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ON GRAPHS WITH ISOMORPHIC, NON-ISOMORPHIC AND CONNECTED N_2-NEIGHBOURHOODS

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Summary. The subgraph $N_2(u, G)$ induced by the edges xy of G for which $\min\{\varrho(x, u), \varrho(y, u)\} = 1$ is called the neighbourhood of the second type of the vertex u. In the paper three questions are studied: existence and properties of graphs with N_2-neighbourhoods isomorphic to a given graph, existence of graphs with non-isomorphic N_2-neighbourhoods and existence and properties of graphs with connected N_2-neighbourhoods.

INTRODUCTION

Let $G = (V(G), E(G))$ be a finite undirected graph without loops and multiple edges. The neighbourhood of an arbitrary vertex $u \in V(G)$ (i.e. the subgraph induced on the set of vertices adjacent to u) will be denoted by $N_1(u)$ and called the *neighbourhood of the first type* of u. Following [2] let us denote by $N_2(u, G)$ (or, briefly, $N_2(u)$), the *neighbourhood of the second type of* u, i.e. the subgraph of G with the set of edges containing all the edges vw of G for which $\min\{\varrho(v, u), \varrho(w, u)\} = 1$ and with the corresponding set of vertices ($\varrho(x, y)$ denotes the distance of vertices x, y). Then the following questions can be formulated:

1. Given a graph G, does there exist $\tilde{G}$ such that for every vertex $u \in V(\tilde{G})$, $N_i(u)$ is isomorphic to G? (For $i = 1$ this is the well-known Trahtenbrot-Zykov problem, see e.g. [1], [3], [4], [5], [6].)

2. Does there exist a graph G such that for every $u, v \in V(G)$ the neighbourhoods $N_i(u)$ and $N_i(v)$ are non-isomorphic? (For $i = 1$ see [2], for 2-neighbourhoods defined as subgraphs induced on the sets of vertices at distance 2 see [7], [8].)

3. What are sufficient conditions for G to be N_i-locally connected and what are the properties of N_i-locally connected graphs? (G is said to be *N_i-locally connected* if for every $v \in V(G)$ the neighbourhood $N_i(v)$ is a connected graph. For $i = 1$ see [9], [10].)

Investigation of these questions for $i = 2$ is the main aim of the present paper.

1. N_2-REALIZABLE GRAPHS

We say that a graph G is *N_2-realizable* if there exists a nonempty graph $\tilde{G}$ (called an *N_2-realization* of G) such that for every vertex $u \in V(\tilde{G})$, $N_2(u, \tilde{G})$ is isomorphic to G. We can assume without loss of generality that $\tilde{G}$ is connected.

An N_2-realizable graph obviously cannot contain isolated vertices. Let us observe some other properties of N_2-realizable graphs. Denote by $\Delta(G)$ ($\delta(G)$) the maximum (minimum) degree of G.

Theorem 1.1. *If $\tilde{G}$ is an N_2-realization of G, then*

$$\Delta(G) \leqq \Delta(\tilde{G}) \leqq \Delta(G) + 1 .$$

If moreover $\delta(G) \geqq 2$, then

$$\delta(\tilde{G}) \geqq \delta(G) + 1 .$$

Proof. 1. Obviously $\Delta(G) \leqq \Delta(\tilde{G})$. Suppose that there exists $u \in V(\tilde{G})$ such that $d_{\tilde{G}}(u) \geqq \Delta(G) + 2$ ($d_{\tilde{G}}(u)$ denotes the degree of u in $\tilde{G}$). Then $N_2(v, \tilde{G})$ for arbitrary v adjacent to u contains a vertex of degree at least $\Delta(G) + 1$ and therefore cannot be isomorphic to G.

2. Suppose that there exists $u \in V(\tilde{G})$ such that $d_{\tilde{G}}(u) \leqq \delta(G)$ and consider again $N_2(v, \tilde{G})$ for an arbitrary $v \in V(\tilde{G})$ adjacent to u. Then the following two possibilities can occur:

a) $N_2(v)$ does not contain u. Then $d_{\tilde{G}}(u) = 1$ and since $\delta(N_2(u)) \geqq 2$, necessarily $d_{\tilde{G}}(v) \geqq 3$. Therefore v is adjacent to another vertex $w \neq u$ and it is easily seen that w has degree 1 in $N_2(u)$ which is a contradiction.

b) $N_2(v)$ contains u. Then

$$\delta(G) \leqq d_{N_2(v)}(u) = d_{\tilde{G}}(u) - 1 \leqq \delta(G) - 1$$

which is again a contradiction.

Corollary. *An N_2-realization of a regular graph of degree $d \geqq 2$ is a regular graph of degree $d + 1$.*

A set $M \subset V(G)$ is said to be a *covering set*, if every edge of G has at least one vertex in M. The minimum number of vertices in a covering set will be denoted by $\alpha(G)$.

Theorem 1.2. *If G is N_2-realizable then $\alpha(G) \leqq \Delta(G) + 1$.*

Proof. If $G = N_2(u)$ then every edge of G has at least one vertex adjacent to u and hence the set of all vertices of G which are adjacent to u is a covering set. The proof is completed by using Theorem 1.1.

Corollaries. 1. *If G is N_2-realizable then*

$$|E(G)| \leqq \Delta(G) \, . \, (\Delta(G) + 1)$$

($|M|$ denotes the number of elements of M).

Proof: One vertex can cover not more $\Delta(G)$ edges, hence $|E(G)| \leqq \alpha(G) \, . \, \Delta(G)$ and we can use Theorem 1.2.

2. *If G is an N_2-realizable regular graph of degree d then $|V(G)| \leqq 2(d + 1)$.*

Proof. Use Corollary 1 for $\Delta(G) = d$, $|E(G)| = \frac{1}{2}|V(G)| \, . \, d$.

3. a) *For $n \geqq 7$ the circuit C_n is not N_2-realizable.*

b) *If G is a cubic (i.e. regular of degree 3) graph and $|V(G)| \geqq 9$ then G is not N_2-realizable.*

c) *For $n \geqq 7$ the path P_n is not N_2-realizable.*

Denote by $g(G)$ the *girth* of G, i.e. the length of the shortest circuit in G (if G contains no circuits, put $g(G) = \infty$).

Theorem 1.3. *Suppose $\tilde{G}$ is an N_2-realization of $G \dotplus C_3$ and $\delta(\tilde{G}) \geqq 3$. Then G contains a path of length 3 if and only if $g(\tilde{G}) \leqq 4$.*

Proof. Let $P \subset \tilde{G}$ be a path of length 3, $P \subset N_2(u)$. Then the vertices of P adjacent to u together with u determine in $\tilde{G}$ a circuit of length at most 4. The converse is evident.

Theorem 1.4. *If G is an N_2-realizable regular graph of degree $d \geqq 2$ then G is 2-connected.*

Proof. 1. Suppose G is disconnected. For every regular graph G' of degree d we obviously have $|V(G')| \geqq d + 1$ which together with Corollary 2 of Theorem 1.2 shows that G has 2 components (each of them on $d + 1$ vertices) and hence $G = 2K_{d+1}$. From $\alpha(K_n) = n - 1$ and from Theorem 1.2 it follows that G is not N_2-realizable.

2. Suppose G has an articulation (cutvertex) x. Since each of the blocks of G has (including x) at least $d + 1$ vertices and $|V(G)| \leqq 2(d + 1)$, necessarily one of the blocks of G has exactly $d + 1$ vertices. Hence the degree-sequence of this block is

$$\underbrace{d, d, \ldots, d}_{d\text{-times}}, \alpha$$

for some $\alpha < d$, which can be easily proved to be impossible.

We shall further use the following simple assertion:

Theorem 1.5. *Suppose $|E(G)| \geqq 1$ and let $\tilde{G}, \tilde{\tilde{G}}$ be N_2-realizations of G such that $\tilde{G} \subset \tilde{\tilde{G}}$. Then $\tilde{G} = \tilde{\tilde{G}}$.*

Proof is easy.

One can easily observe that the unique N_2-realization of the complete graph K_n for $n > 2$ is K_{n+1}. (Here and in the sequel the term "unique" is meant up to isomorphism.) Let us consider N_2-realizability of some other classes of graphs.

Theorem 1.6. *The circuits C_3, C_5, C_6 have a unique N_2-realization while C_4 and C_n for $n \geqq 7$ are not N_2-realizable.*

Proof. $n = 3$: Let $N_2(u) \simeq C_3$ ($\simeq$ denotes isomorphism). We have (up to isomorphism) the following two possibilities: $d_{\tilde{G}}(u) = 2$ or $d_{\tilde{G}}(u) = 3$. In the first case we obtain an N_2-realization of C_3 isomorphic to K_4, in the second case considering $N_2(v)$ of any vertex v adjacent to u we are led again to an N_2-realization isomorphic to K_4.

$n = 4$: Let $N_2(u) \simeq C_4$. Then some two non-adjacent vertices v_1, v_2 of C_4 must be joined with u by an edge, which implies that v_2 has degree 3 in $N_2(v_1)$ – a contradiction.

$n = 5$: If $N_2(u) \simeq C_5$ then there necessarily exist three vertices v_1, v_2, v_3 on C_5 such that (say) v_1 is not adjacent to v_2 and v_3 but v_2 is adjacent to v_3 and all of them are adjacent to u. Considering $N_2(v_1)$ and using Theorem 1.5 we obtain the only possible N_2-realization to be $C_3 \times P_1$, i.e. the graph of the trigonal prism.

$n = 6$: Similarly as in the preceding case it can be proved that the only N_2-realization of C_6 is the graph of the 3-dimensional cube.

For $n \geqq 7$ see Corollary 3a of Theorem 1.2.

A vertex $u \in V(G)$ is said to be *universal* if it is adjacent to all other vertices of G.

Theorem 1.7. *If G has exactly one universal vertex and $|V(G)| = n \geqq 4$, then one of the following possibilities occurs:*

a) *$G \simeq K_{1,n-1}$ and G is uniquely N_2-realizable;*

b) *n is odd, $G \simeq K_{\underbrace{2,2,\dots,2}_{\frac{1}{2}(n-1) \text{ times}},1}$ and G has the unique*

N_2-realization $\tilde{G} \simeq K_{\underbrace{2,2,\dots,2}_{\frac{1}{2}(n+1) \text{ times}}}$;

c) *G is not N_2-realizable.*

Proof. Suppose that $N_2(u_0) \simeq G$ has n vertices $u_1, \dots, u_n$, u_1 is universal in $N_2(u_0)$ and $\tilde{G}$ is an N_2-realization of G.

Case 1. Suppose u_0, u_1 are adjacent in $\tilde{G}$. Then the neighbourhood $N_2(u_1)$ must have a universal vertex and without loss of generality we may assume that it is u_0. If there exists a vertex u_k ($k \neq 0, 1$) which is adjacent to both u_0 and u_1 then an easy consideration shows that both u_0 and u_1 are universal in $N_2(u_k)$ which is a contradiction. Hence no u_k is adjacent to both u_0 and u_1 and by considering $N_2(u_1)$ and using Theorem 1.5 it is seen that the only possble $\tilde{G}$ is the "double-star", i.e. the tree consisting of the edge u_0u_1, $n-1$ edges u_ku_1 for $2 \leqq k \leqq n$ and $n-1$ other edges adjacent to u_0; the resulting graph is an N_2-realization of the star $K_{1,n-1}$.

Case 2. If u_0, u_1 are not adjacent in $\tilde{G}$ then the universality of u_1 in $N_2(u_0)$ implies that u_0 is adjacent to all u_i for $i = 2, \ldots, n$. Now the neighbourhood $N_2(u_i)$ for every $i = 0, 1, \ldots, n$ has exactly n vertices and hence $\tilde{G}$ cannot have any other vertices. We shall prove by induction the following assertion:

Lemma. *Let l be an integer such that $1 \leqq l \leqq \frac{1}{2}(n-1)$. If each of the graphs $N_2(u_i)$, $i = 0, \ldots, 2l-1$ contains exactly one universal vertex then all pairs of vertices u_i, u_j for $0 \leqq i \leqq 2l-1$ are adjacent in G except the pairs u_{2k}, u_{2k+1} for $k = 0, 1, \ldots, l-1$.*

Proof. For $l = 1$ the lemma holds evidently. Suppose that $l \leqq \frac{1}{2}(n-1)$ and the assertion of our lemma is true for $l-1$ – therefore the pairs of vertices u_{2k}, u_{2k+1} are not adjacent for $k = 0, 1, \ldots, l-2$. This implies that none of the vertices u_i for $i < 2l-2$ can be universal in $N_2(u_{2l-2})$; hence this universal vertex must be one of u_i for $2l-1 \leqq i \leqq n$ and we may assume without loss of generality that it is u_{2l-1}. Hence u_{2l-1} is adjacent in $\tilde{G}$ to all u_j for $2l \leqq j \leqq n$ and therefore the vertices u_{2l-2}, u_{2l-1} cannot be adjacent in $\tilde{G}$ (since in the other case u_{2l-1} would be another universal vertex in $N_2(u_0)$. This implies that all the pairs u_{2l-2}, u_j for $2l \leqq j \leqq n$ are adjacent in $\tilde{G}$ and the lemma is proved.

Case 2a. n is odd. Using our lemma for $l = \frac{1}{2}(n-1)$ and observing that the vertices u_{n-1}, u_n cannot be adjacent in $\tilde{G}$ (since otherwise both u_{n-1} and u_1 would be universal in $N_2(u_0)$) it is proved that the only possibility is $G \simeq K_{\underbrace{2,2,\ldots,2}_{\frac{1}{2}(n+1) \text{ times}}}$.

Case 2b. n is even. Then using the lemma for $l = \frac{1}{2}(n-2)$ and considering $N_2(u_{n-2})$ we conclude that one of u_{n-1}, u_n (say u_{n-1}) must be universal in $N_2(u_{n-2})$. Then u_{n-1}, u_n and one of the pairs of vertices u_{n-2}, u_{n-1} and u_{n-2}, u_n must be adjacent. But in the first case u_{n-1} and in the other case u_n is another universal vertex in $N_2(u_0)$. This contradiction proves the non-existence of an N_2-realization.

Corollary. *The wheels W_3 and W_4 are uniquely N_2-realizable while W_n for $n \geqq 5$ is not N_2-realizable (wheel W_n is C_n together with an aditional univesal vertex).*

Proof. $\tilde{W}_3 \simeq K_5$ since $W_3 \simeq K_4$, $\tilde{W}_4 \simeq K_{2,2,2}$ since $W_4 \simeq K_{2,2,1}$, for $n \geqq 5$ use Theorem 1.7.

Theorem 1.8. *Let G be a disjoint union of stars, i.e.*

$$G = \bigcup_{i=1}^{n} K_{k_i,1}, \quad k_i \geqq 2, \quad i = 1, \ldots, n, \quad n \geqq 2.$$

Then G is N_2-realizable if and only if $k_1 = k_2 = \ldots = k_n = n - 1$ and in this case G has infinitely many non-isomorphic N_2-realizations.

Proof. Suppose G is N_2-realizable. First observe that if $\tilde{G}$ is an N_2-realization of G then an arbitrary vertex $u \in V(\tilde{G})$ is adjacent in $\tilde{G}$ to all centers of components of G and to no other vertices: if some end-vertex v of G were adjacent to u in $\tilde{G}$ then its neighbourhood $N_2(v)$ should contain a path of length 3 which is a contradiction. Hence $\tilde{G}$ is a regular graph of degree n and therefore necessarily $k_1 = k_2 = \ldots \ldots = k_n = n - 1$.

Conversely, suppose $k_1 = k_2 = \ldots = k_n = n - 1$. Then $G = nK_{n-1,1}$ and according to Theorem 1.3, G is N_2-realized by an arbitrary regular graph $\tilde{G}$ of degree n such that $g(\tilde{G}) \geqq 5$. Existence of an infinite family of such graphs is proved in [12], Chapter III, Theorem 1.4′.

Denote by P_k, $k \geqq 1$, the path of length k, i.e. with k edges and $k + 1$ vertices.

Theorem 1.9. *Let G be a disjoint union of paths, i.e. $G = \bigcup_{i=1}^{n} P_{k_i}$, $k_i \geqq 1$, $i = 1, \ldots, n$, $n \geqq 1$. Then G is N_2-realizable only in the following cases:*

n (*number of paths*)	k_i $(i = 1, \ldots, n)$ (*lengths of paths*)	number of *non-isomorphic N_2-realizations*
1	1	2
	2	2
	3	1
	6	∞
2	1, 1	∞
	2, 3	∞
	2, 4	∞
3	2, 2, 2	∞

Proof. If $\tilde{G}$ N_2-realizes G then according to Theorem 1.2 necessarily $\alpha(G) \leqq 3$. Hence $n \leqq 3$ and it remains to consider the following possibilities: for $n = 1$: $k = 1, 2, 3, 4, 5, 6$; for $n = 2$: $k_i = 1, 1$; $1, 2$; $1, 3$; $1, 4$; $2, 2$; $2, 3$; $2, 4$; for $n = 3$: $k_i = 1, 1, 1$; $1, 1, 2$; $1, 2, 2$; $2, 2, 2$.

Case $n = 1$. Non-realizability of P_4 and P_5 is proved and examples of N_2-realizations of P_1, P_2, P_3 and P_6 are given in [2]. It remains to prove the assertion concerning the number of N_2-realizations.

a) Let $N_2(u) \simeq P_1$. Then u is adjacent either to one of the vertices of P_1 or to both of them. In virtue of Theorem 1.5 the first case yields C_3 and the second case yields P_3 as the only possible N_2-realizations.

b) Let $N_2(u) \simeq P_2$, let v_1, v_2, v_3 be the three vertices of P_2. We have (up to isomorphism) the following four possibilities: u is adjacent to v_2; u is adjacent to v_1 and v_2; u is adjacent to v_1 and v_3; u is adjacent to v_1, v_2 and v_3. In the first case considering $N_2(v_2)$ we obtain the first N_2-realization of P_2 which is a tree on 6 vertices with exactly 2 of them of degree 3 while in the third case we obtain C_4 as the second possible N_2-realization of P_2. The second and fourth cases imply a contradiction.

c) Let $N_2(u) \simeq P_3$. In a similar manner as in the preceding case it can be proved that the N_2-realization which is shown in [2] (i.e. the circuit C_5 with one diagonal edge) is the only one.

d) In [2] it is shown that P_6 is N_2-realized by the graph of the m-gonal prism $C_m \times P_1$ for arbitrary $m \geqq 5$.

Case $n = 2$. a) C_m N_2-realizes $2P_1$ for an arbitrary $m \geqq 5$.

b) Suppose $N_2(u) \simeq P_1 \cup P_2$, $V(P_1) = \{v_1, v_2\}$, $V(P_2) = \{w_1, w_2, w_3\}$. According to Theorem 1.1 $d_{\tilde{G}}(u) \leqq 3$ and hence we obtain the following three possibilities:

– u is adjacent to w_2 and one of v_i's (say v_1);

– u is adjacent to w_1, w_3 and one of v_i's (say v_1);

– u is adjacent to w_2, v_1 and v_2.

The last two cases immediately imply a contradiction while in the first case the condition $N_2(w_2) \simeq P_1 \cup P_2$ implies that either one of the vertices w_1, w_3 must have degree 1 or they are joined by another path P_2. In both of these cases considering $N_2(w_1)$ we obtain a contradiction.

c) Let $N_2(u) \simeq P_1 \cup P_3$. Then necessarily $d_{\tilde{G}}(u) = 3$. Since $N_2(u) \supset P_1$, the vertex u is adjacent to some vertex v of degree 2 in $\tilde{G}$ and hence $N_2(v)$ cannot be isomorphic to $P_1 \cup P_3$.

d) Non-realizability of $P_1 \cup P_4$ can be proved similarly.

e) Non-realizability of $2P_2 \simeq 2K_{2,1}$ follows from Theorem 1.8.

f) An N_2-realization of the graph $P_2 \cup P_3$ can be constructed by using an arbitrary connected regular graph of degree 3 and replacing each of its vertices by C_3.

g) An N_2-realization of the graph $P_2 \cup P_4$ can be constructed in a similar manner as in the above case by using a connected regular graph of degree 4 and the circuit C_4.

Case $n = 3$. a) If $N_2(u)$ is a graph with 3 components and one of them is P_1 then u is adjacent to some vertex v such that $d_{\tilde{G}}(v) = 2$ and hence $N_2(v)$ cannot be a graph with 3 components. Hence the graphs $3P_1, 2P_1 \cup P_2$ and $P_1 \cup 2P_2$ are not N_2-realizable.

b) $3P_2 \simeq 3K_{2,1}$ has infinitely many N_2-realizations according to Theorem 1.8.

Theorem 1.10. *The complete bipartite graph $K_{m,n}$ is N_2-realizable if and only if either* $\min\{m, n\} = 1$ *or* $|m - n| = 1$. *The graphs $K_{1,1}$ and $K_{1,2} \simeq K_{2,1}$ have exactly two non-isomorphic N_2-realizations while in the other cases the N_2-realization of $K_{m,n}$ is unique.*

Proof. The assertion concerning $K_{1,1} \simeq P_1$ and $K_{1,2} \simeq K_{2,1} \simeq P_2$ follows from Theorem 1.9 while the assertion concerning $K_{1,n} \simeq K_{n,1}$ for $n \geqq 3$ follows from Theorem 1.7.

Let $\tilde{G}$ be an N_2-realization of $G = K_{m,n}$, $u_0 \in V(\tilde{G})$, $N_2(u_0) \simeq K_{m,n}$, $m \geqq 2$, $n \geqq 2$. Let $A = \{a_1, \dots, a_m\}$, $B = \{b_1, \dots, b_n\}$ be the two classes of vertices of $K_{n,m}$. Then u_0 is adjacent either to all a_i's or to all b_j's since otherwise for a pair of vertices a_{i_0}, b_{j_0} such that none of them is adjacent to u_0 the edge $a_{i_0}b_{j_0}$ would not be in $N_2(u_0)$. Further, u_0 is adjacent either to all a_i's and no b_j's or to all b_j's and no a_i's since in the first case for b_{j_0} adjacent to u_0 the neighbourhood $N_2(a_1)$ would contain the circuit of length 3 with vertices a_2, b_{j_0}, u_0; the second case is similar. Consequently, in the first case $a \in A \Rightarrow N_2(a) \simeq K_{m-1,n+1}$, $b \in B \Rightarrow N_2(b) \simeq N_2(u_0) \simeq K_{m,n}$ and hence $m - n = 1$ and $\tilde{G} \simeq K_{m,m}$; in the second case $a \in A \Rightarrow N_2(a) \simeq N_2(u_0) \simeq$ $\simeq K_{m,n}$, $b \in B \Rightarrow N_2(b) \simeq K_{m+1,n-1}$ and hence $n - m = 1$ and $\tilde{G} \simeq K_{n,n}$.

Theorem 1.11. *The only N_2-realizable cubic (i.e. regular of degree 3) graphs are the tetrahedron K_4, the trigonal prism $C_3 \times P_1$ and the 3-dimensional cube Q_3, and each of them has a unique N_2-realization.*

Proof. The only cubic graph with four vertices is the uniquely N_2-realizable tetrahedron K_4. For $|V(G)| = 6$ there exist 2 non-isomorphic cubic graphs, namely

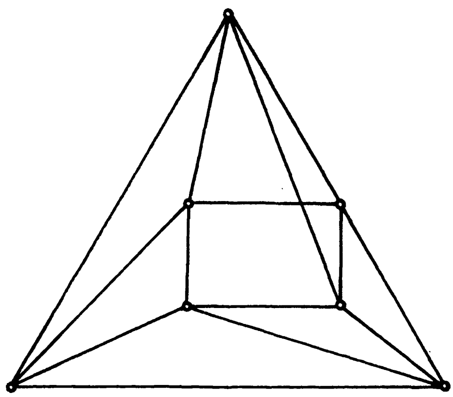

Fig. 1

$K_{3,3}$ and the trigonal prism $C_3 \times P_1$. $K_{3,3}$ is not N_2-realizable according to Theorem 1.10. Suppose $N_2(u) \simeq C_3 \times P_1$, $u_{i,j}$ $(i = 1, 2, 3,\ j = 1, 2)$ being its vertices.

Necessarily $d_{\tilde{G}}(u) = 4$ and hence the only (up to isomorphism) possibility is that $u_{1,1}, u_{2,1}, u_{2,2}$ and $u_{3,2}$ are adjacent to u (these vertices must form a covering set). The condition $N_2(u_{1,1}) \simeq C_3 \times P_1$ then implies that the vertices $u_{3,1}$ and $u_{1,2}$ are adjacent in $\tilde{G}$ and hence we have obtained the unique N_2-realization which is shown in Fig. 1.

If $|V(G)| = 8$ then $\alpha(G) = 4$ according to Theorem 1.2 and hence G is necessarily bipartite. The only bipartite cubic graph with 8 vertices in the 3-dimensional cube the N_2-realization of which is shown in Fig. 2. The proof of uniqueness is similar to the preceding case.

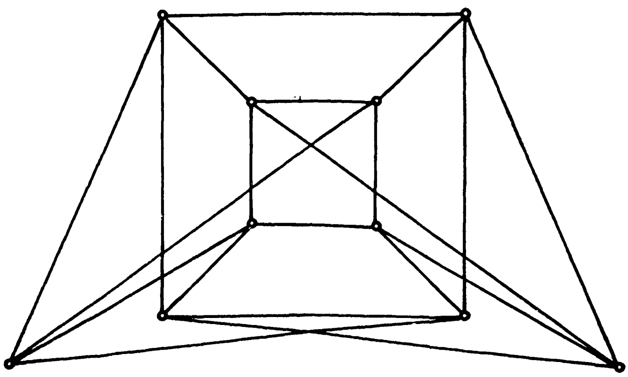

Fig. 2

For $|V(G)| > 8$ see Corollary 3b of Theorem 1.2.

Corollaries.

1. *The only N_2-realizable cube Q_n is the* 3-*dimensional one.*

Proof. For $Q_2 \simeq C_4$ and Q_3 see Theorems 1.6 and 1.11. Q_n is not N_2-realizable for $n \geqq 4$ according to Corollary 2 of Theorem 1.2 since Q_n is regular of degree n and $|V(Q_n)| = 2^n > 2(n + 1)$.

2. *The only N_2-realizable graphs of Platonic bodies are the tetrahedron and the cube, and their N_2-realizations are unique.*

Proof. N_2-realizability of the tetrahedron and the cube and non-realizability of the dodecahedron is established by the preceding theorem. The icosahedron is not N_2-realizable since it has no covering with at most 6 vertices. Suppose G is the graph of the octahedron, $\tilde{G}$ its N_2-realization, $u \in V(\tilde{G})$, $N_2(u) \simeq G$. Necessarily $d_{\tilde{G}}(u) = 5$; hence we may denote by u_1 the vertex of G which is not adjacent to u, by u_6 the only vertex of G which is not adjacent to u_1, and by u_2, u_3, u_4, u_5 the other vertices of G. $\tilde{G}$ is regular of degree 5 and hence u_1 is necessarily adjacent in $\tilde{G}$ either to u_6 or to another vertex v, but it can be shown that both of these possibilities lead to a contradiction.

2. GRAPHS WITH NON-ISOMORPHIC N_2-NEIGHBOURHOODS

Following [2] let us denote by $\mathfrak{G}_2$ the class of graphs with the following property: for every pair of vertices u, v of G the neighbourhoods $N_2(u)$ and $N_2(v)$ are not isomorphic.

Theorem 2.1. *Let n be an integer. Then there exists a connected graph G_n on n vertices belonging to $\mathfrak{G}_2$ if and only if $n \geqq 7$.*

We shall first prove some auxiliary assertions.

Lemma 1. *Let $n \geqq 7$, $G_n \in \mathfrak{G}_2$, suppose that G_n is connected, none of the vertices $u_1, \ldots, u_n$ of G_n is universal and the only vertex which is adjacent to u_n is u_{n-2}. Let us construct a graph G_{n+1} on $n + 1$ vertices from G_n by adding a vertex u_{n+1} and making it universal in G_{n+1}. Then $G_{n+1} \in \mathfrak{G}_2$, G_{n+1} is connected and u_n is adjacent only to u_{n-2} and u_{n+1}.*

Proof. Suppose that $f: N_2(u_\alpha, G_{n+1}) \rightarrow N_2(u_\beta, G_{n+1})$ is an isomorphism. Without loss of generality we may assume that $\alpha \neq n + 1$ and hence $u_{n+1} \in V(N_2(u_\alpha, G_{n+1}))$. If $f(u_{n+1}) = u_{n+1}$ then the partial mapping $f|_{V(G_n)}$ is an isomorphism $N_2(u_\alpha, G_n)$ onto $N_2(u_\beta, G_n)$. Hence $f(u_{n+1}) = u_\gamma$, $\gamma \leqq n$ and u_γ is universal in $N_2(u_\beta, G_{n+1})$. If $\beta = n + 1$ then $N_2(u_\beta, G_{n+1}) = G_n$ and u_γ is universal in G_n. Hence $\beta \leqq n$ and therefore u_{n+1} is the second universal vertex in $N_2(u_\beta, G_{n+1})$. Interchanging these two universal vertices we obtain an isomorphism $f_1: N_2(u_\alpha, G_{n+1}) \rightarrow N_2(u_\beta, G_{n+1})$ such that $f_1(u_{n+1}) = u_{n+1}$, which is a contradiction.

Lemma 2. *Let $n \geqq 7$, $G_n \in \mathfrak{G}_2$, $V(G_n) = \{u_1, \ldots, u_n\}$, suppose that u_n is universal in G_n, the only vertex of degree 1 in $N_2(u_n, G_n)$ is u_{n-1} and u_{n-1} is adjacent only to u_{n-3} and u_n. Let us construct a graph G_{n+1} on $n + 1$ vertices from G_n by adding a vertex u_{n+1} and joining it to u_{n-1} by an edge. Then $G_{n+1} \in \mathfrak{G}_2$, G_{n+1} is connected and has no universal vertex.*

Proof. The vertex u_n is universal in G_n and hence all vertices of G_n have (by assumption, non-isomorphic) N_2-neighbourhoods on $n - 1$ vertices. The only vertices u_i of G_{n+1} for which $N_2(u_i, G_{n+1}) \neq N_2(u_i, G_n)$ are evidently u_{n-3} and u_n (and, of course, u_{n+1}). $N_2(u_{n+1}, G_{n+1})$ has 3 vertices while both $N_2(u_{n-3}, G_{n+1})$ and $N_2(u_n, G_{n+1})$ have n vertices. Suppose that there exists an isomorphism $f: N_2(u_n, G_{n+1}) \rightarrow N_2(u_{n-3}, G_{n+1})$. By assumption, the only vertex of degree 1 in both $N_2(u_n, G_{n+1})$ and $N_2(u_{n-3}, G_{n+1})$ is u_{n+1}. Hence the partial mapping $f|_{V(G_n)}$ is an isomorphism of $N_2(u_n, G_n)$ onto $N_2(u_{n-3}, G_n)$, which is a contradiction.

Proof of Theorem 2.1. The non-existence of $G_n \in \mathfrak{G}_2$ for $n \leqq 6$ can be easily verified by listing all such graphs (see e.g. [13]). For $n \geqq 7$ let us construct a graph G_n using the following construction:

– for $n = 7$ see Fig. 3;

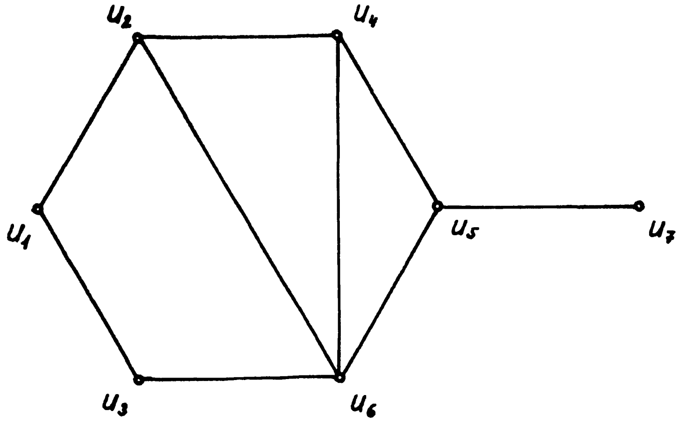

Fig. 3

– having obtained G_n, construct G_{n+1} using Lemma 1 if n is odd and Lemma 2 if n is even.

Then G_{n+1} is connected and $G_{n+1} \in \mathfrak{G}_2$.

Theorem 2.2. *Let n, k be integers, $k \geqq 1$, $n \geqq k^2 + 5k + 1$. Then there exists a graph $G \in \mathfrak{G}_2$ with n vertices and k components.*

Proof. Let us define a graph G using the graphs G_n which are described in the proof of Theorem 2.1:

– the first component of G is G_7,

– the i-th component of G is G_{2i+4}, $i = 2, \ldots, k$.

Then every component of G belongs to $\mathfrak{G}_2$ and since for every pair of vertices u_1, u_2 which belong to different components of G their N_2-neighbourhoods have different numbers of vertices, necessarily $G \in \mathfrak{G}_2$. Further, $n = 7 + \sum_{i=2}^{k}(2i + 4) = k^2 + 5k + 1$ and hence for $n = k^2 + 5k + 1$ the theorem is proved.

For $n > k^2 + 5k + 1$ take the same graph G with the only difference in the k-th component: if we denote $a = n - (k^2 + 5k + 1)$ then it is constructed as G_{2k+4+a} if a is even and as a graph which can be obtained from G_{2k+3+a} by adding a new vertex and joining it to the only universal vertex of G_{2k+3+a} if a is odd.

3. N_2-LOCALLY CONNECTED GRAPHS

Theorem 3.1. *Let G be a connected N_2-locally connected graph, suppose that G*

contains a path of length 4. *Denote by G' the graph which is obtained from G by deleting all vertices of degree* 1 *together with their edges. Then every edge of G' is contained in some circuit of length $m \leqq 4$ and G' is 2-connected.*

Proof. Let h be an edge of G'. Each of its vertices is adjacent to another edge – denote them by h_1, h_2. If h_1, h_2 have a common vertex then h is contained in a triangle h, h_1, h_2. Suppose that h_1, h_2 have no common vertex and that in G there is no circuit of length $m \leqq 4$ containing h. Then the existence of path of length 4 in G and the connectedness of G yield that in G there exists a path of length 4 such that if u_0, u_1, u_2, u_3, u_4 are its vertices then $h = u_1u_2$. The neighbourhood $N_2(u_2, G)$ then contains the edges u_0u_1 and u_3u_4. Suppose that in G there is no circuit of length $m \leqq 4$ containing h. Hence if a vertex v is adjacent to u_1 and w is adjacent ot u_2 then v cannot be adjacent to w and therefore the edges u_0u_1 and u_3u_4 are in different components of $N_2(u_2, G)$.

Let u be an articulation of G'. Then u is an articulation of G and such edges h_1, h_2 can be found that h_1, h_2 are in different blocks of G and none of them is adjacent to u (since otherwise u would not be an articulation of G'). But then $N_2(u, G)$ is disconnected, which is a contradiction.

Obviously, every N_1-locally connected graph G is N_2-locally connected and hence the assertions which are proved in [9], [10] can be used to obtain sufficient conditions for G to be N_2-locally connected. Nevertheless, some of them can be replaced by weaker ones.

Theorem 3.2. *Every graph which contains no path of length* 4 *is N_2-locally connected.*

Proof is easy.

Theorem 3.3. *Let G be a graph such that every pair u, v of non-adjacent vertices satisfies the inequality*

$$d_G(u) + d_G(v) \geqq |V(G)| .$$

Then G is N_2-locally connected.

Proof. Let $u_0 \in V(G)$ and suppose that $N_2(u_0, G)$ is disconnected. Choose vertices u_1, u_2 in different components of $N_2(u)$ so that they are adjacent to u_0. Each of the vertices u_1, u_2 is adjacent to $d_G(u_i) - 1$ vertices (excluding u_0) and these vertices are necessarily different. Hence

$$|V(G)| \geqq (d_G(u_1) - 1) + (d_G(u_2) - 1) + 3$$

which implies

$$d_G(u_1) + d_G(u_2) \leqq |V(G)| - 1 ,$$

a contradiction.

Example. The graph G which can be obtained by taking two disjoint copies of K_n, $n \geqq 2$, and joining their vertices with an additional universal vertex u, is not N_2-locally connected and every pair x, y its of vertices such that $x \neq u$ and $y \neq u$ satisfies $d_G(x) + d_G(y) = 2n < 2n + 1 = |V(G)|$. Hence Theorem 3.3 is the best possible.

Corollary. *If $\delta(u) \geqq \frac{1}{2}|V(G)|$ then G is N_2-locally connected.*

Theorem 3.4. *Let G be a graph without triangles and such that*

$$\sum_{u \in V(P)} d_G(u) \geqq |V(G)| + 2$$

for every path $P \subset G$ of length 2. Then G is N_2-locally connected.

Proof. Let u_0, u_1, u_2 be the same as in the proof of Theorem 3.3. Then u_0 is adjacent to $d_G(u_0)$ vertices and each of the vertices u_1, u_2 is adjacent to another $d_G(u_i) - 1$ vertices. These vertices are different since $N_2(u_0, G)$ is disconnected and G has no triangles. Hence

$$|V(G)| \geqq d_G(u_0) + d_G(u_1) - 1 + d_G(u_2) - 1 + 1$$

which yields

$$\sum_{i=0}^{2} d_G(u_i) \leqq |V(G)| + 1 ,$$

a contradiction.

Corollary. *Suppose that G is a graph without triangles for which one of the following conditions is fulfilled:*

a) *for every pair of vertices u, v,*

$$d_G(u) + d_G(v) \geqq \tfrac{2}{3}(|V(G)| + 2) ;$$

b)
$$\delta(G) \geqq \tfrac{1}{3}(|V(G)| + 2).$$

Then G is N_2-locally connected.

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References

[1] *A. A. Zykov:* Problem 30, Theory of graphs and its applications. Proc. Symp. Smolenice 1963 (*M. Fiedler*, ed.), Prague 1964, 164–165.

[2] *J. Sedláček:* Local properties of graphs. Čas. pěst. mat. *106* (1981), 290–298.

[3] *V. K. Bulitko:* On graphs with given vertex-neighbourhoods. Trudy mat. inst. im. Steklova *133* (1973), 78—94.
[4] *P. Hell:* Graphs with given neighbourhoods I. Problèmes Combinatoires et Théorie des Graphes (Colloq. Orsay 1976), C.N.R.S., Paris 1978, 219—223.
[5] *J. I. Hall:* Locally Petersen graphs. J. Graph Theory *4* (1980), 173—187.
[6] *A. Vince:* Locally homogeneous graphs from groups. J. Graph Theory *5* (1981), 417—422.
[7] *H. Bielak:* On graphs with non-isomorphic 2-neighbourhoods. Čas. pěst. mat. *108* (1983), 294—298.
[8] *H. Bielak, E. Soczewińska:* Some remarks about digraphs with non-isomorphic 1- or 2-neighbourhoods. Čas. pěst. mat. *108* (1983), 299—304.
[9] *G. Chartrand, R. E. Pipert:* Locally connected graphs. Čas. pěst. mat. *99* (1974), 158—163.
[10] *D. W. VanderJagt:* Sufficient conditions for locally connected graphs. Čas. pěst. mat. *99* (1974), 400—404.
[11] *O. Ore:* Theory of graphs. AMS, Providence, R. I. 1962.
[12] *B. Bollobás:* Extremal graph theory. Academic Press 1978.
[13] *F. Harary:* Graph theory. Addison-Wesley, Reading, Mass. 1969.

Souhrn

O GRAFECH S ISOMORFNÍMI, NEISOMORFNÍMI A SOUVISLÝMI N_2-OKOLÍMI

ZDENĚK RYJÁČEK

Podgraf $N_2(u, G)$ grafu G indukovaný množinou hran xy grafu G, pro něž $\min\{\varrho(x, u), \varrho(y, u)\} = 1$, se nazývá okolí 2. druhu uzlu u. V článku jsou vyšetřovány tři otázky: existence a vlastnosti grafů, v nichž N_2-okolí každého uzlu je isomorfní z daným grafem, existence grafů s neisomorfními N_2-okolími uzlů a existence a vlastnosti grafů, v nichž N_2-okolí všech uzlů jsou souvislá.

Резюме

О ГРАФАХ С ИЗОМОРФНЫМИ, НЕИЗОМОРФНЫМИ И СВЯЗНЫМИ N_2-ОКРУЖЕНИЯМИ

ZDENĚK RYJÁČEK

Подграф $N_2(u, G)$, порожденный такими ребрами xy графа G, для которых $\min\{\varrho(x, u), \varrho(y, u)\} = 1$, называется окружением второго типа вершины u. В настоящей статье рассмотрены следующие три вопроса: существование и свойства графов, N_2 — окружения вершин которых изоморфны заданному графу, существование графов, N_2 — окружения вершин которых неизоморфны и существование и свойства графов, N_2 — окружения вершин которых являются связными.

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