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# Partitions of $k$-branching trees and the reaping number of Boolean algebras

Claude Laflamme

*Abstract.* The reaping number $\mathfrak{r}_{m,n}(\mathbb{B})$ of a Boolean algebra $\mathbb{B}$ is defined as the minimum size of a subset $\mathcal{A} \subseteq \mathbb{B} \setminus \{\mathbf{O}\}$ such that for each $m$-partition $\mathcal{P}$ of unity, some member of $\mathcal{A}$ meets less than $n$ elements of $\mathcal{P}$.

We show that for each $\mathbb{B}$, $\mathfrak{r}_{m,n}(\mathbb{B}) = \mathfrak{r}_{\lceil \frac{m}{n-1} \rceil,2}(\mathbb{B})$ as conjectured by Dow, Steprāns and Watson. The proof relies on a partition theorem for finite trees; namely that every $k$-branching tree whose maximal nodes are coloured with $\ell$ colours contains an $m$-branching subtree using at most $n$ colours if and only if $\lceil \frac{\ell}{n} \rceil < \lceil \frac{k}{m-1} \rceil$.

*Keywords:* Boolean algebra, reaping number, partition

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## 1. Introduction.

Given a Boolean algebra $\mathbb{B}$ and an integer $m$, an $m$-partition of $\mathbb{B}$ is a set $\mathcal{P} \in [\mathbb{B}]^m$ such that $\vee\mathcal{P} = \mathbf{1}$ and $a \wedge b = \mathbf{O}$ for each $\{a, b\} \in [\mathcal{P}]^2$. $\mathcal{A} \subseteq \mathbb{B}$ is said to be $(m, n)$-reaped by the $m$-partition $\mathcal{P}$ if

$$(\forall a \in \mathcal{A})|\{b \in \mathcal{P} : a \wedge b \neq \mathbf{O}\}| \geq n.$$

The cardinal invariant $\mathfrak{r}_{m,n}(\mathbb{B})$ can now be defined as the minimum size of a subset $\mathcal{A} \subseteq \mathbb{B} \setminus \{\mathbf{O}\}$ which cannot be $(m, n)$-reaped.

The more standard reaping numbers $\mathfrak{r}_{m,2}(\mathbb{B})$ have been studied in [1], [2] and [3] where they are simply denoted by $\mathfrak{r}_m(\mathbb{B})$; we clearly have $\mathfrak{r}_n(\mathbb{B}) \leq \mathfrak{r}_{n+1}(\mathbb{B})$ for each Boolean algebra $\mathbb{B}$.

In [4], the more general reaping numbers $\mathfrak{r}_{m,n}(\mathbb{B})$ are defined where they are used to prove that for each $n$ there is a Boolean algebra $\mathbb{B}$ such that $\mathfrak{r}_n(\mathbb{B}) < \mathfrak{r}_{n+1}(\mathbb{B})$; they further prove the surprising inequality $\mathfrak{r}_n(\mathbb{B}) \leq \mathfrak{r}_2^+(\mathbb{B})$ which holds for every Boolean algebra $\mathbb{B}$ and integer $n$. In this short note, we prove that for each $\mathbb{B}$, $\mathfrak{r}_{m,n}(\mathbb{B}) = \mathfrak{r}_{\lceil \frac{m}{n-1} \rceil}(\mathbb{B})$ as conjectured by Dow, Steprāns and Watson.

As for terminology, an integer $n$ will often be identified with its predecessors $\{0, ..., n-1\}$. A tree will always mean a finite collection of sequences of integers which are closed under initial segments; it is called $k$-branching if every one of its non-maximal node has at least $k$ immediate successors and $\mu(\mathcal{T})$ will denote the maximal nodes of $\mathcal{T}$. In particular, ${}^n k$ is the full $k$-branching tree of height $n$, and $\chi : \mu(\mathcal{T}) \to n$ is an $n$-colouring of the maximal nodes of $\mathcal{T}$. Finally, $\lceil x \rceil$ denotes as usual the least integer greater than or equal to $x$.

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**2. Partitions of $k$-branching trees.**

In this section, we shall characterize exactly which tuples $k, \ell, m, n$ of integers have the property that every $k$-branching tree whose maximal nodes are coloured with $\ell$ colours contains an $m$-branching subtree using at most $n$ colours, a property that will be denoted by $\mathcal{P}(k,\ell,m,n)$. The answer, conjectured in [4], is given by the following:

**Theorem 1.** *$\mathcal{P}(k,\ell,m,n)$ holds if and only if $\lceil \frac{\ell}{n} \rceil < \lceil \frac{k}{m-1} \rceil$.*

PROOF: We first put $a = \lceil \frac{\ell}{n} \rceil$, $b = \lceil \frac{k}{m-1} \rceil$ and assume that $a < b$; we shall prove that $\mathcal{P}(k,\ell,m,n)$ holds.
Since $\ell \leq an$, partition $\ell$ into at most $a$ sets $<s_i : i < a>$, each of size at most $n$. Given a $k$-branching tree $\mathcal{T}$ and a colouring $\chi : \mu(\mathcal{T}) \to \ell$ of its maximal nodes, define a new colouring $\overline{\chi} : \mu(\mathcal{T}) \to a$ by $\overline{\chi}(\sigma) = i$ iff $\chi(\sigma) \in s_i$. Since $k \geq b(m-1) - (m-2)$, we get $a \leq b - 1 \leq \frac{k-1}{m-1}$; but $\mathcal{P}(k, \frac{k-1}{m-1}, m, 1)$ can easily be verified to hold and therefore $\mathcal{T}$ contains an $m$-branching subtree $\mathcal{T}'$ using only one $\overline{\chi}$-colour, say $i$. Thus $\mathcal{T}'$ is an $m$-branching subtree of $\mathcal{T}$ using at most $n$ $\chi$-colours, namely those from $s_i$, and we are done.

For the other direction, we shall show that $\mathcal{P}(k,\ell,m,n)$ fails whenever $\lceil \frac{\ell}{n} \rceil \geq \lceil \frac{k}{m-1} \rceil$. This will be done by induction on $n$, the case $n = 1$ being straightforward. Assume now the result true for $n$ and we prove it for $n+1$. Fix $k, \ell, m$ such that $\lceil \frac{\ell}{n+1} \rceil \geq \lceil \frac{k}{m-1} \rceil$ and we must show that $\mathcal{P}(k,\ell,m,n+1)$ fails.
Let $a = \lceil \frac{\ell}{n+1} \rceil$, $b = \lceil \frac{k}{m-1} \rceil$, and choose $\ell'$ as small as possible such that $a = \lceil \frac{\ell'}{n} \rceil$, namely $\ell' = an - (n-1)$. We know by induction that $\mathcal{P}(k,\ell',m,n)$ fails and therefore fix for each $s \in [\ell]^{\ell'}$ a $k$-branching tree $\mathcal{T}_s$ with a colouring $\chi_s : \mu(\mathcal{T}_s) \to s$ such that every $m$-branching subtree uses at least $n+1$ colours from $s$. The counterexample $\mathcal{T}$ to $\mathcal{P}(k,\ell,m,n+1)$ will be obtained by tagging a tree $\mathcal{T}_{s_\sigma}$ to each maximal node $\sigma$ of the tree ${}^{an-2n+1}k$.

We will now label each node down the tree ${}^{an-2n+1}k$ with a "root" $r_\sigma \subseteq \ell$ of size at most $\ell'$ such that if an $m$-branching subtree of $\mathcal{T}$ contains $\sigma$, then it will either use at least $n+2$ colours as desired or else use at least $n+1$ colours from $r_\sigma$. We let $r_\sigma = s_\sigma$ if $\sigma$ is a maximal node, but by shrinking the size of $r_\sigma$ by one each time we go down the tree, its size will be $n$ by the time we arrive at the bottom because $n + (an - 2n + 1) = \ell'$ and therefore the only alternative then is that any $m$-branching subtree of $\mathcal{T}$ will use at least $n+2$ colours.

To ensure that the size of the roots can be reduced, let $\tau$ be a non-maximal node of ${}^{an-2n+1}k$ and assume by induction that $|r_\sigma| = i+1$ is fixed for each immediate successor $\sigma$ of $\tau$ and that any $m$-branching subtree containing $\sigma$ uses at least $n+2$ colours or else uses at least $n+1$ colours from $r_\sigma$. Assume further that at most $m-1$ of the $r_\sigma$'s are equal and that their pairwise intersections is $r_\tau$ if different, with $|r_\tau| = i$. By a simple calculation, any $m$-branching subtree containing $\tau$ uses at least $n+2$ colours or else uses at least $n+1$ colours from $r_\tau$. That this strategy can be worked out is where the particular values of $k, \ell', \ell, m$ and $n$ play a role.
An exact description of the $r_\sigma$ can be obtained as follows. We construct a 1-1 function $f_\sigma : \{1, ..., \ell'\} \to \{1, ..., \ell\}$ for each maximal node $\sigma$ of ${}^{an-2n+1}k$. To start

with, $f_\sigma \restriction \{1, ..., n\}$ is the identity function. Now having obtained $f_\sigma \restriction \{1, ..., n+i\}$, for $i \leq an - 2n$, put $t = \{1, ..., \ell\} \setminus f''_\sigma\{1, ..., n+i\}$ and fix $\pi : t \to \{1, ..., \ell - n - i\}$ the unique order preserving bijection. Finally define $f_\sigma(n+i+1) = \pi^{-1}(\lfloor \frac{\sigma(i)}{m-1} \rfloor + 1)$. This can be done since $\ell \geq a(n+1) - n$, $k \leq a(m-1)$ and therefore $\lfloor \frac{\sigma(i)}{m-1} \rfloor + 1 \leq \ell - n - i$ for any $i \leq an - 2n$. Now for $\tau$ a node of ${}^{an-2n+1}k$ of height $i$ say, pick any maximal node $\sigma$ extending $\tau$ and label $\tau$ with the root $f''_\sigma\{1, ..., n+i\}$. It can now be verified that the above strategy can be implemented with these roots. □

## 3. Reaping numbers of Boolean algebras.

In [4], the ordering of the reaping numbers in Boolean algebras has been characterized in terms of the property $\mathcal{P}(k, \ell, m, n)$ as follows:

**Theorem 2** ([4]). *$\mathfrak{r}_{k,\ell}(\mathbb{B}) \leq \mathfrak{r}_{m,n}(\mathbb{B})$ for every Boolean algebra $\mathbb{B}$ if and only if $\mathcal{P}(k, m, \ell, n-1)$ fails.*

In particular, it follows from this theorem the existence for each $n$ of a Boolean algebra such that $\mathfrak{r}_n(\mathbb{B}) < \mathfrak{r}_{n+1}(\mathbb{B})$.

Further, it follows from Theorem 1 that for each Boolean algebra, each reaping number $\mathfrak{r}_{m,n}(\mathbb{B})$ is equal to the standard $\mathfrak{r}_{\lceil \frac{m}{n-1} \rceil}(\mathbb{B})$; thus the ordering of the reaping numbers is completely described.

**Theorem 3.** *For each Boolean algebra $\mathbb{B}$, $\mathfrak{r}_{m,n}(\mathbb{B}) = \mathfrak{r}_{\lceil \frac{m}{n-1} \rceil}(\mathbb{B})$.*

Proof: Since both $\mathcal{P}(m, \lceil \frac{m}{n-1} \rceil, n, 1)$ and $\mathcal{P}(\lceil \frac{m}{n-1} \rceil, m, 2, n-1)$ fail by Theorem 1, we get $\mathfrak{r}_{m,n}(\mathbb{B}) \leq \mathfrak{r}_{\lceil \frac{m}{n-1} \rceil}(\mathbb{B}) \leq \mathfrak{r}_{m,n}(\mathbb{B})$ for each Boolean algebra $\mathbb{B}$ by Theorem 2. □

## References

[1] Balcar B., Simon P., *On minimal $\pi$-character of points in extremally disconnected spaces*, Topology Appl. **41** (1991), 133-145.
[2] ———, *Reaping number and $\pi$-character of Boolean algebras*, preprint, 1991.
[3] Beslagić A., van Douwen E.K., *Spaces of nonuniform ultrafilters in spaces of uniform ultrafilters*, Topology Appl. **35** (1990), 253-260.
[4] Dow A., Steprāns, Watson S., *Reaping numbers of Boolean algebras*, preprint, 1992.

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