Pavel Pták; Josef Tkadlec
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A NOTE ON DETERMINACY OF MEASURES

PAVEL PTÁK, JOSEF TKADLEC, Praha

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Summary. In the article it is shown that the Cramér-Wold theorem implies a stronger form of the Christensen theorem.

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Let $\mathscr{B}(R^n)$ denote the collection of all Borel subsets of R^n an let $\mathscr{C}$ be a subset of $\mathscr{B}(R^n)$. Let $\mathscr{C}$ be called *determining* when the following statement holds: If μ_1, μ_2 are two probability measures on $\mathscr{B}(R^n)$ which agree on $\mathscr{C}$ then they are necessarily identical. The theorem of Christensen ([3]) says that the collection of all open balls is determining and the theorem of Cramér and Wold ([2]) says that the collection of all open half-spaces is determining. In this note we observe that the Cramér-Wold theorem implies a stronger form of the Christensen theorem. (As a by-product we obtain another proof of the Christensen theorem. For further discussion on the determinacy of measures, the reader is referred to [1], [4], [5], [6] and [7].)

Theorem. *Let p be a point in R^n $(n \in N)$ and let $\mathscr{C}$ denote the collection of all open balls having p on the boundary. Then $\mathscr{C}$ is determining.*

Proof. Let μ_1, μ_2 agree on $\mathscr{C}$. Applying a suitable transformation and multiple if necessary, we may assume that $p = 0 \in R^n$ and $\mu_1\{0\} = \mu_2\{0\} = 0$. Let $\mathscr{C}_1$ denote the collection of all open half-spaces which have 0 on the boundary. Put $\mathscr{D} = \mathscr{C} \cup \mathscr{C}_1$. Then μ_1, μ_2 agree on $\mathscr{D}$. Indeed, each open half-space in $\mathscr{C}_1$ can be obtained as a union of an increasing sequence of balls in $\mathscr{C}$. Hence μ_1, μ_2 have to agree on $\mathscr{C}_1$ in view of their monotone continuity.

Let now $\varphi: R^n \to R^n$ be a mapping such that $\varphi(0) = 0$ and $\varphi(x) = x/\|x\|^2$ otherwise. Then φ is obviously a Borel isomorphism. One can easily show that $\varphi(\mathscr{D})$ is exactly the collection of all open half-spaces in R^n. By our assumption, the measures $\mu_1\varphi^{-1}$, $\mu_2\varphi^{-1}$ agree on $\varphi(\mathscr{D})$ and therefore $\mu_1\varphi^{-1} = \mu_2\varphi^{-1}$ (the Cramér-Wold theorem). This means that $\mu_1 = \mu_2$ and the proof is complete.

References

[1] *P. Billingsley:* Convergence of Probability Measures. Willey, New York 1968.
[2] *H. Cramér, H. Wold:* Some theorems on distribution function. J. London Math. Soc. *11* (1936), 290—295.
[3] *J. P. R. Christensen:* On some measures analogous to Haar measure. Math. Scand. *26* (1970), 103—106.
[4] *R. O. Davies:* Can two different measures agree on balls? (Good news and bad news). Mimeographed notes, 1972.
[5] *R. O. Davies:* Measures not approximable or not specifiable by means of balls. Mathematika *18* (1971), 157—160.
[6] *T. Neubrunn:* A note on quantum probability spaces. Proc. Amer. Math. Soc. *25* (1970), 672—675.
[7] *F. H. Ruymgaart:* A note on the concept of joint distributions of pairs of observables. The Indian Journal of Statistics, Ser. A, Vol. *45* (1983), 38—43.

Souhrn

POZNÁMKA O URČENOSTI MĚR

PAVEL PTÁK, JOSEF TKADLEC

V článku je ukázáno, že Cramérova-Woldova věta implikuje silnější verzi Christensenovy věty.

Резюме

ЗАМЕЧАНИЕ ОБ ОПРЕДЕЛЕННОСТИ МЕР

PAVEL PTÁK, JOSEF TKADLEC

В работе показано, что теорема Крамэра-Волда влечет за собой более сильный вариант теоремы Христенсена.

Authors' address: Katedra matematiky FEL ČVUT, Suchbátarova 2, 166 27 Praha 6.