way to commutative algebra. One can easily see that the condition $\mathscr{F}in$ is equivalent to the existence of the so called *good filtration*

(3) $\Omega^*: \ldots \subset \Omega^{-1} = \{0\} \subset \Omega^0 \subset \Omega^1 \subset \ldots \subset \Omega^l \subset \Omega^{l+1} \subset \ldots \subset \Omega = \cup \Omega^l$

of the Ψ_0-module Ω by a nondecreasing system of finitely generated submodules Ω^l. The term *good filtration* means that every module Ω^{l+1} includes (besides the module Ω^l) all forms of the type $\mathscr{L}_X\omega$ ($X \in \mathscr{H}$, $\omega \in \Omega^l$) and moreover, the Ψ_0-module Ω^{l+1} is generated by the mentioned forms ω, $\mathscr{L}_X\omega$ ($X \in \mathscr{H}$, $\omega \in \Omega^l$) for all l sufficiently large. Symbolically:

$(4)_{1,2}$ $\qquad \Omega^{l+1} \subset \Omega^l + \mathscr{L}_{\mathscr{H}}\Omega^l$ (all l), $\quad \Omega^{l+1} = \Omega^l + \mathscr{L}_{\mathscr{H}}\Omega^l$ (l large enough).

The existence of such a filtration is clear, one can take for Ω^l the sum of all summands (2) with $s \leqq l$. (Note that the choice of a filtration (3) corresponds to a specification of the family of dependent variables (unknown functions) in the classical theory of differential equations. We shall see later on that such a filtration is canonically determined if J is a space of ∞-jets of sections of a fibered manifold. This is the common approach to the theory (cf. [3], [4]) but almost a trivial case from our point of view. We find it necessary to stress the fact that these filtrations are viewed as *auxiliary and accidental objects* and we wish to derive some results which do not depend on them.)

Let a filtration (3) possessing the property $(4)_1$ be given. Introducing the factor-modules $\mathscr{G}^l = \Omega^l/\Omega^{l-1}$, one can easily observe that every operator $\mathscr{L}_X$ ($X \in \mathscr{H}$) induces a Ψ_0-linear mapping $\mathscr{G}^l \to \mathscr{G}^{l+1}$. This mapping is of fundamental importance and will be simply denoted by the single letter X. We summarize: If a form $\omega \in \Omega^l$ determines a class $\hat{\omega} \in \mathscr{G}^l$, then the form $\mathscr{L}_X\omega$ determines a certain class from $\mathscr{G}^{l+1}$ denoted $X\hat{\omega}$, briefly, $(\mathscr{L}_X\omega)^\wedge = X\hat{\omega} \in \mathscr{G}^{l+1}$ ($\omega \in \Omega^l$, $\hat{\omega} \in \mathscr{G}^l$).

It is evident that we may also let the tensor space

$$\otimes\mathscr{H} = \Psi_0 \oplus \mathscr{H} \oplus (\mathscr{H} \otimes \mathscr{H}) \oplus (\mathscr{H} \otimes \mathscr{H} \otimes \mathscr{H}) \oplus \ldots$$

operate on the direct sum $\mathscr{G} = \mathscr{G}^0 \oplus \mathscr{G}^1 \oplus \ldots$ by the rule

$$(X_1 \otimes \ldots \otimes X_s)\,\hat{\omega} = X_1(\ldots X_s\hat{\omega} \ldots) = (\mathscr{L}_{X_1} \ldots \mathscr{L}_{X_s}\omega)^\wedge \in \mathscr{G}^{l+s} \quad (\hat{\omega} \in \mathscr{G}^l).$$

But the identity

$$(\mathscr{L}_X\mathscr{L}_Y - \mathscr{L}_Y\mathscr{L}_X)\,\omega = \mathscr{L}_{[X,Y]}\omega \in \Omega^{l+1} \quad (X, Y \in \mathscr{H},\ \omega \in \Omega^l),$$

clearly implies $(X \otimes Y - Y \otimes X)\,\hat{\omega} = 0 \in \Omega^{l+1}$. It follows that $\mathscr{G}$ is in reality a $\odot\mathscr{H}$-module, where by

$$\odot\mathscr{H} = \Psi_0 \oplus \mathscr{H} \oplus (\mathscr{H} \odot \mathscr{H}) \oplus (\mathscr{H} \odot \mathscr{H} \odot \mathscr{H}) \oplus \ldots$$

we denote the graded free commutative Ψ_0-algebra over the Ψ_0-module $\mathscr{H}$. The above results may be reformulated as follows: Let

(5) $\qquad \mathscr{Z} = \Sigma g_{i_1 \ldots i_s}\mathscr{L}_{X_{i_1}} \ldots \mathscr{L}_{X_{i_s}} + (\ldots)\ (g_{i_1 \ldots i_s} \in \Psi_0;\ X_{i_1}, \ldots, X_{i_s} \in \mathscr{H})$

be an operator of an exact order s, where dots denote some lower order summands of the type $g\mathscr{L}_X \ldots \mathscr{L}_Y$ ($X, \ldots, Y \in \mathscr{H}$, less than s Lie derivatives). Consider the tensor

(6) $\qquad Z = \Sigma g_{i_1 \ldots i_s}X_{i_1} \odot \ldots \odot X_{i_s} \in \odot^s\mathscr{H}$.

Then $Z\hat{\omega} = (\mathscr{Z}\omega)^\wedge \in \mathscr{G}^{l+s}$ for every $\omega \in \Omega^l$ with the class $\hat{\omega} \in \mathscr{G}^l$. (Note, besides, that the set of operators (5) is identical with the familiar enveloping algebra $U\mathscr{H}$ of the Lie algebra $\mathscr{H}$ so that the above interrelation between the family of operators (5) and the corresponding tensor (6) is well-known.)

If we fix a point $z \in J$, then the Ψ_0-algebra $\odot\mathscr{H}$ restricted at the point z turns into the common polynomial algebra

$$(7) \qquad \odot\mathscr{H}_z = \Psi_{0z} \oplus \mathscr{H}_z \oplus (\mathscr{H}_z \odot \mathscr{H}_z) \oplus \ldots = \mathbb{R} \oplus \mathscr{H}_z \oplus (\mathscr{H}_z \odot \mathscr{H}_z) \oplus \ldots$$

over the n-dimensional linear space $\mathscr{H}_z$. Quite analogously we obtain the graded homogeneous $\odot\mathscr{H}_z$-module $\mathscr{G}_z = \mathscr{G}_z^0 \oplus \mathscr{G}_z^1 \oplus \ldots$. (The homogeneity means that $Z_z\hat{\omega} \in \mathscr{G}_z^{l+s}$ whenever $Z_z \in \odot^s\mathscr{H}_z$ and $\hat{\omega} \in \mathscr{G}_z^l$.) The condition $(4)_2$ implies that $\mathscr{G}$ is a finitely generated $\odot\mathscr{H}$-module. It follows that $\mathscr{G}_z$ is a Noetherian $\odot\mathscr{H}_z$-module and the machinery of the commutative algebra can be put in motion. (The last reasonings are not quite correct, see the note at the page 383).

4. A way to homological algebra. Let $\Psi_{r,s}$ be the Ψ_0-submodule of the module Ψ_{r+s} consisting of all exterior forms $\psi \in \Psi_{r+s}$ of the type

$$(8) \qquad \psi = \Sigma\omega_{k_1} \wedge \ldots \wedge \omega_{k_r} \wedge \psi_k \quad (\omega_{k_1}, \ldots, \omega_{k_r} \in \Omega,\ \psi_k \in \Psi_s)\,;$$

we also put $\Psi_{r,s} = \{0\}$ whenever $r < 0$ or $s < 0$. Clearly, $\Psi_{0,0} = \Psi_0$, $\Psi_{1,0} = \Omega$, $\Psi_{0,s} = \Psi_s$. The $\mathscr{C}\ell o\omega$ assumption clearly implies $\mathrm{d}\Psi_{1,0} = \mathrm{d}\Omega \subset \Psi_{1,1}$ and one can see that $\mathrm{d}\Psi_{r,s} \subset \Psi_{r,s+1}$.

The forms $\psi \in \Psi_{r+1,s-1}$ may be characterized by the property $X_1 \lrcorner \ldots X_s \lrcorner \psi \equiv \equiv 0$ for all $X_1, \ldots, X_s \in \mathscr{H}$. It follows that every factor $\mathscr{G}_{r,s} = \Psi_{r,s}/\Psi_{r+1,s-1}$ may be identified with the Ψ_0-module of all skewsymmetric s-forms defined on the Ψ_0-module $\mathscr{H}$ with values in the space $\Psi_{r,0}$. That means $\mathscr{G}_{r,s} = \Psi_{r,0} \otimes (\wedge^s\mathscr{H}^*)$ and the class corresponding to the form (8) is expressed by the sum

$$(9) \qquad \tilde{\psi} = \Sigma\omega_{k_1} \wedge \ldots \wedge \omega_{k_r} \otimes \tilde{\psi}_k\,,$$

where the class $\tilde{\psi}_k \in \mathscr{G}_{0,s} = \wedge^s\mathscr{H}^*$ is nothing else but a simple restriction on $\mathscr{H}$:

$$\tilde{\psi}_k(X_1, \ldots, X_s) \equiv \psi_k(X_1, \ldots, X_s) \quad (X_1, \ldots, X_s \in \mathscr{H})\,.$$

As a result, we get a naturally induced differential

$$\partial\colon \mathscr{G}_{r,s} \to \mathscr{G}_{r,s+1}\,, \quad \partial\tilde{\psi} = (\mathrm{d}\psi)^\sim \in \mathscr{G}_{r,s+1} \quad (\psi \in \mathscr{G}_{r,s})\,.$$

But the filtration (3) may be also taken into account. To this aim, let $\mathscr{G}_{r,s}^l$ $(r > 0)$ be the of all classes (8) specified by the additional requirement

$$(10) \qquad \omega_{k_1} \in \Omega^{l_1}, \ldots, \omega_{k_r} \in \Omega^{l_r} \quad \text{and} \quad l_1 + \ldots + l_r \leqq l\,.$$

Then the condition (4) ensures $\partial\mathscr{G}_{r,s}^l \subset G_{r,s+1}^{l+1}$. Finally, we look at the class in the factor-space $\Gamma_{r,s}^l = \mathscr{G}_{r,s}^l/\mathscr{G}_{r,s}^{l-1}$. Since $\Gamma_{1,0}^l = \mathscr{G}_{1,0}^l = \mathscr{G}^l$, the class of the form (9) is expressed by the sum

$$(11) \qquad \hat{\psi} = \Sigma\hat{\omega}_{k_1} \wedge \ldots \wedge \omega_{k_r} \otimes \tilde{\psi}_k \in \Gamma_{r,s}^l \quad (l_1 + \ldots + l_r = l)\,,$$

where the summands with $l_1 + \ldots + l_r < l$ were deleted by the factorization. We obtain a naturally induced differential $\partial\colon \Gamma^l_{r,s} \to \Gamma^{l+1}_{r,s+1}$. (See the next Section 5 for explicit formulae.)

Altogether, we have the following complexes and homologies:

$$(12)_r \qquad \ldots \to \Psi_{r,s-1} \xrightarrow{\mathrm{d}} \Psi_{r,s} \xrightarrow{\mathrm{d}} \Psi_{r,s+1} \to \ldots, \quad H(\Psi)_{r,s},$$

$$(13)_r \qquad \ldots \to \mathscr{G}_{r,s-1} \xrightarrow{\partial} \mathscr{G}_{r,s} \xrightarrow{\partial} \mathscr{G}_{r,s+1} \to \ldots, \quad H(\mathscr{G})_{r,s},$$

$$(14)_r \qquad \ldots \to \mathscr{G}^{l-1}_{r,s-1} \xrightarrow{\partial} \mathscr{G}^l_{r,s} \xrightarrow{\partial} \mathscr{G}^{l+1}_{r,s+1} \to \ldots, \quad H(\mathscr{G})^l_{r,s},$$

$$(15)_r \qquad \ldots \to \Gamma^{l-1}_{r,s-1} \xrightarrow{\partial} \Gamma^l_{r,s} \xrightarrow{\partial} \Gamma^{l+1}_{r,s+1} \to \ldots, \quad H(\Gamma)^l_{r,s}.$$

The first complexes $(12)_r$ play an important role in geometry and analysis and the other ones serve as a more simple approximations for their predecessors. The last complex $(15)_r$ is already a purely algebraic object. Note that these complexes were studied only in the particular case when J is a subspace of the space of all ∞-jets of sections of a fibered manifold, see [6]. In this case we have a fixed hierarchy of independent and dependent variables on J and the homologies $H(\Gamma)^l_{g,s}$ are found to be Poincaré-dual to the familiar Spencer homologies of systems of differential equations, see [7]. Note that the Spencer homologies were defined only in this rather particular case in current literature and the original definition essentially employs the above mentioned hierarchy of variables. But the homologies $(15)_r$ *do not depend* on the choice of *independent variables*, hence the Spencer homologies *do not depend on it*, either. This result is rather surprising if compared with the original Spencer's definition, see [8].

5. Explicit formulae. A point $z \in J$ is expressed by an infinite array of the type

$$z^0 = j^1_0(z^1), \ldots, z^l = j^{l+1}_l(z^{l+1}), \ldots \quad (z^l \in J^l).$$

There exist many local coordinate systems on the space J adapted to the representation (1), each being defined as a sequence $f_1, f_2, \ldots \in C^\infty(J) = \Psi_0$ satisfying the requirement that every part $f_1, \ldots, f_{n^l}$ $(l = 0, 1, \ldots)$ of the sequence lies already in $C^\infty(J^l) = \Psi^{[l]}_0$ and determines a local coordinate system on J^l near the point z^l.

Every form $\psi \in \Psi$ can be locally expressed by a finite number of coordinates while a vector field $X \in TJ$ is identified with an infinite formal series of the type $\Sigma g_k\, \partial/\partial f_k$, where $g_k = Xf_k$ are quite arbitrary functions.

Without any loss of generality, we may assume $\xi_1 = \mathrm{d}f_1, \ldots, \xi_n = \mathrm{d}f_n$ in the $\mathscr{D}im$ condition. Then, using the more advantageous notation $x^1 = f_1, \ldots, x^n = f_n$, $y^1 = f_{n+1}$, $y^2 = f_{n+2}, \ldots$ for the coordinates, and abbreviations

$$\omega_K = \omega_{k_1} \wedge \ldots \wedge \omega_{k_r}, \quad \mathrm{d}x^I = \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_s}$$

for the exterior products, the formulae (8), (9), (11) acquire the following more explicit expression:

$$\psi = \Sigma f_K^I \omega_K \wedge \mathrm{d}x^I + \varphi \quad (f_K^I \in \Psi_0,\ \omega_K \in \Psi_{r,0},\ \mathrm{d}x^I \in \Psi_{0,s},\ \varphi \in \Psi_{r+1,s-1}),$$
$$\tilde{\psi} = \Sigma f_K^I \omega_K \otimes (\mathrm{d}x^I)^\sim,$$
$$\hat{\psi} = \Sigma f_K^I \hat{\omega}_K \otimes (\mathrm{d}x^I)^\sim \quad (l_1 + \ldots + l_r \equiv l,\ \omega_K \in \Psi_{r,0}^l).$$

Let

$$(16) \qquad \partial_i = \partial/\partial x^i + \Sigma y_i^k\, \partial/\partial y^k \quad (y_i^k \equiv \partial_i y^k)$$

be the local basis of $\mathscr{H}$ dual to the basis $(\mathrm{d}x^1)^\sim, \ldots, (\mathrm{d}x^n)^\sim$ and explicitly defined by the identity

$$(17) \qquad \mathrm{d}f = \Sigma \partial_i f \,.\, \mathrm{d}x^i + \omega \quad (f \in \Psi_0,\ \omega \in \Omega)$$

(cf. the $\mathscr{D}im$ condition). Abbreviating $iI = ii_1 \ldots i_s$ and using decompositions like (17), the differential ∂ can be concisely recorded by the formulae

$$\partial\tilde{\psi} = (-1)^r \Sigma\, \partial_i f_K^I \,.\, \omega_K \otimes (\mathrm{d}x^{iI})^\sim + \Sigma f_K^I\, \partial\omega_K \otimes (\mathrm{d}x^I)^\sim,$$
$$\partial\hat{\psi} = \Sigma f_K^I\, \partial\hat{\omega}_K \otimes (\mathrm{d}x^I)^\sim,$$

where the forms $\partial\omega_K \in \Psi_{r,1}$, $\partial\hat{\omega}_K \in \Psi_{r,1}^{l+1}$ may be determined by the usual rule of derivation of an exterior product provided the case of the 1-forms is already known.

As a result, the calculations are reduced to the case $\omega \in \Omega^l \subset \Psi_{1,0}$, $\hat{\omega} \in \mathscr{G}_{1,0}^l = \mathscr{G}^l$ and we still have to seek the differentials $\partial\omega \in \Psi_{1,1}$, $\partial\hat{\omega} \in \mathscr{G}_{1,1}^{l+1}$. But according to the $\mathscr{D}im$ assumption, we may write $\mathrm{d}\omega = \Sigma\omega_i \wedge \mathrm{d}x^i + \varphi$ with certain $\omega_i = -\partial_i \lrcorner\, \mathrm{d}\omega \in \Omega^{l+1}$ and $\varphi \in \Psi_{2,0}$ (cf. condition $(4)_1$) and the definition of the above mentioned differentials easily yields the final result $\partial\omega = \Sigma\omega_i \otimes (\mathrm{d}x^i)^\sim$, $\partial\hat{\omega} = \Sigma\hat{\omega}_i \otimes (\mathrm{d}x^i)^\sim$ with $\hat{\omega}_i \in \mathscr{G}^{l+1}$. This completely concludes the calculations.

NOTES ON COMMUTATIVE ALGEBRA

6. Terminology and several basic results. Although the commutative algebra was invented for quite different purposes, the common setting of the theory requires only some relatively mild readjustments. Having this in mind, we shall deal with *homogeneous graded Noetherian A-modules* $M = M^0 \oplus M^1 \oplus \ldots$ (occasionally we understand $M^l \equiv \{0\}$ if $l < 0$, for technical reasons) over the graded ground ring $A = A^0 \oplus A^1 \oplus \ldots$ which, however, stands for a mere abbreviation of the familiar polynomial algebra $\odot\mathscr{H}_z^*$ (cf. Section 3) so that $A^0 = \mathbb{R}$, $A^1 = \mathscr{H}_z = H$ (a further abbreviation), $A^2 = \mathscr{H}_z \odot \mathscr{H}_z = H \odot H$, and so on (see (7)). Moreover, following the above idea, *we consider only homogeneous submodules* $N = N^0 \oplus N^1 \oplus \ldots$ $(N^l \equiv N \cap N^l)$ of M and the related factor-modules $M/N = (M/N)^0 \oplus (M/N)^1 \oplus \ldots$ $((M/N)^l \equiv M^l/N^l)$ without further explicit warning. In particular, we wish to emphasize that *only the homogeneous ideals* $\mathfrak{a} = \mathfrak{a}^0 \oplus \mathfrak{a}^1 \oplus \ldots$ $(\mathfrak{a}^l \equiv \mathfrak{a} \cap A^l)$ are admissible, the *maximal ideal* $\mathfrak{m} = \{0\} \oplus A^1 \oplus A^2 \oplus \ldots = \{0\} \oplus H \oplus (H \odot H) \oplus \ldots$ being one of them.

Following [9], [10], we recall some classical concepts slightly adapted for our needs. Ann M is the ideal of all $Z \in A$ satisfying $ZM = \{0\}$, Nil M is the ideal of all $Z \in A$ satisfying $Z^c M = \{0\}$ for all exponents c large enough, Supp M is the set of all prime ideals $\mathfrak{p}$ satisfying $\mathfrak{p} \supset \text{Ann } M$, Ass M is the set of all prime ideals $\mathfrak{p}$ for which there exists a submodule of M isomorphic to the A-module $A/\mathfrak{p}$. Note that Ass M is a subset of Supp M but the minimal ideals of the set Supp M (ordered by the relation of inclusion) lie in Ass M. The set of all minimal ideals of either Supp M or Ass M will be denoted by Min M. Given $\mathfrak{p} \in \text{Ass } M$, we denote by $M(\mathfrak{p})$ a maximal submodule of M possessing the property $\mathfrak{p} \notin \text{Ass } M(\mathfrak{p})$. Note that it is uniquely determined by $\mathfrak{p}$, provided $\mathfrak{p} \in \text{Min } M$. Given a subset $Q \subset \text{Ass } M$, we denote by $M(Q)$ a maximal submodule of M for which Ass $M(Q)$ is disjoint with Q. Note that one can take $M(Q) = \cap M(\mathfrak{q})$ (intersection over all $\mathfrak{q} \in Q$). Finally, if $\ell(L)$ denotes the dimension (in the elementary sense of linear algebra) of an arbitrary $\mathbb{R}$-linear space L, it is well-known that the number $\ell(M^0 \oplus \ldots \oplus M^l)$ is a polynomial function of the variable l (the *Hilbert polynomial*), for all l large enough. Omitting the trivial case $M = \{0\}$, we shall write

$$\ell(M^0 \oplus \ldots \oplus M^l) \sim \chi(M, l) = \mu\binom{l}{\nu} + (\ldots) = \mu l^\nu/\nu! + (\ldots),$$

where $\mu = \mu(M) > 0$ (the *multiplicity* of M) and $\nu = \nu(M)$ (the *dimension* of M) are certain integers, $\binom{l}{\nu} = l!/\nu!\,(l-\nu)!$ is the binomial coefficient and the dots denote some inessential summands of lower degree than the leading term. If $\nu(M) > 0$, then obviously

$$(18) \qquad \ell(M^l) \sim \chi(M, l) - \chi(M, l-1) = \mu l^{\nu-1}/(\nu-1)! + (\ldots).$$

We conclude with the *Krull-Chevalley-Samuel theorem*

$$(19) \qquad \nu(M) = \sup \nu(M/M(\mathfrak{p})) = \sup \nu(A/\mathfrak{p}) = \sup \sigma(\mathfrak{p}).$$

The supremum is taken over all $\mathfrak{p} \in \text{Ass } M$ (or over all $\mathfrak{p} \in \text{Min } M$) and $\sigma(\mathfrak{p})$ denotes the length of any longest *strongly increasing* chain of the type $\mathfrak{p} \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_\sigma$ ($\sigma = \sigma(\mathfrak{p})$) consisting of prime ideals.

In order to make the following exposition easy, we state several simple results numbered by Roman numerals for quick references. I: If $M \neq \{0\}$, then Ass M is a finite and nonempty set. II: Sup M = Supp $N \cup$ Supp M/N. III: Supp $M \otimes \overline{M}$ = = Supp $M \cap$ Supp $\overline{M}$, in particular Supp $M/\mathfrak{a}M$ = Supp $M \otimes A/\mathfrak{a}$ = Supp $M \otimes$ $\otimes$ Supp $A/\mathfrak{a}$. IV: Ass $M/M(Q) = Q$, Ass $M(Q) = M - Q$. V: Nil $M = \cap \mathfrak{p}$ ($\mathfrak{p} \in$ $\in$ Ass M, equivalently, $\mathfrak{p} \in \text{Min } M$). VI: The multiplication $Z: M \to M$ is injective if and only if $Z \notin \cup \mathfrak{p}$ ($\mathfrak{p} \in \text{Ass } M$). VII: If $\mathfrak{p}, \mathfrak{q}, \overline{\mathfrak{q}}$ are prime ideals and $\mathfrak{p} \supset \mathfrak{q}$, $\mathfrak{p} \supset \overline{\mathfrak{q}}$, $\mathfrak{q} \neq \overline{\mathfrak{q}}$, then $\sigma(\mathfrak{p}) < \sigma(\mathfrak{q})$, $\sigma(\mathfrak{p}) < \sigma(\overline{\mathfrak{q}})$. VIII: If $N, \overline{N}$ are submodules of M and $\mathfrak{p} \in \text{Ass } M/(N + \overline{N})$, then $\mathfrak{p} \supset \text{Nil } M/(N + \overline{N}) \supset \text{Nil } M/N \cup \text{Nil } M/\overline{N}$, hence $\mathfrak{p} \supset \mathfrak{q}$, $\mathfrak{p} \supset \overline{\mathfrak{q}}$ with an appropriate $\mathfrak{q} \in \text{Ass } M/N$, $\overline{\mathfrak{q}} \in \text{Ass } M/N$. IX: The familiar exact

Mayer-Vietoris sequence $0 \to M/N \cap \bar{N} \to M/N \oplus M/\bar{N} \to M/(N \oplus \bar{N}) \to 0$ yields

(20) $$\chi(M/N \cap N, l) = \chi(M/N, l) + \chi(M/\bar{N}, l) - \chi(M/(N + \bar{N}), l).$$

X: The short exact sequence $0 \to N \to M \to M/N \to 0$ yields $\chi(M, l) = \chi(M/N, l) + \chi(N, l)$.

After more or less standard preliminaries we turn to other topics from commutative algebra specially invented for future use.

7. The concept of regularity. Dealing with a module $M = M^0 \oplus M^1 \oplus \dots$, we may introduce the module $M^+ = \{0\} \oplus M^1 \oplus M^2 \oplus \dots$ with the obvious A-module structure determined by a simple restriction so that M^+ is a submodule of M, in particular $A^+ = \mathfrak{m}$. A little more intricate A-module $M^{+c} = (M^0 \oplus \dots \oplus M^c) \oplus M^{c+1} \oplus \dots$ is defined for $c = 0, 1, \dots$ by the following multiplication rule:

$Z\xi = 0$ whenever $Z \in \mathfrak{m}$ and $\xi : M^0 \oplus \dots \oplus M^{c-1} \subset (M^{+c})^0$,

the product $Z\xi$ is retained if $Z \in R$ or $\xi \in M^c \oplus M^{c+1} \oplus \dots$.

The module M^{+c} is not (in general) a submodule of M. The products $Z\xi$ may be different if $Z \in \mathfrak{m}$ and $\xi \in M^0 \oplus \dots \oplus M^{c-1}$. Note that $(M^{+c})^{+d} = M^{+(c+d)}$ and the module $M^{+c+} = M^{c+1} \oplus M^{c+2} \oplus \dots$ already is a submodule of M.

We shall often deal with ordered sequences of the type $X_* = X_1, \dots, X_v$ ($v = 0, 1, \dots$) with the terms $X_1, \dots, X_v$ lying in H. In this case, $(X)_k$ denotes the ideal generated by the family $X_1, \dots, X_k$ ($k = 0, \dots, v$; we put $(X)_0 = \{0\}$). Slightly modifying the common concepts from the commutative algebra, we call the mentioned sequence *M^c-regular* if the (naturally induced) multiplications

(21) $$X_k: (M/(X)_{k-1}M^c) \to (M/(X)_{k-1}M^{c+1}) \quad (k = 1, \dots, v)$$

are injective mappings. The above sequence X_* is called *M-quasiregular* (cf. [11]), if the multiplications (21) are injective for every $c = 1, 2, \dots$. Finally, X is called an *M-general sequence* (a new concept) if the multiplications (21) are injective for all c large enough. (Unfortunately, the classical concept of an *M-regular* sequence X_* for which (21) are always injections is almost useless here. We wish to mention that, unlike the classical case, the order in the sequence X_* is important and cannot be changed.)

8. Existence questions. (i) $X_* = X_1, \dots, X_v$ is an M-quasiregular sequence if and only if $X_k \notin \cup \mathfrak{p}$ ($\mathfrak{p} \in \operatorname{Ass} M^+/(X)_{k-1}M^+$) for all $k = 1, \dots, v$. The same sequence is M-general if and only if $X_k \notin \cup \mathfrak{p}$ ($\mathfrak{p} \in \operatorname{Ass} M^{+c+}/(X)_{k-1}M^{+c+}$) for all c large enough.

(ii) Let $M_1, M_2, \dots$ be A-modules and $X_* = X_1, \dots, X_v$ an M_i-quasiregular sequence for all $i = 1, 2, \dots$. This sequence may be enlarged to a simultaneously M_i-quasiregular sequence $X_1, \dots, X_v, X_{v+1}$ if and only if $\mathfrak{m} \notin \operatorname{Ass} M_i^+/(X)_v M_i^+$ for all $i = 1, 2, \dots$. (Trivial, since $H \cap (\cup \mathfrak{p})$ (union over all $\mathfrak{p} \in \operatorname{Ass} M_i^+/(X)_v M_i^+$, $i = 1, 2, \dots$ and $\mathfrak{p} \neq \mathfrak{m}$) is a proper subset of H, cf. I.)

(iii) With the possible exception of the maximal ideal $\mathfrak{m}$, the sets Ass M, Ass M^+, Ass M^{+c} coincide. (Directly from the definition of Ass and the fact $\sigma(\mathfrak{p}) = 0$ if and only if $\mathfrak{p} = \mathfrak{m}$, see (19).)

(iv) $\mathfrak{m} \notin$ Ass M^{+c+} for all c large enough. Proof: If $\mathfrak{m} \notin$ Ass M, then $\mathfrak{m} \notin$ Ass M^{+c+} according to the definition of Ass. Let $\mathfrak{m} \in$ Ass M. Then $\nu(M/M(\mathfrak{m})) = \sigma(\mathfrak{m}) = 0$ (cf. (19)), hence $\ell(M^l/M(\mathfrak{m})^l) = 0$ for all l large, say, for $l \geqq c$ (cf. (18)). It follows that $M(\mathfrak{m})^l \supset M^l$ $(l \geqq c)$, hence $\mathfrak{m} \notin$ Ass $(M^c \oplus M^{c+1} \oplus \ldots)$ according to the definition of $M(\mathfrak{m})$.

(v) The multiplication $Z: M^{+c+} \to M^{+c+}$ is injective for all c large enough if and only if $Z \notin \cup\mathfrak{p}$ ($\mathfrak{p} \in$ Ass M, $\mathfrak{p} \neq \mathfrak{m}$). A consequence of (i), (iii), (iv).

(vi) $X_* = X_1, \ldots, X_v$ is an M-general sequence if and only if $X_k \notin \cup\mathfrak{p}$ ($\mathfrak{p} \in$ Ass $M/(X)_{k-1}M$, $\mathfrak{p} \neq \mathfrak{m}$) for all $k = 1, \ldots, v$. (See (v).)

(vii) Let $M_1, M_2, \ldots$ be given A-modules. A simultaneously M_i-general sequence $X_* = X_1, \ldots, X_v$ (in particular, the empty sequence for the case $v = 0$) can be always enlarged to a sequence $X_1, \ldots, X_v, X_{v+1}$ which is simultaneously M_i-general as well. Indeed, the aditional term should be chosen from the complement of the set $\cup(\mathfrak{p} \cap H)$ (union over $\mathfrak{p} \in$ Ass $M_i/(X)_v M_i$, $i = 1, 2, \ldots$; $\mathfrak{p} \neq \mathfrak{m}$) in the space H (cf. (vi)). This is always possible since the complement is nonempty (cf. I).

9. A property of M/N-general sequence. We begin with elementary considerations. Let $\mathfrak{a}$ be an ideal and N a submodule of M, as usual. Look at the exact commutative diagram (22) but without the dotted arrows and the term W (for a moment).

(22)
$$\begin{array}{ccccccccccc}
 & & & & & & & & 0 & & \\
 & & & & & & & & \downarrow & & \\
 & & & & 0 & & 0 & & U & & \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \to & \mathfrak{a}N & \to & \mathfrak{a}M & \to & \mathfrak{a}M/\mathfrak{a}N & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \to & N & \to & M & \to & M/N & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \vdots & & \\
0 & \to & V & \to & N/\mathfrak{a}N & \to & M/\mathfrak{a}M & \dashrightarrow & W & \dashrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \vdots & & \\
 & & & & 0 & & 0 & & 0 & &
\end{array}$$

A simple diagram chasing shows that the kernels U, V are isomorphic. Moreover, the four properties

(23) $$U = \{0\}, \quad V = \{0\}, \quad N \cap \mathfrak{a}M = \mathfrak{a}N, \quad \mathfrak{a}M/\mathfrak{a}N = \mathfrak{a}(M/N)$$

(the last property means a natural isomorphism) can be found equivalent. (Indeed, the modules U and V are isomorphic to $N \cap \mathfrak{a}M/\mathfrak{a}N$ and there are isomorphisms $\mathfrak{a}(M/N) = (N + \mathfrak{a}M)/N = \mathfrak{a}M/N \cap \mathfrak{a}M$.) If (23) is satisfied, then

(24) $$\begin{aligned} (M/\mathfrak{a}M)/N/\mathfrak{a}N) &= (M/\mathfrak{a}M)/((N + \mathfrak{a}M)/\mathfrak{a}M) = M/(N + \mathfrak{a}M) = W, \\ (M/N)/(\mathfrak{a}M/\mathfrak{a}N) &= (M/N)/((N + \mathfrak{a}M)/N) = M/(N + \mathfrak{a}M) = W, \end{aligned}$$

and the diagram (22) with $U = V = \{0\}$ may be completed by the above mentioned module W.

In particular, let $N = \mathfrak{b}M$, where $\mathfrak{b}$ is an ideal. Then the third condition (23) may be read as $\mathfrak{a}M \cap \mathfrak{b}M = \mathfrak{a}\mathfrak{b}M$ so that the role of the ideals $\mathfrak{a}$, $\mathfrak{b}$ is symmetrical and the last condition (23) yields the identities

$$(25) \qquad \mathfrak{a}M/\mathfrak{a}\mathfrak{b}M = \mathfrak{a}(M/\mathfrak{b}M), \quad \mathfrak{b}M/\mathfrak{a}\mathfrak{b}M = \mathfrak{b}(M/\mathfrak{a}M).$$

One can observe that the above reasoning is valid also for every fixed level l of the gradations. That means, we may substitute the components U^l, $(\mathfrak{a}N)^l$, N^l, $(N/\mathfrak{a}N)^l, \ldots$ for the corresponding modules $U, N, \mathfrak{a}N, N/\mathfrak{a}N, \ldots$, for every fixed degree l. If the properties $U^l = \{0\}, \ldots, (\mathfrak{a}M/\mathfrak{a}N)^l = (\mathfrak{a}(M/N))^l$ corresponding to (23) are satisfied, we obtain a linear space W^l (instead of the module W) from the relations (24) and the identities (25) for the degree l. Now, according to Lemma 10 below, this modification may be accomplished if we choose $\mathfrak{a} = (X)_k$ $(k = 1, \ldots, v)$, where $X_* = X_1, \ldots, X_v$ is an $(M/N)^{l-1}$-regular sequence. Then, abbreviating $M_* = M/(X)_v M$ (for any module M) the isomorphisms (23), (24) yield the rule $M_*^l/N_*^l = (M/N)_*^l$. At the same time, the second isomorphism (25) can be read as $(\mathfrak{b}M)_*^l = (\mathfrak{b} \, . \, M_*)^l$ so that we have the identities $(M_*/\mathfrak{b} \, . \, M_*)^l = (M_*/(\mathfrak{b}M)_*)^l = (M/\mathfrak{b}M)_*^l$.

We shall, however, need only a weaker (but more suggestive) variant of these results. To this aim, let us write $M \cong \overline{M}$ $(M \gtrsim \overline{M})$ for every pair of modules $M = M^0 \oplus M^1 \oplus \ldots$ and $\overline{M} = \overline{M}^0 \oplus \overline{M}^1 \oplus \ldots$ such that $M^l = \overline{M}^l$ $(M^l \subset \overline{M}^l)$ for all l sufficiently large. One can verify that $M \cong (\gtrsim) M$ implies $\mathfrak{a}M \cong (\gtrsim) \mathfrak{a}\overline{M}$ for any ideal $\mathfrak{a}$, and $\chi(M, l) = (\leqq) \chi(\overline{M}, l) +$ const. The above considerations can be easily reformulated in terms of the relation $\cong$. In particular, if $X_* = X_1, \ldots, X_v$ is an M/N-general sequence then, according to Lemma 10, we have $N \cap \mathfrak{a}M \cong \mathfrak{a}N$ $(\mathfrak{a} = (X)_k)$, the modified conditions (23) are satisfied and we obtain the following result:

$$(26) \qquad N_* \gtrsim M_*, \quad (M/N)_* \cong M_*/N_*, \quad (\mathfrak{b}M)_* \cong \mathfrak{b} \, . \, M_*;$$

the last identity corresponds to the special case $N = \mathfrak{b}M$.

10. Lemma. *Let $X_* = X_1, \ldots, X_v$ be an $(M/N)^{c-1}$-regular sequence. Then $(M \cap (X)_k M)^c = ((X)_k N)^c$ for $k = 1, \ldots, v$.*

Proof. The inclusion $\supset$ is trivial. Passing to the opposite case $\subset$, assume that $\xi \in (N \cap (X)_k M)^c$, that means, $\xi = X_1\xi_1 + \ldots + X_k\xi_k \in N^c$ with $\xi_1, \ldots, \xi_k \in M^{c-1}$. It follows that $X_k\xi_k = \xi - X_1\xi_1 - \ldots - X_{k-1}\xi_{k-1}$ and the regularity assumption implies $\xi_k \in (N + (X)_{k-1}M)^{c-1}$, that is, $\xi_k = \eta + X_1\eta_1 + \ldots + X_{k-1}\eta_{k-1}$ with $\eta \in N^{c-1}$ and $\eta_1, \ldots, \eta_{k-1} \in M^{c-2}$. (Recall the formal convention $M^l \equiv \{0\}$ whenever $l < 0$.) Hence

$$\xi = X_1(\xi_1 + X_k\eta_1) + \ldots + X_{k-1}(\xi_{k-1} + X_k\eta_{k-1}) + X_k\eta = \zeta + X_k\eta,$$

where $\zeta = \xi - X_k\eta \in N^c \cap ((X)_{k-1}M)^c = (N \cap (X)_{k-1}M)^c$. Since the case $k = 1$

(or even $k = 0$) is easy, we may proceed by induction on k. So we conclude $\zeta \in \in ((X)_{k-1}N)^c$, hence

$$\xi = \zeta + X_k\eta \in ((X)_{k-1}N)^c \cup (X_kN)^c) \subset ((X)_kN)^c$$

and the proof is complete.

11. Several results on multiplicities. (i) Let $Z \in A^s$ ($s \geqq 1$), $Z \notin \cup \mathfrak{p}$ ($\mathfrak{p} \in \operatorname{Ass} M$, $\mathfrak{p} \neq \mathfrak{m}$). Then the multiplication $Z\colon M^l \to M^{l+s}$ is an injective mapping for l large enough (cf. (v) Section 8) and the exact sequence $0 \to M^l \xrightarrow{Z} M^{l+s} \to (M/ZM)^{l+s} \to 0$ yields the relation $\chi(M, l + s) = \chi(M/ZM, l + s) + \chi(M, l) + \text{const.}$, see X and (18). Assuming $\nu(M) \geqq 1$, we obtain $\nu(M/ZM) = \nu(M) - 1$, $\mu(M/ZM) = s\,\mu(M)$, by simply equating the leading coefficients.

(ii) Let $X_* = X_1, \ldots, X_v$ be an M-general sequence. Successively applying (i), we obtain $\nu(M/(X)_k M) = \nu(M) - k$, $\mu(M/(X)_k M) = \mu(M)$ for the case $0 \leqq k \leqq \leqq \nu(M)$ and obviously $\nu(M/(X)_k M) = 0$ if $k \geqq \nu(M)$.

(iii) We are going to prove that $\nu(M/ZM) \geqq \nu(M) - 1$ for every $Z \in A^s$ ($s \geqq 1$). The case when Z lies in some ideal $\mathfrak{q} \in \operatorname{Ass} M$ with $\sigma(\mathfrak{q}) = \nu(M)$ is easy. Indeed, in this case $\mathfrak{q} \in \operatorname{Supp} M/ZM = \operatorname{Supp} M \cap \operatorname{Supp} A/ZA$ and it follows that $\nu(M/ZM) = = \sup \sigma(\mathfrak{q}) \geqq \sigma(\mathfrak{q}) = \nu(M)$, according to (19). So, denoting by Q the set of all ideals $\mathfrak{q} \in \operatorname{Ass} M$ with $\sigma(\mathfrak{q}) = \nu(M)$, assume that $Z \notin \bigcup \mathfrak{q}$ ($\mathfrak{q} \in Q$). Clearly $\nu(M/ZM) \geqq \geqq \nu(M/(ZM + M(Q))$ and we have the isomorphisms

$$M/(ZM + M(Q)) = (M/M(Q))/((ZM + M(Q)) = (M/M(Q))/Z(M/M(Q)).$$

Since $Z \notin \bigcup \mathfrak{q}$ ($\mathfrak{q} \in Q = \operatorname{Ass} M/M(Q)$, cf IV, point (i) applied to the module $\overline{M} = = M/M(Q)$ yields $\nu(M/ZM) \geqq \nu(\overline{M}/Z\overline{M}) = \nu(\overline{M}) - 1$, where $\nu(\overline{M}) = \nu(M/M(Q)) = = \sup \sigma(\mathfrak{q}) = \nu(M)$. This concludes the proof.

(iv) Let Q, $\overline{Q}$ be disjoint and nonempty subsets of $\operatorname{Ass} M$. Inserting $N = M(Q)$, $\overline{N} = M(\overline{Q})$ into (20), we obtain the additivity formula

$$\chi(M/M(Q \cup \overline{Q}), l) = \chi(M/M(Q), l) + \chi(M/M(\overline{Q}), l) + (\ldots), \tag{27}$$

since the last term $\chi(M/M(Q) + M(\overline{Q}), l) = (\ldots)$ is inessential (cf. VIII, VII, (19)). The formula (27) can be generalized to the case of more disjoint subsets $Q, \overline{Q}, \overline{\overline{Q}}, \ldots$ of the set $\operatorname{Ass} M$, of course.

(v) Continuing the preceding point, let us choose $Q = \operatorname{Min} M$, $\overline{Q} = \operatorname{Ass} M - - \operatorname{Min} M$. In this case $M(Q \cup \overline{Q}) = M(\operatorname{Ass} M) = \{0\}$ (cf. I) and $\chi(M/M(Q), l) = = (\ldots)$ is an inessential summand (cf. VIII, VII, (19)) so that (27) reduces to the relation $\chi(M, l) = \chi(M/M(Q), l) + (\ldots)$. But $\chi(M/M(Q), l) = \Sigma\chi(M/M(\mathfrak{q}), l) + + (\ldots)$ (sum over $\mathfrak{q} \in \operatorname{Min} M$) as follows from the above additivity formula. We have the final result

$$\mu(M) = \Sigma\mu(M/M(\mathfrak{q})) \quad (\mathfrak{q} \in \operatorname{Min} M \text{ and } \sigma(\mathfrak{q}) = \nu(M)), \tag{28}$$

since the summands $\chi(M/M(\mathfrak{q}), l)$ with $\sigma(\mathfrak{q}) < \nu(M)$ are inessential. (We also refer to [12].)

(vi) Let $Q \subset \operatorname{Ass} M$ be a nonempty subset, $\bar{Q} = \operatorname{Ass} M - Q$, $\mathfrak{a} = \bigcap \mathfrak{q}$ $(\mathfrak{q} \in Q)$. According to IV, V, we have $\mathfrak{a}^c M \subset M(Q)$ for all exponents c large enough. In this case we may write $\chi(M/\mathfrak{a}^c M, l) = \chi(M/M(Q), l) + \chi(M(Q)/\mathfrak{a}^c M, l)$ (cf. X) and soon we shall prove that the last summand in this identity is inessential. So, comparing the higher order coefficients, we have

$$(29) \qquad \nu(M/\mathfrak{a}^c M) = \nu(M/M(Q)) = \sup \sigma(\mathfrak{q}) \quad (\text{supremum over } \mathfrak{q} \in Q),$$

$$\mu(M/\mathfrak{a}^c M) = \mu(M/M(Q), l) = \Sigma \mu(M/M(\mathfrak{q})) \quad (\mathfrak{q} \in Q,\ \sigma(\mathfrak{q}) = \nu(M/M(Q)),$$

(cf. IV, (19), (28)). Let us return back to the summand mentioned above. Let $\mathfrak{p} \in \operatorname{Min} M(Q)/\mathfrak{a}^c M$ be arbitrary. Then, on the one hand, there exists an ideal $\bar{\mathfrak{q}} \in \operatorname{Ass} M(Q) = \bar{Q}$ such that $\mathfrak{p} \supset \bar{\mathfrak{q}}$ (cf. IV, II) but, on the other hand,

$$\operatorname{Min} M(Q)/\mathfrak{a}^c M \subset \operatorname{Ass} M/\mathfrak{a}^c M \subset \operatorname{Supp} M/\mathfrak{a}^c M = \operatorname{Supp} M \cap \operatorname{Supp} A/\mathfrak{a}^c$$

(cf. III), hence $\mathfrak{p} \in \operatorname{Supp} A/\mathfrak{a}^c$, $\mathfrak{p} \supset \mathfrak{a}^c$, $\mathfrak{p} \supset \mathfrak{a}$ and $\mathfrak{p} \supset \mathfrak{q}$ for an appropriate $\mathfrak{q} \in Q$. It follows that $\nu(M(Q)/\mathfrak{a}^c M) = \sup \sigma(\mathfrak{p}) \leqq \sigma(\mathfrak{q}) \leqq \nu(M/M(Q))$, according to (19) and VII. This concludes the proof.

(vii) Continuing the preceding point, we mention the particular case of a fixed ideal $\mathfrak{a} = \mathfrak{q} \in \operatorname{Min} M$ for which (29) reduces to the equation

$$(30) \qquad \mu(M/\mathfrak{q}^c M) = \mu(M/M(\mathfrak{q})) = \sigma(\mathfrak{q})\,.$$

We assume $\mathfrak{q}^c M \subset M(\mathfrak{q})$, of course.

INVARIANCE OF ASSOCIATED IDEALS

12. Preliminary results. Let (3) and

$$(31) \qquad \bar{\Omega}^*: \ldots \subset \bar{\Omega}^{-1} = \{0\} \subset \bar{\Omega}^0 \subset \ldots \subset \bar{\Omega}^k \subset \bar{\Omega}^{k+1} \subset \ldots \subset \Omega = \bigcup \bar{\Omega}^k$$

be two good filtrations of a diffiety Ω. According to $\mathscr{F}in$, for every integer l there exist $K = K(l)$, $L = L(l)$ satisfying $\Omega^l \subset \bar{\Omega}^K \subset \Omega^L$ and it follows that

$$(32) \qquad \Omega^{l+c} \subset \bar{\Omega}^{K+c} \subset \bar{\Omega}^{L+c} \quad (c \geqq 0)$$

for all l large enough, see $(4)_2$. Recall the A-module $\mathscr{G}_z$ from Section 3 and let $\bar{\mathscr{G}}_z = \bar{\mathscr{G}}_z^0 \oplus \bar{\mathscr{G}}_z^1 \oplus \ldots$ $(\bar{\mathscr{G}}^k = \bar{\Omega}^k/\bar{\Omega}^{k-1})$ be the analogous objects for the second filtration (31). Since $\ell(\Omega_z^l) = \ell(\mathscr{G}_z^0 \oplus \ldots \oplus \mathscr{G}_z^l)$, $\ell(\bar{\Omega}^k) = \ell(\bar{\mathscr{G}}_h^0 \oplus \ldots \oplus \bar{\mathscr{G}}_z^k)$, we have the inequality $\chi(\mathscr{G}_z, l + c) \leqq \chi(\bar{\mathscr{G}}_z, K + c) \leqq \chi(\mathscr{G}_z, L + c)$, that is,

$$\mu(l + c)^\nu/(l + c)! + (\ldots) \leqq \bar{\mu}(K + c)^{\bar{\nu}}/(K + c)! + (\ldots) \leqq$$
$$\leqq \mu(L + c)^\nu/(L + c)! + (\ldots)$$

valid for all nonnegative c. It follows that *the constants* $\nu = \nu(\mathscr{G}_z) = \bar{\nu} = \nu(\bar{\mathscr{G}}_z)$ *and* $\mu = \mu(\mathscr{G}_z) = \bar{\mu} = \mu(\bar{\mathscr{G}}_z)$ *are independent of the choice of the filtrations* and are invariants of the diffiety Ω itself. (Note that analogous constants are known in differential algebra and play the role of a trancendence degree and multiplicity of a differential field. They belong to the most important achievements of the theory. For the analytical case of the classical theory of exterior differential systems, the meaning of these constants is also well-known: The general solution of the system depends on μ functions of ν variables, see [12].) Without referring to this preliminary result, we are going to prove the following assertion:

13. Theorem. *Let Ω^*, $\bar{\Omega}^*$ be good filtrations of a diffiety Ω and let a point $z \in J$ be fixed. Then* $\operatorname{Min} \mathscr{G}_z = \operatorname{Min} \bar{\mathscr{G}}_z$ *and* $\mu(\mathscr{G}_z/\mathscr{G}_z(\mathfrak{p})) = \mu(\bar{\mathscr{G}}_z/\bar{\mathscr{G}}_z(\mathfrak{p}))$ *for every ideal* $\mathfrak{p} \in \operatorname{Min} \mathscr{G}_z = \operatorname{Min} \bar{\mathscr{G}}_z$.

In accordance with the above invariance, we propose the expressive (but a little formal) notation $\nu(\Omega_z) = \nu(\mathscr{G}_z)$, $\mu(\Omega_z) = \mu(\mathscr{G}_z)$, $\operatorname{Min} \Omega_z = \operatorname{Min} \mathscr{G}_z$, $\mu(\Omega_z/\Omega_z(\mathfrak{p})) = \mu(\mathscr{G}_z/\mathscr{G}_z(\mathfrak{p}))$, replacing the letter $\mathscr{G}$ related to the accidental object, to the filtration, by the letter Ω corresponding to the absolute one, to the diffiety itself.

14. Double filtrations. Before passing to the part proper of the proof, we shall consider some interrelations between good filtrations (3) and (31). We begin with introducing the Ψ_0-modules

$$\mathscr{G}^{l,k} = \Omega^l \cap \bar{\Omega}^k / \Omega^{l-1} \cap \bar{\Omega}^k = (\Omega^l \cap \bar{\Omega}^k + \Omega^{l-1})/\Omega^{l-1}\,,$$
$$\bar{\mathscr{G}}^{l,k} = \Omega^l \cap \bar{\Omega}^k / \Omega^l \cap \bar{\Omega}^{k-1} = (\Omega^l \cap \bar{\Omega}^k + \bar{\Omega}^{k-1})/\bar{\Omega}^{k-1}\,,$$

appearing in the filtrations

$$\ldots \subset \mathscr{G}^{l,k} \subset \mathscr{G}^{l,k+1} \subset \ldots \subset \mathscr{G}^l = \bigcup \mathscr{G}^{l,k}\,,$$
$$\ldots \subset \bar{\mathscr{G}}^{l,k} \subset \bar{\mathscr{G}}^{l+1,k} \subset \ldots \subset \bar{\mathscr{G}}^k = \bigcup \bar{\mathscr{G}}^{l,k}\,.$$

One can observe that the corresponding bigradations are isomorphic,

$$\mathscr{G}^{l,k}/\mathscr{G}^{l,k-1} = G^{l,k} = \bar{\mathscr{G}}^{l,k}/\bar{\mathscr{G}}^{l-1,k}\,,$$

so that the corresponding graded Ψ_0-modules

$$\ldots \oplus G^{l,k} \oplus G^{l,k+1} \oplus \ldots, \quad \ldots \oplus G^{l,k} \oplus G^{l+1,k} + \ldots$$

consist of the same components.

A rather important fact is that these gradations are of a finite length. Indeed, recall the inclusions (32). They may be rewritten in the form $\Omega^{k-(K-l)} \subset \bar{\Omega}^k \subset \Omega^{k+(L-K)}$ with l fixed and $k = c + K \geqq K$. It follows that

$$\Omega^{k-c} \subset \bar{\Omega}^k \subset \Omega^{k+c} \quad \text{(all } k, c \text{ fixed and large)}\,. \tag{33}$$

In particular, $\Omega^{k-d} \subset \Omega^{k-c-1} \subset \bar{\Omega}^{k-1}$ $(d \geqq c + 1)$, hence $\Omega^{k-d} \cap \bar{\Omega}^k \subset \Omega^{k-1}$ and consequently $\mathscr{G}^{k-d,k} \equiv \{0\}$, provided $d \geqq c + 1$. Quite analogously, the inclusion $\bar{\Omega}^k \subset \Omega^{k+c} \subset \Omega^{k+d}$ $(d \geqq c)$ implies $\Omega^{k+d} \cap \bar{\Omega}^k = \bar{\Omega}^k$ and consequently $\bar{\mathscr{G}}^{k+d,k} =$

$= (\bar{\Omega}^k + \bar{\Omega}^{k-1})/\bar{\Omega}^{k-1} = \bar{\mathcal{G}}^k$. So we have $G^{k-d,k} = G^{k+d,k} \equiv \{0\}$ and the above filtrations and gradations take on the following form:

$$(34) \qquad \mathcal{G}^{l,l-c-1} = \{0\} \subset \mathcal{G}^{l,l-c} \subset \ldots \subset \mathcal{G}^{l,l+c} = \mathcal{G}^l\,,$$

$$\bar{\mathcal{G}}^{k-c-1,k} = \{0\} \subset \bar{\mathcal{G}}^{k-c,k} \subset \ldots \subset \bar{\mathcal{G}}^{k+c,k} = \bar{\mathcal{G}}^k\,,$$

$$(35) \qquad \ldots \oplus \{0\} \oplus G^{l,l-c} \oplus \ldots \oplus G^{l,l+c} \oplus \{0\} \oplus \ldots\,,$$

$$\ldots \oplus \{0\} \oplus G^{k-c,k} \oplus \ldots \oplus G^{k+c,k} \oplus \{0\} \oplus \ldots\,,$$

where c is a fixed constant.

Our next aim is to introduce several $\odot\mathscr{H}$-module structures. This may be accomplished by means of the operators (5). For example, every such operator of degree s yields a mapping $\mathscr{Z}: \Omega^l \to \Omega^{l+s}$ and the naturally induced mapping $Z: \mathcal{G}^l \to \mathcal{G}^{l+s}$ is unambigously determined already by the tensor (6). We have only repeated the reasoning of Section 3, of course, but an analogous procedure can be done with $\bar{\mathcal{G}}^k$, $\mathcal{G}^{l,k}$, $\bar{\mathcal{G}}^{l,k}$ and $G^{l,k}$ as well. As a result, we obtain the multiplications

$$Z: \bar{\mathcal{G}}^k \to \bar{\mathcal{G}}^{k+s},\ \mathcal{G}^{l,k} \to \mathcal{G}^{l+s,k+s},\ \bar{\mathcal{G}}^{l,k} \to \bar{\mathcal{G}}^{l+s,k+s},\quad G^{l,k} \to G^{l+s,k+s}$$

and the $\odot\mathscr{H}$-module structures on the graded spaces

$$\mathcal{G} = \oplus\mathcal{G}^l,\ \bar{\mathcal{G}} = \oplus\bar{\mathcal{G}}^k,\ \mathcal{G}^{(l-k)} = \oplus\mathcal{G}^{l+s,k+s},\ \bar{\mathcal{G}}^{(l-k)} = \oplus\bar{\mathcal{G}}^{l+s,k+s},$$

$$G^{(l-k)} = \oplus G^{l+s,k+s}\,.$$

They are closely interrelated: $\mathcal{G}^{(t)}$ and $\bar{\mathcal{G}}^{(t)}$ are submodules of $\mathcal{G}$ and $\bar{\mathcal{G}}$, respectively, we have the filtrations

$$(36) \qquad \ldots \subset \mathcal{G}^{(t)} \subset \mathcal{G}^{(t-1)} \subset \ldots \subset \mathcal{G} = \bigcap\mathcal{G}^{(t)}\,,$$

$$\ldots \subset \bar{\mathcal{G}}^{(t)} \subset \bar{\mathcal{G}}^{(t+1)} \subset \ldots \subset \bar{\mathcal{G}} = \bigcap\bar{\mathcal{G}}^{(t)}\,,$$

and the corresponding graded modules are isomorphic to the module $G^{(t)}$:

$$(37) \qquad \mathcal{G}^{(t)}/\mathcal{G}^{(t+1)} = G^{(t)} = \bar{\mathcal{G}}^{(t)}/\bar{\mathcal{G}}^{(t-1)}\,.$$

The above inclusions cannot be taken literally but only up to certain isomorphisms. For instance, the inclusions $\mathcal{G}^{(l-k)} \subset \mathcal{G}^{(l-k-1)}$ are direct sums of the injections

$$\mathcal{G}^{l,k} = (\Omega^l \cap \bar{\Omega}^k + \Omega^{l-1})/\Omega^{l-1} \to (\Omega^l \cap \bar{\Omega}^{k+1} + \Omega^{l-1})/\Omega^{l-1} = \Omega^{l,k+1}\,.$$

Note that

$$(38) \qquad \mathcal{G}^{(t)} \equiv \{0\}\ (t > c)\,,\quad \mathcal{G}^{(t)} \equiv \mathcal{G}\ (t \leqq -c)\,,\quad \bar{\mathcal{G}}^{(t)} \equiv \{0\}\ (t < -c)\,,$$

$$\bar{\mathcal{G}}^{(t)} \equiv \bar{\mathcal{G}}\ (t \geqq c)\,,$$

as follows from (34).

Inserting the lower index z, we can localize the above objects at a point $z \in J$. The $\odot\mathscr{H}$-modules turn into Noetherian A-modules interrelated by various filtrations, gradations and A-homomorphisms. Since the reasonings are obvious, we need not go into details. See the note at the page 383.

15. Proof for the ideals. Recall that our aim is to prove that the minimal ideals of the sets Ass $\mathscr{G}_z$ and Ass $\bar{\mathscr{G}}_z$ are the same. Owing to the symmetry of assumptions, it is sufficient to verify that an arbitrary ideal $\bar{\mathfrak{q}} \in \text{Ass}\, \mathscr{G}_z$ contains a certain ideal $\mathfrak{q} \in \text{Ass}\, \bar{\mathscr{G}}_z$. One can also observe that this happens if and only if the intersection $\bigcap \mathfrak{p}$ ($\mathfrak{p} \in \text{Ass}\, \mathscr{G}_z$) is contained in the intersection $\bigcap \bar{\mathfrak{p}}$ ($\bar{\mathfrak{p}} \in \text{Ass}\, \mathscr{G}_z$). In other words, assuming $Z \in \bigcap \mathfrak{p} = \text{Nil}\, \mathscr{G}_z$, we wish to prove that $Z \in \bigcap \bar{\mathfrak{p}} = \text{Nil}\, \bar{\mathscr{G}}_z$. Note that it is sufficient to deal with a homogeneous element $Z \in A^s$ under the stronger assumption $Z \in \text{Ann}\, \mathscr{G}_z$, otherwise we take an appropriate power of Z instead of Z.

We come to the main part of the proof. The assumption $Z \in \text{Ann}\, \mathscr{G}_z$ clearly implies $Z \in \text{Ann}\, \mathscr{G}_z^{(t)}$, hence $Z \in \text{Ann}\, G_z^{(t)}$ for all t. Then, passing to the dashed filtration, the last inclusion means that the multiplication by Z turns every module $\bar{\mathscr{G}}_z^{(t)}$ into the module $\bar{\mathscr{G}}_z^{(t-1)}$ and it follows (see (38)) that the iterated multiplication by the product $Z \ldots Z = Z^{2c}$ sends every module $\bar{\mathscr{G}}_z^{(t)}$, in particular the module $\bar{\mathscr{G}}_z^{(c)} = \bar{\mathscr{G}}_z$, into zero. So we have $Z \in \text{Nil}\, \bar{\mathscr{G}}_z$ and the proof is complete.

16. Proof for the multiplicities. Recall that our aim is to verify the equality $\mu(\mathscr{G}_z/\mathscr{G}_z(\mathfrak{q})) = \mu(\bar{\mathscr{G}}_z/\bar{\mathscr{G}}_z(\mathfrak{q}))$ for every ideal $\mathfrak{q} \in \text{Min}\, \mathscr{G}_z = \text{Min}\, \bar{\mathscr{G}}_z$. The connection between these multiplicities is provided by the usual double filtrations, of course, but the ideal $\mathfrak{q}$ under consideration must be isolated and taken under control.

Following this strategy, the above modules $\mathscr{G}_z^{(t)}$ and $\bar{\mathscr{G}}_z^{(t)}$ will be replaced by the more complicated

$$G^{(t)} = \mathscr{G}_z^{(t)}/\mathscr{G}_z^{(t)}(\mathfrak{q}) \;(\text{if } \mathfrak{q} \in \text{Ass}\, \mathscr{G}_z^{(t)}), \quad G^{(t)} = \{0\} \;(\text{if } \mathfrak{q} \notin \text{Ass}\, \mathscr{G}_z^{(t)})$$

and analogous modules $\bar{G}^{(t)}$ for the second filtration. Within the same filtration, these modules are closely related since

$$(39) \qquad \mathscr{G}_z^{(t)}(\mathfrak{q}) = \mathscr{G}_z^{(t)} \cap \mathscr{G}_z(\mathfrak{q}) \;(\mathfrak{q} \in \text{Ass}\, \mathscr{G}_z^{(t)}), \quad \mathscr{G}_z^{(t)} \subset \mathscr{G}_z(\mathfrak{q}) \;(\mathfrak{q} \notin \text{Ass}\, \mathscr{G}_z^{(t)})$$

and analogously for the modules $\bar{\mathscr{G}}_z^{(t)}(\mathfrak{q})$. It follows that there are filtrations

$$(40) \qquad \ldots \subset G^{(t)} \subset G^{(t-1)} \subset \ldots, \quad \ldots \subset \bar{G}^{(t)} \subset G^{(t+1)} \subset \ldots$$

exactly corresponding to the original filtrations (36). But unfortunately, the module $\mathscr{G}_z(\mathfrak{q})$ seems to be unrelated to $\bar{\mathscr{G}}_z(\mathfrak{q})$ and the existence of isomorphisms which should correspond to the relations (37) is not clear. For these reasons, we introduce simpler modules

$$\Gamma^{(t)} = \mathscr{G}_z^{(t)}/\mathfrak{q}^c\mathscr{G}_z^{(t)}, \quad \bar{\Gamma}^{(t)} = \bar{\mathscr{G}}_z^{(t)}/\mathfrak{q}^c\bar{\mathscr{G}}_z^{(t)}$$

with c large enough, ensuring

$$(41) \qquad \mathfrak{q}^c\mathscr{G}_z^{(t)} \subset \mathscr{G}_z^{(t)}(\mathfrak{q}), \quad \mathfrak{q}^c\bar{\mathscr{G}}_z^{(t)} \subset \mathscr{G}_z^{(t)}(\mathfrak{q})$$

for the cases when $\mathfrak{q} \in \text{Ass}\, \mathscr{G}_z^{(t)}$, $\mathfrak{q} \in \text{Ass}\, \bar{\mathscr{G}}_z^{(t)}$, respectively (the existence of c follows from IV, V). One can verify that these modules $\Gamma^{(t)}$, $\bar{\Gamma}^{(t)}$ approximate the modules $G^{(t)}$, $\bar{G}^{(t)}$ in the following sense: If $\mathfrak{q} \in \operatorname{Ass} \mathscr{G}_z^{(t)}$, then $\nu(G^{(t)}) = \nu(\Gamma^{(t)}) = \sigma(\mathfrak{q})$ (a consequence of III and (19)) and $\mu(G^{(t)}) = \mu(\Gamma^{(t)})$ (see (vii) Section 11). If $\mathfrak{q} \notin \operatorname{Ass} \mathscr{G}_z^{(t)}$, then $\nu(\Gamma^{(t)}) < \sigma(\mathfrak{q})$ as follows from III and VII (this is a trivial case). Analogous results are valid for the dashed modules $\bar{G}^{(t)}$, $\bar{\Gamma}^{(t)}$.

All modules needed for the proof are now available. In terms of these modules we wish to equate the multiplicites $\mu(\mathscr{G}_z/\mathscr{G}_z(\mathfrak{q})) = \mu(G^{(-c)}) = \mu(\Gamma^{(-c)})$ (cf. (38)) with the dashed multiplicities $\mu(\bar{\mathscr{G}}_z/\bar{\mathscr{G}}_z(\mathfrak{q})) = \mu(\bar{G}^{(c)}) = \mu(\bar{\Gamma}^{(c)})$ or, briefly, we wish to prove that $\mu(\Gamma^{(-c)}) = \mu(\bar{\Gamma}^{(c)})$.

A promising way consists in decomposing the modules $\Gamma^{(-c)}$, $\bar{\Gamma}^{(c)}$ into certain isomorphic factors, in other words, in establishing some substitutes for the relations (36) and (37) but with modules $\Gamma^{(t)}$, $\bar{\Gamma}^{(t)}$. But unfortunately, the behaviour of the last modules is not so simple and it calls for a more intricate and less straightforward methods. We shall employ the achievements of Section 9 introducing a *sufficiently general* sequence $X_* = X_1, \ldots, X_v$ of the length $v = \sigma(\mathfrak{q}) - 1$. (The term *sufficiently general* means that X_* should be an M-general sequence for a certain broad family of various modules M appearing in the course of proof; see (vii) Section 8, for the existence of X_*.) We shall mention several properties of such a sequence X_*, successively directed towards the proof of the above equality of multiplicities.

We begin with the following reformulation of the approximation property: If $\mathfrak{q} \in \operatorname{Ass} \mathscr{G}_z^{(t)}$, then $\nu(G_*^{(t)}) = \nu(\Gamma_*^{(t)}) = 1$ (cf. (ii) Section 11) and the dimensions of the homogeneous components of the gradations are $\ell((G_*^{(t)})^l) = \mu(G^{(t)}) = \mu(\Gamma^{(t)}) = \mu(\Gamma_*^{(t)}) = \ell((\Gamma_*^{(t)})^l)$ (cf. (18) and (ii) Section 11). If $\mathfrak{q} \notin \operatorname{Ass} \mathscr{G}_z^{(t)}$, then $\nu(G_*^{(t)}) = \nu(\Gamma_*^{(t)}) = 0$ and $\ell((G_*^{(t)})^l) = \ell((\Gamma_*^{(t)})^l) = 0$ for all l large enough (the same reasoning). Analogous results are valid for the modules $\bar{G}^{(t)}$, $\bar{\Gamma}^{(t)}$.

We continue with the observation that the filtrations (40) lead to the inclusions

$$(42) \qquad \ldots \mathrel{\underset{\sim}{\subset}} G_*^{(t)} \mathrel{\underset{\sim}{\subset}} G_*^{(t-1)} \mathrel{\underset{\sim}{\subset}} \ldots, \quad \ldots \mathrel{\underset{\sim}{\subset}} \bar{G}_*^{(t)} \mathrel{\underset{\sim}{\subset}} G_*^{(t+1)} \mathrel{\underset{\sim}{\subset}} \ldots,$$

as follows from (26) and the assumption that X_* is a $G^{(t-1)}/G^{(t)}$- and $G^{(t+1)}/G^{(t)}$-general sequence. An analogous reasoning applied to (41) yields the inclusions

$$(43) \qquad \mathfrak{q}^c \mathscr{G}_{z*}^{(t)} \mathrel{\underset{\sim}{\subset}} \mathscr{G}_z^{(t)}(\mathfrak{q})_*, \quad \mathfrak{q}^c \bar{\mathscr{G}}_{z*}^{(t)} \mathrel{\underset{\sim}{\subset}} \bar{\mathscr{G}}_z^{(t)}(\mathfrak{q})_*$$

provided $\mathfrak{q} \in \operatorname{Ass} \mathscr{G}_z$, $\mathfrak{q} \in \operatorname{Ass} \bar{\mathscr{G}}_z$, respectively. But we already know that the dimensions of the factor-spaces

$$(\mathscr{G}_{z*}^{(t)}/\mathscr{G}_z^{(t)}(\mathfrak{q})_*)^l = (G_*^{(t)})^l, \quad (\mathscr{G}_{z*}^{(t)}/\mathfrak{q}^c \mathscr{G}_{z*}^{(t)})^l = (\Gamma_*^{(t)})^l$$

are equal for l large enough. It follows that the inclusions in (43) may be replaced by the equalities $\cong$. Then, according to (39), we have

$$G_*^{(t)} = \mathscr{G}_{z*}^{(t)}/\mathscr{G}_z(\mathfrak{q})_* \cong \mathscr{G}_{z*}^{(t)}/\mathfrak{q}^c \mathscr{G}_{z*}^{(t)} = \Gamma_*^{(t)}.$$

One can observe that the relations $G_*^{(t)} \cong \Gamma_*^{(t)}$ are (trivially) satisfied also for the case $\mathfrak{q} \notin \operatorname{Ass} \mathscr{G}_z^{(t)}$. Analogous results are valid for the dashed modules, of course.

We come to the final part of the proof. According to the last result, (42) may be rewritten as

$$\ldots \supsetneqq \Gamma_*^{(t)} \supsetneqq \Gamma_*^{(t-1)} \supsetneqq \ldots, \quad \ldots \supsetneqq \bar{\Gamma}_*^{(t)} \supsetneqq \bar{\Gamma}_*^{(t+1)} \supsetneqq \ldots .$$

The existence of such a filtration is not a self-evident fact and is equivalent to the relations

$$(44) \qquad \mathfrak{q}^c \mathscr{G}_{z*}^{(t)} \cong \mathscr{G}_{z*}^{(t)} \cap \mathfrak{q}^c \mathscr{G}_{z*}^{(t-1)}, \quad \mathfrak{q}^c \bar{\mathscr{G}}_{z*}^{(t)} \cong \bar{\mathscr{G}}_{z*}^{(t)} \cap \mathfrak{q}^c \bar{\mathscr{G}}_{z*}^{(t+1)} .$$

These relations lead to the crucial isomorphisms

$$\Gamma_*^{(t)}/\Gamma_*^{(t+1)} \cong \bar{\Gamma}_*^{(t)}/\Gamma_*^{(t-1)}$$

exactly corresponding to (37). (Besides an elementary verification of the last assertion, an approach based on Section 9 is possible, too. Indeed, let us put $M = \mathscr{G}_{z*}^{(t)}$, $N = \mathscr{G}_{z*}^{(t+1)}$, $\mathfrak{a} = \mathfrak{q}^c$. Then (44) is identified with (23) and the isomorphisms (24) together with the rule (26) and the isomorphisms (37) lead to the result

$$\Gamma_*^{(t)}/\Gamma_*^{(t+1)} = M/\mathfrak{a}M)/(N/\mathfrak{a}N) \cong (M/N)/\mathfrak{a}(M/N) = G_{z*}^{(t)}/\mathfrak{q}^c G_{z*}^{(t)} .$$

Analogously $\bar{\Gamma}_*^{(t)}/\bar{\Gamma}_*^{(t-1)} = \bar{G}_{z*}^{(t)}/\mathfrak{q}^c \bar{G}_{z*}^{(t)}$ from the symmetry of assumptions.)

The proof is concluded by the equation

$$\mu(\Gamma^{(-c)}) = \ell((\Gamma_*^{(-c)})^l) = \Sigma \ell((\Gamma_*^{(t)}/\Gamma_*^{(t+1)})^l) =$$
$$= \Sigma \ell((\bar{\Gamma}_*^{(t+1)}/\bar{\Gamma}_*^{(t)})^l) = \ell((\bar{\Gamma}_*^{(c)})^l) = \mu(\bar{\Gamma}^{(c)}) .$$

We employ the finiteness (38) rewritten for the modules $\Gamma^{(t)}$ and $\bar{\Gamma}^{(t)}$ and the obvious inclusions $\mathfrak{q} \in \operatorname{Ass} \mathscr{G}_z^{(-c)} = \operatorname{Ass} \mathscr{G}_z$, $\mathfrak{q} \in \operatorname{Ass} \bar{\mathscr{G}}_z^{(c)} = \operatorname{Ass} \bar{\mathscr{G}}_z$.

17. Corollary. *The ideals* $\mathfrak{q} \in \operatorname{Ass} \mathscr{G}_z$ *with* $\sigma(\mathfrak{q}) = \nu(\mathscr{G}_z)$ *are invariants of the diffiety* Ω *and the number* $\mu(\mathscr{G}_z) = \Sigma \mu(\mathscr{G}_z/\mathscr{G}_z(\mathfrak{q}))$ (*sum over* $\mathfrak{q} \in \operatorname{Ass} \mathscr{G}_z$, $\sigma(\mathfrak{q}) = \nu(\mathscr{G}_z)$) *does not depend on the used filtration, either* (See (28) for the last sum.)

18. A formal concept of prolongation. Consider filtrations (3) and (31) of a diffiety Ω. The filtration $\bar{\Omega}^*$ is called a *prolongation* of the filtration Ω^* if $\Omega^l \subset \bar{\Omega}^l$ for all l. In particular, if only the filtration Ω^* is given, then the special filtration $\bar{\Omega}^*$ defined by $\bar{\Omega}^l \equiv \{0\}$ $(l < 0)$, $\bar{\Omega}^l \equiv \Omega^{l+c}$ $(l \geqq 0)$ is called a *normal c-prolongation* of Ω^* and is denoted by $\Omega^* = \Omega^{*+c}$. Usually we assume $c = 1, 2, \ldots$ and the case $c = 1$ is named a *normal prolongation.*

According to $\mathscr{F}in$ condition, for every pair Ω^*, $\bar{\Omega}^*$ of (good) filtrations of Ω there exists a constant d such that $\bar{\Omega}^{*+d}$ is a prolongation of Ω^*. (Indeed, look at the inclusion (32) which implies $\Omega^c \subset \Omega^{l+c} \subset \bar{\Omega}^{d+c}$ $(d = K(l) + l)$ with l fixed and large enough.) One can easily see that the graded $\odot\mathscr{H}$-module derived from the normal c-prolongation Ω^{*+c} is identical with the module $\mathscr{G}^{+c}$ (which explains the seemingly artificial definition of multiplication for the last module, cf. Section 7.) According to (iii) Section 8, all associated prime ideals different from the maximal ideal $\mathfrak{m}$ are preserved under all normal prolongations. On the other hand, non-minimal ideals (the so called *embedded ideals*) need not be preserved. Since the case of the maximal ideal is easy, we shall state another and less trivial example.

19. Example. We wish to construct certain filtrations Ω^*, $\bar{\Omega}^*$ satisfying $\bigcup \mathfrak{p} \neq \bigcup \bar{\mathfrak{p}}$ (the unions are over $\mathfrak{p} \in \operatorname{Ass} \mathscr{G}_z$ and $\mathfrak{p} \neq \mathfrak{m}$, $\bar{\mathfrak{p}} \in \operatorname{Ass} \bar{\mathscr{G}}_z$ and $\bar{\mathfrak{p}} \neq \mathfrak{m}$). Indeed, then the sets $\operatorname{Ass} \mathscr{G}_z$, $\operatorname{Ass} \bar{\mathscr{G}}_z$ surely differ by a non-maximal ideal.

Let an arbitrary (good) filtration Ω^* of a diffiety Ω be given. We shall assume $\ell(H) = \nu(A) = \nu(\mathscr{G}_z) = 2$. Let $X, Y \in \mathscr{H}$ be such vector fields that the sequence $X_* = X_z, Y_z$ is $\mathscr{G}_z$-quasiregular. (Note that the last assumption is inessential since it is satisfied if X_z, Y_z is a $\mathscr{G}_z$-general sequence and we take an appropriate c-normal prolongation Ω^{*+c} instead of Ω^*.) Moreover, we assume $\Omega^{l+1} = \Omega^l + \mathscr{L}_X\Omega^l + \mathscr{L}_Y\Omega^l$ $(l = 0, 1, \ldots)$ which is again an inessential restriction, see $(4)_2$.

We come to the part proper of the construction. By virtue of $\nu(\mathscr{G}_z) > 1$ we have $X_z\mathscr{G}_z^0 \neq . \mathscr{G}_z^1$ and there exists $\omega^0 \in \Omega^0$ such that $Y_z\hat{\omega}_z^0 \notin X_z\mathscr{G}_z^0$. Put recurrently $\omega^{l+1} = \mathscr{L}_Y\omega^l$ $(l \geqq 0)$. Clearly $\omega^{l+1} \in \Omega^{l+1}$ and $\hat{\omega}_z^{l+1} = Y_z\hat{\omega}_z^l \in \mathscr{G}_z^{l+1}$ for the classes. One can verify that $\hat{\omega}_z^{l+1} = Y_z\hat{\omega}_z^l \notin \mathscr{G}_z^l$ due to the quasiregularity assumption. Now, consider the filtration $\bar{\Omega}^*$ defined by $\bar{\Omega}^l \equiv \{0\}$ $(l < 0)$, $\bar{\Omega}^l \equiv \Omega^l + \mathscr{L}_X\Omega^l$ $(l \geqq 0)$. It is possible to prove that $\bar{\Omega}^*$ is a good filtration. Nevertheless, we have $\omega^l \in \bar{\Omega}^l$, $\mathscr{L}_X\omega^l \in \bar{\Omega}^l$, $\mathscr{L}_Y\omega^l = \omega^{l+1} \notin \Omega^l$ and, temporarily denoting by dash the classes corresponding to the second filtration, we have $\bar{\omega}_z^l \in \bar{\mathscr{G}}_z^l$, $X_z\bar{\omega}_z^l \in \bar{\mathscr{G}}_z^l$, $Y_z\bar{\omega}_z^l \notin \bar{\mathscr{G}}_z^l$. It follows that $X_z \in \bigcup \bar{\mathfrak{p}}$, but clearly $X_z \notin \bigcup \mathfrak{p}$ since X_z, Y_z is a $\mathscr{G}_z$-quasiregular sequence.

CHARACTERISTICS AND COMMUTATIVE ALGEBRA

20. Definition and classification. Let a filtration Ω^* of a diffiety be given and let a point $z \in J$ be fixed. A linear subspace L of the space $H = \mathscr{H}_z$ is called a *characteristic subspace* (to the filtration Ω^*, at the point z), if there exists no $\mathscr{G}_z$-general sequence $X_* = X_1, \ldots, X_v$ of length $v = \ell(L)$ with terms $X_1, \ldots, X_v$ lying in L. More in detail, let us denote by $\Delta_w^u(\mathscr{G}_z)$ (in short, Δ_w^u, where $u = 1, 2, \ldots$ and $w = 0, 1, \ldots$) the family of all linear subspaces $L \subset H$ of dimension $\ell(L) = u + w$ such that there exists a $\mathscr{G}_z$-general sequence $X_* = X_1, \ldots, X_w$ in L which is a maximal $\mathscr{G}_z$-general sequence in L, i.e., the enlarged sequence $X_1, \ldots, X_w, X_{w+1}$ cannot be $\mathscr{G}_z$-general for any $X_{w+1} \in L$.

We shall prove later on by homological methods that *every* maximal $\mathscr{G}_z$-general sequence lying in the space $L \in \Delta_w^u$ is necessarily of length w. Taking this fact for granted, it is obvious that the sets Δ_w^u are mutually disjoint and may be determined by the following construction:

Δ_0^u with $u \geqq 1$: A subspace $L \subset H$ belongs to Δ_0^u if and only if there exists no $X_1 \in L$ such that the multiplication $X_1: \mathscr{G}_z^l \to \mathscr{G}_z^{l+1}$ is injective for l large enough. That means, $X: \mathscr{G}_z^{+c+} \to \mathscr{G}_z^{+c+}$ (c is large) cannot be an injection for any $X \in L$ or, equivalently, $L \subset \bigcup(\mathfrak{p} \cap H)$; the union is over $\mathfrak{p} \in \operatorname{Ass} \mathscr{G}_z$, $\mathfrak{p} \neq \mathfrak{m}$ (see (v) Section 8). It follows that a characteristic subspace $L \in \Delta_0^u$ is determined by the choice of an ideal $\mathfrak{p} \in \operatorname{Ass} \mathscr{G}_z$, $\mathfrak{p} \neq \mathfrak{m}$, and some linearly independent vectors $X_1, \ldots, X_u \in \mathfrak{p} \cap H$. If this choice cannot be realized, then Δ_0^u is empty.

Δ^u_w with $u \geqq 1$ and $0 < w < \nu(\mathscr{G}_z)$: We begin with an arbitrary $\mathscr{G}_z$-general sequence $X_1, \ldots, X_w$. Note that these vectors are linearly independent since $\nu(\mathscr{G}_z/(X)_1 \mathscr{G}_z) = \nu(\mathscr{G}_z) - 1, \ldots, \nu(\mathscr{G}_z/(X)_w \mathscr{G}_z) = \nu(\mathscr{G}_z) - w > 0$ (cf. (ii) Section 11). Then we choose an ideal $\mathfrak{p} \in \operatorname{Ass} \mathscr{G}_z/(X)_w \mathscr{G}_z$, $\mathfrak{p} \neq \mathfrak{m}$, and some vectors $X_{w+1}, \ldots, X_{w+u}$ such that the total sequence $X_1, \ldots, X_{w+u}$ consists of linearly independent vectors. The last sequence spans the desired space $L \in \Delta^u_w$. The family Δ^u_w may be empty, of course.

Δ^u_w with $u \geqq 1$ and $w \geqq \nu(\mathscr{G}_z)$: If $X_* = X_1, \ldots, X_w$ is a $\mathscr{G}_z$-general sequence of length $w \geqq v = \nu(\mathscr{G}_z)$, then $\nu(\mathscr{G}_z/(X)_v \mathscr{G}_z) = 0$ (cf. (ii) Section 11), hence $\nu(\mathscr{G}_z/(X)_w \mathscr{G}_z) = 0$ as well, and $\ell(\mathscr{G}^l_z/(X)_w \mathscr{G}^{l-1}_z) = 0$ for all l large enough. It follows that the sequence X_* may be enlarged by an arbitrary vector $X_{w+1} \in L$, it is never maximal and Δ^u_w is empty.

The general concept of characteristics was clarified in E. Cartan's work, but he explicitly considered only the particular case denoted here by Δ^1_w which exactly corresponds to the unpleasant boundary value problem to which the Cauchy-Kowalewska theorem cannot be applied (cf. [13]), and his definition is reasonable only for the involutive case (see Section 35 below). Note that unlike the classical differential geometers, the contemporary experts on partial differential equations usually deal only with rather special problems and, as the general concept of characteristics is concerned, many ambiguities are present in current literature.

Note, finally, that a linear subspace L of H is called *regular* (to the filtration Ω^*, at the point z), if L is not a characteristic subspace, that means, if there exists a $\mathscr{G}_z$-general sequence $X_* = X_1, \ldots, X_w$ of length $w = \ell(L)$ in L. Note that in this case any maximal $\mathscr{G}_z$-general sequence in L is of length at least $\ell(L)$. The set of all regular subspaces L is denoted $\Delta^0_w(\mathscr{G}_z)$ (Δ^0_w, in short) and is very rich (cf. (vii) Section 8).

21. Invariance of characteristic subspaces. Repeating once more the construction of Δ^u_w ($u \geqq 1$, $0 \leqq w < \nu(\mathscr{G}_z)$), let us choose a $\mathscr{G}_z$-general sequence $X_* = X_1, \ldots, X_w$ and look for the ideals from the set $\operatorname{Ass} \mathscr{G}_z/(X)_w \mathscr{G}_z$. Little can be said about the general case, but the minimal ideals constituting the set $\operatorname{Min} \mathscr{G}_z/(X)_w \mathscr{G}_z$ can be described more easily since they coincide with the minimal ideals of the set

$$\operatorname{Supp} \mathscr{G}_z/(X)_w \mathscr{G}_z = \operatorname{Supp} \mathscr{G}_z \cap \operatorname{Supp} A/(X)_w$$

(cf. III). It follows that $\mathfrak{q} \in \operatorname{Min} \mathscr{G}_z$ if and only if there exists an ideal $\mathfrak{q} \in \operatorname{Min} \mathscr{G}_z$ such that $\mathfrak{q}$ is a minimal ideal in the set of all ideals including both $\mathfrak{p}$ and $(X)_w$. According to the first assertion of Theorem 13, the set $\operatorname{Min} \mathscr{G}_z$ does not depend on the auxiliary filtration Ω^* and we conclude that the same is true for the set $\operatorname{Min} \mathscr{G}_z/(X)_w \mathscr{G}_z$ of all above mentioned ideals $\mathfrak{q}$. As a final result, if we choose some vectors $X_{w+1}, \ldots, X_{w+u} \in \mathfrak{q} \cap H$ linearly independent of $X_1, \ldots, X_w$, then the linear span L of the family $X_1, \ldots, X_{w+u}$ is a characteristic subspace for *all* good filtrations of the diffiety Ω.

The second assertion of Theorem 3 concerning the multiplicities cannot be employed for the theory of characteristic subspaces in such a simple manner. It is nevertheless true that

$$(45)\qquad \mu((\mathscr{G}_z/(X)_w\mathscr{G}_z)/(\mathscr{G}_z/(X)_w\mathscr{G}_z)(\mathfrak{q})) = \mu((\bar{\mathscr{G}}_z/(X)_w\bar{\mathscr{G}}_z)/(\bar{\mathscr{G}}_z/(X)_w\bar{\mathscr{G}}_z)(\mathfrak{q}))$$

for *any* two good filtrations Ω^* and $\bar{\Omega}^*$ of the diffiety Ω, provided $X_* = X_1, \ldots, X_w$ is a *sufficiently general sequence.* The above constant (45) may be taken for the multiplicities of the above mentioned characteristic subspaces derived from the ideal $\mathfrak{q}$ by the above procedure. (Note that the understanding of the term *sufficiently general sequence* X_* is a little different than in the previous sections. The sequence X_* is apriori given, of course, and the above statement means that the multiplicities of the characteristic subspaces can be regularly calculated if we use certain, not too special, filtrations of Ω.)

The invariant sense of characteristic subspaces is a rather important fact for the classification theory of general systems of partial differential equations. Eventually, the concept of an elliptical, hyperbolical, etc. system is proved invariant for a very wide category of transformations. Some modifications permit to include the case of complex characteristic subspaces, but we do not follow this (rather obvious) way.

22. On the multiplicity of characteristics. Our aim is to verify the equality (45) for any ideal $\mathfrak{q} \in \operatorname{Min} \mathscr{G}_z/(X)_w\mathscr{G}_z = \operatorname{Min} \bar{\mathscr{G}}_{z*}/(X)_w\mathscr{G}_{z*}$ and a sufficiently general sequence $X_* = X_1, \ldots, X_w$ of length $w < \nu(\mathscr{G}_z)$. One can observe that (45) is a generalization of the second assertion of Theorem 13 which formally arises for the particular case $w = 0$. It follows that the strategy of the proof should consist in replacing all modules M appearing in Section 16 by the factors $M_* = M/(X)_w M$.

As an example, we obtain the inclusions

$$\ldots \supseteq \mathscr{G}_{z*}^{(t)} \supseteq \mathscr{G}_{z*}^{(t-1)} \supseteq \ldots, \quad \ldots \supseteq \bar{\mathscr{G}}_{z*}^{(t)} \supseteq \bar{\mathscr{G}}_{z*}^{(t+1)} \supseteq \ldots$$

standing at the place of the original filtration (36). (We suppose that X_* is $\mathscr{G}_z^{(t-1)}/\mathscr{G}_z^{(t)}$- and $\bar{\mathscr{G}}_z^{(t+1)}/\bar{\mathscr{G}}_z^{(t)}$-general here. This is the only assumption on the sequence X_* needed in the proof.) Then the ideal $\mathfrak{q}$ is isolated by considering the modules

$$G^{(t)} = \mathscr{G}_{z*}^{(t)}(\mathfrak{q}) \;(\text{if } \mathfrak{q} \in \operatorname{Ass} \mathscr{G}_{z*}^{(t)}), \quad G^{(t)} = \{0\} \;(\text{if } \mathfrak{q} \notin \operatorname{Ass} \mathscr{G}_{z*}^{(t)})$$

and the auxiliary approximating modules

$$\Gamma^{(t)} = \mathscr{G}_{z*}^{(t)}/\mathfrak{q}^c\mathscr{G}_{z*}^{(t)} \quad \text{with} \quad \mathfrak{q}^c\mathscr{G}_{z*}^{(t)} \subset \mathscr{G}_{z*}^{(t)} \;(\text{if } \mathfrak{q} \in \operatorname{Ass} \mathscr{G}_{z*}^{(t)}).$$

After these mild arrangements, the other part of the proof runs quite analogously as in Section 16 and therefore is omitted.

23. A particular case. The above construction of characteristic subspaces cannot usually start with an arbitrary (not $\mathscr{G}_z$-general)sequence $X_* = X_1, \ldots, X_v$ but there is the following important exception: Let $X_1, \ldots, X_v \in H$ be a sequence satisfying

$$(46)\qquad \nu(\mathscr{G}_z/(X)_v\mathscr{G}_z) > \nu(\mathscr{G}_z) - \ell(L) \geqq 0,$$

where L denotes the linear span of the family $X_1, \ldots, X_v$. One can prove by contradiction (cf. (ii) Section 11) that such an L is always a characteristic subspace (to all filtrations). Moreover, adjoining some quite arbitrary vectors $X_{v+1}, \ldots, X_{v+u}$ whose number is $u < v(\mathscr{G}_z) - \ell(L)$ to the sequence $X_1, \ldots, X_v$, the linear subspace L' spanned by the total family $X_1, \ldots, X_{v+u}$ is characteristic as well since

$$v(\mathscr{G}_z/(X)_{v+u}\,\mathscr{G}_z) = v((\mathscr{G}_z/(X)_v\,\mathscr{G}_z)/(X)_{v+u}\,\mathscr{G}_z) >$$
$$> v(\mathscr{G}_z) - \ell(L) - u > v(\mathscr{G}_z) - \ell(L') \geqq 0$$

and the criterion (46) is satisfied.

It is interesting that this reasoning can be related to the construction of Section 21 for the particular case of the ideal $\mathfrak{p} \in \operatorname{Min} \mathscr{G}_z$ satisfying $\sigma(\mathfrak{p}) = v(\mathscr{G}_z)$. Indeed, if $X_* = X_1, \ldots, X_w$ is a $\mathscr{G}_z$-general sequence of length $w < v(\mathfrak{q}) = v(\mathscr{G}_z)$, then $v(\mathscr{G}_z/(X)_w\,\mathscr{G}_z) - w > 0$ and the linear subspace L spanned by the family $X_1, \ldots, X_w$ together with some additional and linearly independent vectors $X_{w+1}, \ldots, X_{w+u} \in$ $\in \mathfrak{p} \cap H$ $(1 \leqq u \leqq v(\mathscr{G}_z) - w)$ satisfies the inequality

$$v(\mathscr{G}_z/(X)_{w+u}\,\mathscr{G}_z) = v(\mathscr{G}_z/(X)_w\,\mathscr{G}_z) > v(\mathscr{G}_z) - w > v(\mathscr{G}_z) - \ell(L) \geqq 0\,,$$

identical with the criterion (46) above.

24. The Cauchy characteristics. We begin with a fixed filtration Ω^*, as usual. Let $\mathscr{C}^l$ be the set of all vector fields $X \in \mathscr{H}$ satisfying $\mathscr{L}_X \Omega^l \subset \Omega^l$. One can see that $\mathscr{C}^l$ is a Ψ_0-submodule and a Lie subalgebra of $\mathscr{H}$. If $(4)_2$ is satisfied for a certain degree l, then $\mathscr{C}^l \subset \mathscr{C}^{l+1}$. It follows that $\mathscr{C}^l = \mathscr{C}^{l+1} = \ldots$ for all l large enough.

Passing from analysis to algebra, let us fix a point $z \in J$ and consider the linear subspaces $\mathscr{C}^l_z$ of $H = \mathscr{H}_z$. Clearly $X_z \mathscr{G}^l_z = \{0\}$ for every $X \in \mathscr{C}^l$. Denote (for a moment) by C^l_z the linear subspace of H consisting of all vectors $Y_z \in H$ satisfying $Y_z \mathscr{G}^l_z = \{0\}$. Clearly $\mathscr{C}^l_z \subset C^l_z$, but if the dimension $\ell(C^l_z)$ is a constant independent of z, then every vector $Y_z \in C^l_z$ can be realized by a vector field $X \in \mathscr{C}^l$ (i.e., $Y_z = X_z$) and this implies the converse inclusion $\mathscr{C}^l_z \supset C^l_z$. Hence $\mathscr{C}^l_z = C^l_z$ and both the analytic and algebraic approaches are equivalent.

Obviously $\mathscr{C}_z = \cap \mathscr{C}^l_z = H \cap \operatorname{Ann} \mathscr{G}_z$, hence $\mathscr{C}_z \in \Delta^u_0$ $(u = \ell(\mathscr{C}_z))$. The space $\mathscr{C}_z$ may be called the *Cauchy characteristic subspace* (to the filtration Ω^*, at the point z). Unfortunately, it need not be preserved even after standard prolongation. The subspaces $\mathscr{N}_z = H \cap \operatorname{Nil} \mathscr{G}_z \in \Delta^u_0$ $(u = \ell(\mathscr{N}_z))$ are, however, independent of the choice of the auxiliary filtration of Ω,

$$\mathscr{N}_z = H \cap \operatorname{Nil} \mathscr{G}_z = H \cap \operatorname{Nil} \bar{\mathscr{G}}_z = \bar{\mathscr{N}}_z\,, \tag{47}$$

as follows from V and from the first part of Theorem 13. One can find it interesting that a straightforward proof of the equality (47) avoiding the double filtration from Section 14 is possible under very mild additional assumptions. (Note that this approach yields another proof of the first part of Theorem 13.)

25. An outline of the invariance proof. Consider two filtrations Ω^*, $\bar{\Omega}^*$ of Ω. We are going to prove the equality $\operatorname{Nil} \mathscr{G}_z = \operatorname{Nil} \bar{\mathscr{G}}_z$. Assume $X_z \in \operatorname{Nil} \mathscr{G}_z$. Our aim is to verify that $X_z \in \operatorname{Nil} \mathscr{G}_z$, too. The homogeneous case of X_z is quite sufficient and then $Z_z = (X_z)^l \in \operatorname{Ann} \mathscr{G}_z \cap A^s$ for an exponent l large enough. *Assume in addition* that there exists a tensor (6) such that $Z_z \in \operatorname{Ann} \mathscr{G}_z$ for every point $z \in J$. (One can see that this assumption is fulfilled if the dimension $\ell(\operatorname{Ann} \mathscr{G}_z \cap A^s)$ is independent of the choice of the point z.) Then any of the corresponding operators (5) satisfies $\mathscr{X}\Omega^l \subset \Omega^{l+s-1}$, hence $\mathscr{X}^r\Omega^l \subset \Omega^{l+rs-r}$, by iteration. Now, recall the inclusion $\bar{\Omega}^{l-c} \subset \Omega^l \subset \bar{\Omega}^{l+c}$ (cf. (33)), where l is arbitrary and c large enough. It follows that

$$\mathscr{X}^r\bar{\Omega}^{l-c} \subset \mathscr{X}^r\Omega^l \subset \Omega^{l+rs-r} \subset \bar{\Omega}^{l+rs-r+c}.$$

The inclusion between the extreme terms may be rewritten as

$$\mathscr{X}^r\bar{\Omega}^l \subset \bar{\Omega}^{l+rs-r+2c}, \quad \text{hence} \quad \mathscr{X}^r\bar{\Omega}^l \subset \bar{\Omega}^{l+rs-1}$$

for every exponent $r \geqq 2c + 1$. It follows that $Z_z^r \in \operatorname{Nil} \bar{\mathscr{G}}_z$, hence $Z_z \in \operatorname{Nil} \bar{\mathscr{G}}_z$ and the proof is complete.

CHARACTERISTICS AND HOMOLOGICAL ALGEBRA

25. Definitions and notation. The fundamental homological concepts will be introduced only for the usual case of a homogeneous graded and Noetherian A-module $M = M^0 \oplus M^1 \oplus \ldots$ over the polynomial algebra $A = A^0 \oplus A^1 \oplus \ldots$ ($A^0 = \mathbb{R}$, $A^1 = H$, $A^2 = H \odot H, \ldots$). We shall nevertheless need a relative variant of the theory. To this aim, we choose a fixed linear subspace L of H, $\ell(L) = m \leqq \leqq \ell(H) = n$, and introduce the polynomial algebra $B = B^0 \oplus B^1 \oplus \ldots$ ($B^0 = \mathbb{R}$, $B^1 = L$, $B^2 = L \odot L, \ldots$) freely generated by L. Since B is a subalgebra of A, M may be considered both as A- and B-module. It is of great importance that the module B will play the role of the ground ring while A will be considered as a mere ring of certain automorphisms. (Note that M need not be a Noetherian B-module. We shall simplify the situation by identifying $L = H$, $B = A$ later on.)

Now let $\boldsymbol{K}$ be the familiar Koszul complex

(48) $$0 \to B \otimes \bigwedge^m L \to \ldots \to B \otimes \bigwedge^2 L \to B \otimes L \to B \to \mathbb{R} \to 0$$

(the tensor products are taken over $\mathbb{R}$). Introducing the complex $M \otimes_B \boldsymbol{K}$:

(49) $$0 \to M \otimes \bigwedge^m L \to \ldots \to M \otimes \bigwedge^2 L \to M \otimes L \to M \to \mathbb{R} \to 0$$

and deleting the last but one term $\mathbb{R}$, the homology of the arising complex at the place $M \otimes \bigwedge^s L$ will be denoted by $\operatorname{Tor}_B(\mathbb{R}, M)_s$. Quite analogously, introducing the complex $\operatorname{Hom}_B(\boldsymbol{K}, M)$:

(50) $$0 \leftarrow M \otimes \bigwedge^m L^* \leftarrow \ldots \leftarrow M \otimes \bigwedge^2 L^* \leftarrow M \otimes L^* \leftarrow L^* \leftarrow \operatorname{Hom}_B(\mathbb{R}, M) \leftarrow 0$$

and deleting the last but one term $\operatorname{Hom}_B(\mathbb{R}, M)$, the homology at the place $M \otimes \bigwedge^s L^*$ is denoted by $\operatorname{Ext}_B(\mathbb{R}, M)_s$. (Note that we use the isomorphisms $\operatorname{Hom}_B(B \otimes \bigwedge^s L, M) = M \oplus \bigwedge^k L^*$ where L^* is the dual space to the $\mathbb{R}$-linear space L. The B-module structure on $\mathbb{R}$ is determined by the condition $B^+\mathbb{R} = \{0\}$, i.e., we identify $\mathbb{R} = \mathbb{R}^0 \oplus \mathbb{R}^1 \oplus \ldots = \mathbb{R}^0 \oplus \{0\} \oplus \ldots$ as a graded B-module.) The gradation of M gives rise to graded homologies, of course. Namely, (49) leads to the family of complexes $M \otimes K^{l+s}$:

$$(49)^{l+s} \qquad \ldots \to M^{l-1} \otimes \Lambda^{s+1}L \to M^l \otimes \Lambda^s L \to M^{l+1} \otimes \Lambda^{s-1}L \to \ldots$$

with the homology $\mathrm{Tor}_B(\mathbb{R}, M)^1_s$, and the complex (50) is decomposed into the family $\mathrm{Hom}_B(K, M)^{l+n-s}$:

$$(50)^{l+n-s} \qquad \ldots \leftarrow M^{l+1} \otimes \Lambda^{s+1}L^* \leftarrow M^l \otimes \Lambda^s L^* \leftarrow M^{l-1} \otimes \Lambda^{s-1}L^* \leftarrow \ldots$$

determining the homology $\mathrm{Ext}_B(\mathbb{R}, M)^l_s$. The differentials in the complexes (49) are expressed by

$$(51) \qquad \partial(\omega \otimes (X_0 \wedge \ldots \wedge X_s)) = \Sigma(-1)^i X_i\omega \otimes (X_0 \wedge \ldots \hat{\imath} \ldots \wedge X_s)$$

where the roof denotes the omitted i-th factor X_i. The differentials in the complexes (50) are explicitly expressed by the dual formula

$$(52) \qquad \partial(\omega \otimes (\xi_1 \wedge \ldots \wedge \xi_s)) = \Sigma(-1)^i Z_i\omega \otimes (\zeta_i \wedge \xi_1 \wedge \ldots \wedge \xi_s)$$

where $Z_1, \ldots, Z_m$ and $\zeta_1, \ldots, \zeta_m$ are some dual bases in L and L^*, respectively. (That is, we suppose $\zeta_i(Z_j) \equiv \delta_{ij}$ and the last differential does not depend on the choice of these bases. We refer to [7] for an intrinsical definition of these differentials.)

Finally, note that using the mentioned explicit formulae, one can verify the commutativity of the diagram

$$(53) \qquad \begin{array}{ccc} M^l \otimes \Lambda^s L^* & \xrightarrow{\text{identity} \otimes (\lrcorner|L|)} & M^l \otimes \Lambda^{m-s}L \\ \downarrow \partial & & \downarrow \partial \\ M^{l+1} \otimes \Lambda^{s+1}L^* & \xrightarrow{\text{identity} \otimes (\lrcorner|L|)} & M^l \otimes \Lambda^{m-s-1}L \end{array}$$

where we denote $|L| = Z_1 \wedge \ldots \wedge Z_m$. Since the vertical arrows may be inverted by the mapping identity$\otimes \lrcorner\, \zeta_1 \wedge \ldots \wedge \zeta_m$, we obtain the *Poincaré isomorphisms* $\mathrm{Ext}_B(\mathbb{R}, M)^l_s = \mathrm{Tor}_B(\mathbb{R}, M)^l_{m-s}$, cf. [9]. Owing to this duality, it will be sufficient to deal with the homologies $\mathrm{Tor}_B(\mathbb{R}, M)$, and the notation is abbreviated by writing $\mathrm{Tor}_B(\mathbb{R}, M)^l_s = H(M)^l_s$. (Note that the homologies $\mathrm{Ext}_B(\mathbb{R}, M)$ will be useful later on since they form a natural link between analytic and algebraic areas of the theory of diffieties, see [7] for a preliminary account.)

26. A survey of fundamental results. (i) $H(M)^l_0 = M^l/LM^{l-1}$ (easy), hence $H(M)^c_0 = 0$ if and only if $M^c = LM^{c-1}$ (c is fixed here) and the identity $H(M)^l_0 \equiv 0$ ($l \geqq s$) means that the B-module M is generated by the subspace $M^0 \oplus \ldots \oplus M^{c-1}$.

(ii) $H(M)^c_1 = 0$ means that for every identity $Z_1\omega_1 + \ldots + Z_m\omega_m = 0$ (where $\omega_1, \ldots, \omega_m \in M^c$ and $Z_1, \ldots, Z_m$ is a basis of L) there exist $\omega_{ij} \in M^{c-1}$ ($i, j = 1, \ldots, m$) such that $\omega_{ij} = -\omega_{ji}$ and $\omega_j \equiv \Sigma Z_i\omega_{ij}$.

(iii) $H(M)^c_{m-1} = 0$ if and only if every system of identities $Z_i\omega_j \equiv Z_j\omega_i$ $(i, j = 1, \ldots, m; \omega_1, \ldots, \omega_m \in M^c; Z_1, \ldots, Z_m$ is a basis of $L)$ is satisfied only in the case when $\omega_i \equiv Z_i\omega$ with an appropriate $\omega \in M^{c-1}$.

(iv) The space $H(M)^l_m$ consists of all products $\omega \otimes (Z_1 \wedge \ldots \wedge Z_m)$ with $\omega \in M^l$ satisfying $X\omega \equiv 0$ $(X \in L)$, hence $H(M)^c_m = 0$ if and only if the conditions $\omega \in M^c$, $X\omega \equiv 0$ $(X \in L)$ imply $\omega = 0$.

(v) If $\nu(M) = 0$ and $H(M)^l_m \equiv 0$ $(l \neq 0)$ then $M^+ = \{0\}$. Proof. $\ell(M^0 \oplus \ldots \oplus M^l) \sim \mu(M) =$ const., hence $M^l \equiv \{0\}$ for l large enough. Assume $M^c \neq \{0\}$ but $M^{c+1} = \{0\}$ and let $\omega \in M^c$, $\omega \neq 0$. Then $X\omega = 0$ for all $X \in L$, hence $H(M)^c_m \neq 0$ (cf. (iv)) and we conclude that $c = 0$.

(vi) Multiplication by Z $(Z \in B)$ naturally induces a chain mapping of the Koszul complex K, hence a B-module structure on every space $H(M)_s = \oplus H(M)^l_s$. But $Z \in B^+ = \{0\} \oplus L \oplus (L \odot L) \oplus \ldots$ operates trivially on the B-module $\mathbb{R}$, hence on the above mentioned module $H(M)_s = \oplus H(M)^l_s = \mathrm{Tor}_B(\mathbb{R}, M)^l_s$, since the multiplication behaves functorially with respect to the first argument $\mathbb{R}$ of the functor Tor. At the same time, A operates on the module $H(M)_s = \mathrm{Tor}_B(\mathbb{R}, M)$ through the second argument M and one can verify that the former B-module structure is a mere restriction of the latter A-module structure. So it follows that $H(M)_s$ is in reality a $\odot(H/L)$-module. We shall express this result in elementary terms by saying that $X \,.\, [h] \in H(M)^{l+1}_s$ for every $X \in A^1 = H$, $[h] \in H(M)^l_s$ and $X \,.\, [h] = 0$ in the particular case $X \in L$. We refer to [14] for general principles and to [11] for analogous results; nonetheless, a straightforward and elementary verification of all the above assertions is also possible.

(vii) Let $X \in H$ and assume that the multiplication $X: M^c \to M^{c+1}$ (c is fixed) is injective. By the most general principles of homological algebra, the exact sequence

$$0 \to M^c \xrightarrow{X} M^{c+1} \to M^{c+1}/XM^c \to 0$$

determines the connecting homomorphism δ of the sequence

$$(54) \qquad H(M)^{c-1}_s \xrightarrow{X\cdot} H(M)^c_s \longrightarrow H(M/XM)^c_s \xrightarrow{\delta}$$

$$\xrightarrow{\delta} H(M)^c_{s-1} \xrightarrow{X\cdot} H(M)^{c+1}_{s-1} \longrightarrow H(M/XM)^{c+1}_{s-1}$$

and the remaining mappings are obvious. (Note for certainty that the homology class $[h] \in H(M/XM)^c_s$ is represented by $h \in M^c \otimes \bigwedge^s L$ with the boundary $\partial h \in XM^c \otimes \bigwedge^{s-1} L$. Then, owing to the injectivity, the last space may be identified with the space $M^c \otimes \bigwedge^{s-1} L$ so that the boundary ∂h represents a cycle from the space $M^c \otimes \bigwedge^{s-1} L$, that is, an element from $H(M)^c_{s-1}$. This is the desired δh.) The sequence (54) is exact. We omit the (standard) verification (see [11]) specifying only the following fact needed in the proof: Assume $\omega \in M^{c-1}$, $X\omega = 0$. Since $X: M^c \to M^{c+1}$ is an injection and $XY\omega = YX\omega = 0$ for every $Y \in H$, we have $Y\omega \equiv 0$ $(Y \in H)$. As a result, the space M^{c-1} admits a direct decomposition $M^{c-1} = N \oplus \bar{N}$, where the subspace N consists of all $\omega \in M^{c-1}$ satisfying $Y\omega \equiv 0$ $(Y \in H)$ and $\bar{N}$ is a (not uniquely determined) complementary subspace with an injective mapping $X: N \to M^c$.

(viii) Continuing the preceding point, assume in addition $X \in L$. Owing to (v), we have the exact sequences

$$(55) \qquad 0 \to H(M)^c_s \to H(M/XM)^c_s \xrightarrow{\delta} H(M)^c_{s-1} \to 0$$

and $0 \to H(M)^{c+1}_{s-1} \to H(M/XM)^{c+1}_{s-1}$.

27. Lemma. *Let $w \geqq 1$. If $H(M)^c_s \equiv 0$ (c is fixed, $s = m - w, + 1, \ldots, m$), then every maximal M^c-regular sequence $X_* = X_1, \ldots, X_v$ in L is of length $v \geqq w$.*

Proof. The assumption $H(M)^c_m = 0$ implies the existence of $Y_1 \in L$ such that $Y_1: M^c \to M^{c+1}$ is an injection. (In particular, we may choose $Y_1 = X_1$ if $v \geqq 1$.)

Let $w \geqq 2$. Then the sequence (55) together with the assumptions yield the relations $H(M/Y_1M)^c_m = \ldots = H(M/Y_1M)^c_{m-w-2} = 0$ and the first of them implies the existence of $Y_2 \in L$ such that $Y_2: M_c/Y_1M_c^{-1} \to M_c^{+1}/Y_1M_c$ is an injection. (In particular, we may choose $Y_2 = X_2$ if $v \geqq 2$.)

Let $w \geqq 3$. Then the sequence (55) together with the preceding relations yield $H(M/Y_1, Y_2)\, M)^c_m = \ldots = H(M/(Y_1, Y_2)\, M)^c_{m-w-3} = 0$ and the first of them implies the existence of $Y_3 \in L$ such that $Y_3: M^c/(Y_1, Y_2)\, M^{c-1} \to M^{c+1}/(Y_1, Y_2)\, M^c$ is an injection. (In particular, we choose $Y_3 = X_3$ if $v \geqq 3$.)

Following this way, the proof is concluded after w steps.

28. Lemma. *Let $w \geqq 1$. If there exists an M_c-regular sequence $X_* = X_1, \ldots, X_w$ of length w in L, then $H(M)^c_s = 0$ for every $s = m - w + 1, \ldots, m$.*

Proof. Point (iv) of Section 26 yields

$$H(M)^c_m = H(M/X_1M)^c_{m-1} = \ldots = H(M/(X)_{w-1}M)^c_m = 0\,,$$

hence

$$H(M)^c_{m-1} = H(M/X_1M)^c_{m-1} = \ldots = H(M/(X)_{w-2}M) = 0\,, \ldots, H(M)^c_{m-w+1} = 0\,.$$

29. Theorem. *The following three assertions are equivalent:* (i) $H(M)^l_s \equiv 0$ ($l \geqq 1$, $s = m - w + 1, \ldots, m$). (ii) *Every maximal M-quasiregular sequence in L is of length at least* w. (iii) *There exists an M-quasiregular sequence in L of length* w.

30. Theorem. *The following three assertions are equivalent:* (i) $H(M)^l_s = 0$ *for all l large enough and $s = m - w + 1, \ldots, m$.* (ii) *Every maximal M-general sequence in L is of length at least w.* (iii) *There exists an M-general sequence in L of length w.*

Both theorems easily follow from Lemmas 27, 28. Note that the equivalence of points (ii) and (iii) fills the gap in Section 20 concerning the classification of characteristic subspaces. Now we pass to the last important concept discussed in this part of the paper, to the concept of involutiveness.

31. Going-down lemma. *Let* $w \geqq 1$ *and assume* $H(M)^c_m \equiv 0$ (c *is fixed*, $s = m - w + 1, \ldots, m$). *Then every* M^{c+1}*-regular sequence* $X_* = X_1, \ldots, X_w$ *in* L *is* M_c*-regular as well.*

Proof (by contradiction). Assume $X_1\omega = 0$ with $\omega \in M^c$, $\omega \neq 0$. Since $H(M)^c_m = 0$, there exists $Y \in L$ such that $\bar{\omega} = Y\omega \neq 0$. Then $X_1\bar{\omega} = YX_1\omega = 0$, which is impossible. So we conclude that $\omega \in M_c$, $\omega \neq 0$ implies $X_1\omega \neq 0$, that means, the one-term sequence X_1 is M^c-regular. Note that the assumptions of the lemma and (55) imply $H(M/X_1M)^c_m = \ldots = H(M/X_1M)^c_{m-w+2} = 0$.

Owing to the last identities, we may continue the reasoning with the module $M/(X)_1 M = M/X_1 M$ instead of M. Quite analogously as above, the assumptions $X_2\omega \in X_1M^c$, $\omega \in M_c$, $\omega \neq 0$ prove to be contradictory to the presumed M^{c+1}-regularity. It follows that X_1, X_2 is a two-term M_c-regular sequence and $H(M/(X)_2 M)^c_m = \ldots = H(M/(X)_2 M)^c_{m-w+3} = 0$.

Continuing in this way, one can verify the lemma.

32. Going-up lemma. *Assume* $H(M)^{c+1}_0 = H(M)^{c+1}_1 = 0$ *and let* $Z_1, \ldots, Z_m$ *be a basis of* L. *If the sequence* $Z_* = Z_1, \ldots, Z_m$ *is* M^c*-regular, then* Z_* *is* M^{c+1}*-regular as well.*

Proof. Our aim is to verify that the inclusions $\omega \in M^{c+1}$, $Z_k\omega \in (Z)_{k-1} M^{c+1}$ imply $\omega \in (Z)_{k-1} M_c$ for every $k = 1, \ldots, m$. We begin with the case $k = 1$, that means, we wish to prove that $\omega = 0$ provided $\omega \in M^{c+1}$ and $Z_1\omega = 0$.

First of all, according to (ii) Section 26, there exist $\omega_{ij} \in M_c$, $\omega_{ij} = -\omega_{ji}$ ($i, j = 1, \ldots, m$) for which

$$\omega = \Sigma Z_i\omega_{i1}\,, \quad 0 \equiv \Sigma Z_i\omega_{ij} \quad (j = 2, \ldots, m)\,. \tag{56}$$

In particular, $0 = \Sigma Z_i\omega_{im}$ with $\omega_{mm} = 0$, and applying the presumed M_c-regularity, we conclude that $\omega_{m-1,m} = \Sigma Z_i\pi_{im}$ (sum over $i = 1, \ldots, m-2$) for certain $\pi_{im} \in M^{c-1}$. One can then eliminate $\omega_{m-1,m}$ and $\omega_{m,m-1} = -\omega_{m-1,m}$ from the relations (56) with the following result:

$$\omega = \sum_1^{m-2} Z_i\omega_{i1} + Z_{m-1}\bar{\omega}_{m-1,1} + Z_m\bar{\omega}_{m,1}\,,$$

$$0 = \sum_1^{m-2} Z_i\omega_{ij} + Z_{m-1}\bar{\omega}_{m-1,j} + Z_m\bar{\omega}_{m,j} \quad (j = 2, \ldots, m-2)\,,$$

$$0 = \sum_1^{m-2} Z_i\bar{\omega}_{i,m-1}\,, \quad 0 = \sum_1^{m-2} Z_i\bar{\omega}_{im}\,,$$

where we denote

$$\bar{\omega}_{i,m-1} = -\bar{\omega}_{m-1,i} = \omega_{i,m-1} - Z_m\pi_{im}\,,$$

$$\bar{\omega}_{im} = -\bar{\omega}_{mi} = \omega_{im} + Z_{m-1}\pi_{im} \quad (i = 1, \ldots, m-2)\,.$$

So we have the same relations as (56) but without the terms corresponding to $\omega_{m-1,m}$

and $\omega_{m,m-1}$. Repeating this procedure, one can successively eliminate $\omega_{m,m-2} = -\omega_{m-2,m}, \ldots, \omega_{m,1} = -\omega_{1,m}$, then $\omega_{m-1,m-2} = -\omega_{m-2,m-1}, \ldots, \omega_{m-1,1} = -\omega_{1,m-1}$, and so on up to $\omega_{ij} \equiv 0$. Hence $\omega = \Sigma Z_i \omega_{i,1} = 0$ and the case $k = 1$ is concluded.

Look at the case $k = 2$. Assuming $\omega \in M^{c+1}$, $Z_2\omega \in Z_1 M^{c+1}$, we wish to prove the inclusion $\omega \in Z_1 M_c$. This is achieved by applying the preceding construction to the module $M/Z_1 M$ instead of M. It is only necessary to verify the modified assumptions $H(M/Z_1M)_0^{c+1} = H(M/Z_1M)_1^{c+1} = 0$ but they follow from (55) and from the just proved M^{c+1}-regularity of the one-term sequence Z_1.

The remaining cases $k = 3, \ldots, m$ follow quite analogously by considering the module $M/(X)_{k-1} M$ instead of M. It is necessary to verify the modified assumption $H(M/(X)_{k-1} M)_0^{c+1} = H(M/(X)_{k-1} M)_c{}^{+1} = 0$, of course.

33. Corollary. *If* $H(M)_s^c \equiv 0$ *(c is fixed,* $s = 1, \ldots, m$*) and* $H(M)_0^{c+1} = H(M)_1^{c+1} = 0$, *then* $H(M)_s^{c+1} = 0$ $(s = 0, \ldots, m)$.

Proof. Lemma 27 ensures an M^c-regular sequence $X_* = X_1, \ldots, X_m$. Slightly modifying this sequence, the M^c-regularity is preserved and $X_1 = Z_1, \ldots, X_m = Z_m$ is basis of L. According to Lemma 32, X_* is M^{c+1}-regular and Lemma 28 concludes the proof.

34. The involutive case. We shall deal only with the particular case $L = H$, $B = A$ from now on, so that M is an A-module and $H(M)_s^l = \mathrm{Tor}_A(\mathbb{R}, M)_s^l$. The module M is called *involutive* if $H(M)_s^l \equiv 0$ for every $l \geqq 1$ and arbitrary s. (Compare with [8] and [11].) Repeating use of Corollary 33 yields the result that M is an involutive module if only $H(M)_s^1 \equiv 0$ (s arbitrary) and $H(M)_1^l = H(M)_0^l \equiv 0$ for $l \geqq 1$. (This is a rather interesting result which provides a connecting link with a little different classical understanding of involutiveness, see [13], but we postpone this theme to another place.)

The equivalence of points (i), (iii) in Theorem 29 claims M to be an involutive module if and only if there exists an M-quasiregular sequence $X_* = X_1, \ldots, X_n$ of length $n = \ell(H)$. Since every M-general sequence is (M^{+c})-quasiregular for all c large enough (trivial) and M-general sequences of arbitrary length surely exist (cf. (vii) Section 9), the module M^{+c} is involutive for all c large enough. The last assertion corresponds to the classical theorem of prolongation of a system of partial equations to an involutive system; it is almost a tautology in our approach. (Note that there exists quite another proof based on homological algebra, see [11]: Since $H(M) = \oplus H(M)_s^l$ is a trivial $\mathfrak{m}$-module (cf. (vi) Section 26) and a Noetherian A-module at the same time, we have $\ell(H(M)) < \infty$, hence $H(M)_s^l \neq 0$ only for a finite number of l and s, hence $H(M)_s^{l+c} = H(M^{+c})_s^l \equiv 0$ for all $l \geqq 1$ and c large enough.)

The involutive case is rather important for the practice of calculations since they may be reduced to a few initial terms of the filtration. We shall deal with a typical example of such a reduction concerning the characteristic subspaces and M-general sequences (but there exist other examples, cf. [7]).

35. Theorem. *Let M be an involutive module, L a linear subspace of H of dimension $\ell(L) = w + u$, $w \geqq 1$. The following assertions are equivalent*: (i) $L \in \Delta^u_w(M)$. (ii) *Every maximal M-general sequence $X_* = X_1, \ldots, X_v$ in L is of length $v = w$.* (iii) *There a maximal M-general sequence in L of length w.* (iv) *The same as* (ii) *but with M-quasiregular sequence.* (v) *The same as* (iii) *but with M-quasiregular sequence.* (vi) *The same as* (ii) *but with M^1-regular sequence.* (vii) *The same as* (iii) *but with M^1-regular sequence.*

Proof. We already know that (i)–(iii) are equivalent assertions (even without the assumption of involutiveness) and Lemma 31 ensures that the concepts of an M-general and an M-quasiregular sequence are equivalent for the involutive case. Since (v) clearly implies (vii) and the points (vi), (vii) are equivalent (cf. Lemmas 27, 28), it is sufficient to verify that (vii) implies (v).

So let $X_* = X_1, \ldots, X_w$ be a maximal M^1-regular sequence in L. Slightly modifying the vectors $X_1, \ldots, X_w$ (if necessary), we may assume that they are linearly independent. According to Lemma 27, the above mentioned sequence X_* may be enlarged into an M^1-regular sequence $Z_* = Z_1\ (=X_1), \ldots, Z_w\ (=X_w),\ Z_{w+1}, \ldots, Z_n$ in the space H. We may again assume that $Z_1, \ldots, Z_n$ are linearly independent vectors. Repeating use of Lemma 32 in the particular case $L = H$ shows that Z_* is an M-quasiregular sequence. It follows that X_* is quasiregular, too, and one can easily prove that X_* is a maximal sequence of this type.

36. Theorem. *If M is an involutive module, then the concepts of M-general, M-quasiregular and M^1-regular sequence coincide.*

Proof. Lemma 31 ensures that a sequence is M-general if and only if it is M-quasiregular. Since every M-quasiregular sequence is M^1-regular, it is sufficient to prove the converse inclusion.

So let $X_* = X_1, \ldots, X_v$ be an M^1-regular sequence and abbreviate $\nu(M) = \nu$. According to (iii) Section 11, we have $\nu(M/(X)_k M) \geqq \nu - k$ for every $k \leqq \nu - 1$. Since the module M is generated by M^0 (cf. (i) Section 26), it follows that $M^1/(X)_k M^0 \neq \{0\}$ for these values k, and the M^1-regularity then implies that $X_1, \ldots, X_\nu$ (or, $X_1, \ldots, X_v$ for the simpler case $\nu > v$) are linearly independent vectots. Then, slightly modifying the reasoning of the last part of proof of Theorem 35, one can verify that $X_1, \ldots, X_\nu$ (or $X_1, \ldots, X_v$ for the case $\nu > v$) is an M-quasiregular sequence. (We repeat this reasoning for reader's convenience: According to Lemma 27, there exists an enlargement $Z_* = Z_1\ (=X_1), \ldots, Z_n$ of the sequence $X_1, \ldots, X_\nu$ (or X_v) into an M^1-regular sequence. Then, slightly modifying the last terms (if necessary), we may assume that $Z_1, \ldots, Z_n$ is a basis of H. Lemma 32 shows that Z_* is an M-quasiregular sequence and the same must be true for the original sequence $X_1, \ldots, X_\nu$ (or $X_1, \ldots, X_v$).)

The proof is concluded if $v \leqq \nu$, so assume $v > \nu$. In this case, using the exact sequence (55) together with the just proved M-quasiregularity, one can find the homologies $H(M/(X)_\nu M)^l_s \equiv 0\ (l \geqq 1)$. Point (vi) Section 26 applied to the module

$M/(X)_\nu M$ then yields the result $(M/(X)_\nu M)^+ = \{0\}$. (Note that $\nu(M/(X)_\nu M) = 0$ according to (ii) Section 11). So it is clear that *every* enlargement of the sequence $X_1,,,,,,X_\nu$ (in particular, the original seqnence X_*) is M-quasiregular.

Correction added in proof. The $\mathcal{O}\mathcal{H}$-module structure on the space $\mathcal{G}$ induces the $\mathcal{O}\mathcal{H}_z$-module structure on the localized space $\mathcal{G}_z$ only if the implication $\hat{\omega}_z = 0 \Rightarrow (X\hat{\omega})_z = 0$ is valid for every $X \in \mathcal{H}$, $\hat{\omega} \in \mathcal{G}$. Indeed, only then we may correctly define the product $X_z . \hat{\omega}_z = (X\hat{\omega})_z$. An analogous trouble appears at the end of Section 14. Fortunately, it may be removed as follows.

Lemma. *Let $\mathcal{M}$ be the Ψ_0-module of all cross-sections of a vector bundle over the base space J with finite dimensional fibers. Let $L: \mathcal{M} \to \mathcal{N}$ be a Ψ_0-homomorphism into a Ψ_0-module $\mathcal{N}$. Then $\mu_z = 0$ implies $(L\mu)_z = 0$, for any $\mu \in \mathcal{M}$, $Z \in J$.* (Hint: Consider L acting on the free generators of $\mathcal{M}$.)

The Lemma may be applied with $\mathcal{M} = \mathcal{G}^l$ (l fixed) and $L = X \in \mathcal{H}$. If the dimension of the $\mathbb{R}$-linear spaces $\mathcal{G}^1_y$ is constant for all $y \in J$ lying near z (such a point z is called *$\mathcal{G}^l$-regular*), then the assumptions are locally satisfied and the existence of the multiplication by X_z is proved. Existence of $\mathcal{G}_z$-module structure is ensured if z is $\mathcal{G}^l$-regular for all l, i.e., on a set of second Baire category in J. The localized multiplication appearing in Section 14 can be analyzed quite analogously.

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