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Spaces not distinguishing convergences

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Abstract. In the present paper we introduce a convergence condition (Σ') and continue the study of "not distinguish" for various kinds of convergence of sequences of real functions on a topological space started in [2] and [3]. We compute cardinal invariants associated with introduced properties of spaces.

Keywords: P-, QN-, Σ-, Σ'-, Σ^*-, Σ_c-convergence, a space not distinguishing convergences

Classification: 54G99, 54C30, 03E17

0. Introduction

In [3] the authors classified spaces according to which of the convergences P, QN, Σ, Σ^* (defined below) do not distinguish, or according to the property that "every" sequence of functions convergent in some sense has a subsequence convergent in the other sense. This investigation began in [2] for pointwise and quasi-normal convergences. In the present paper we introduce another convergence condition (Σ') which is between (Σ) and (Σ^*) and we enlarge the study of "not distinguish" for all five convergences.

a) Let us recall some definitions from [3]. Let $\mathcal{F}$ be a class of real-valued functions such that

(i) all constant functions are in $\mathcal{F}$,
(ii) $(\forall f, g \in \mathcal{F})\ f - g \in \mathcal{F}$,
(iii) $(\forall f \in \mathcal{F})\ |f| \in \mathcal{F}$,
(iv) $(\forall f \in \mathcal{F})(\forall n > 0)\ f/n \in \mathcal{F}$.

For a space X we set $\mathcal{F}(X) = \mathcal{F} \cap {}^X\mathbb{R}$. For functions $f, f_n : X \to \mathbb{R}$ for $n \in \omega$ we consider the following four kinds of convergence of the sequence $\{f_n\}_{n=0}^\infty$ to f:

(P) pointwise convergence,
(QN) quasi-normal convergence, i.e. for some sequence of positive reals $\{\varepsilon_n\}_{n=0}^\infty$ with $\lim_{n\to\infty} \varepsilon_n = 0$ we have $(\forall x)(\forall^\infty n)\ |f_n(x) - f(x)| \le \varepsilon_n$,
(Σ) $\sum_{n=0}^\infty |f_n(x) - f(x)| < \infty$,
(Σ^*) pseudo-normal convergence, i.e. for some sequence of positive reals $\{\varepsilon_n\}_{n=0}^\infty$ with $\sum_{n=0}^\infty \varepsilon_n < \infty$ we have $(\forall x)(\forall^\infty n)\ |f_n(x) - f(x)| \le \varepsilon_n$.

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The sequence of functions in the above conditions may satisfy any of the following hypotheses ($\mathcal{F}$ is a given class of functions):

($\mathcal{F}$) $f_n \in \mathcal{F}(X)$ for $n \in \omega$, $f = 0$,
($\overline{\mathcal{F}}$) $f_n \in \mathcal{F}(X)$ for $n \in \omega$ (f is arbitrary),
($\mathcal{F}^{\downarrow}$) $f_{n+1} \leq f_n$, $f_n \in \mathcal{F}(X)$ for $n \in \omega$, $f = 0$,
($\overline{\mathcal{F}}^{\downarrow}$) $f_{n+1} \leq f_n$, $f_n \in \mathcal{F}(X)$ for $n \in \omega$ (f is arbitrary).

Definition 0.1. *Let α be any of the hypotheses $\mathcal{F}$, $\overline{\mathcal{F}}$, $\mathcal{F}^{\downarrow}$, $\overline{\mathcal{F}}^{\downarrow}$ and let β, γ be any of the convergences* P, QN, Σ, Σ^*.

(1) *A space X is an $\alpha\beta\gamma$-space if whenever functions $f_n, f : X \to \mathbb{R}$, $n \in \omega$ satisfy condition α, and the sequence $\{f_n\}_{n=0}^{\infty}$ β-converges to f, then the sequence $\{f_n\}_{n=0}^{\infty}$ γ-converges to f.*
(2) *A space X is a weak $\alpha\beta\gamma$-space (shortly* w$\alpha\beta\gamma$*-space) if whenever functions $f_n, f : X \to \mathbb{R}$, $n \in \omega$ satisfy condition α, and the sequence $\{f_n\}_{n=0}^{\infty}$ β-converges to f, then some subsequence $\{f_{n_k}\}_{k=0}^{\infty}$ γ-converges to f.*

From all the properties introduced by this definition only these are non-trivial:

$\mathcal{F}$PQN	$\mathcal{F}\Sigma$QN	$\mathcal{F}\Sigma\Sigma^*$	w$\mathcal{F}$PQN	w$\mathcal{F}$PΣ	w$\mathcal{F}$PΣ^*	w$\mathcal{F}\Sigma$QN	w$\mathcal{F}\Sigma\Sigma^*$
$\overline{\mathcal{F}}$PQN	$\overline{\mathcal{F}}\Sigma$QN	$\overline{\mathcal{F}}\Sigma\Sigma^*$	w$\overline{\mathcal{F}}$PQN	w$\overline{\mathcal{F}}$PΣ	w$\overline{\mathcal{F}}$PΣ^*	w$\overline{\mathcal{F}}\Sigma$QN	w$\overline{\mathcal{F}}\Sigma\Sigma^*$
$\mathcal{F}^{\downarrow}$PQN		$\mathcal{F}^{\downarrow}\Sigma\Sigma^*$	w$\mathcal{F}^{\downarrow}$PQN	w$\mathcal{F}^{\downarrow}$PΣ	w$\mathcal{F}^{\downarrow}$PΣ^*		
$\overline{\mathcal{F}}^{\downarrow}$PQN		$\overline{\mathcal{F}}^{\downarrow}\Sigma\Sigma^*$	w$\overline{\mathcal{F}}^{\downarrow}$PQN	w$\overline{\mathcal{F}}^{\downarrow}P\Sigma$	w$\overline{\mathcal{F}}^{\downarrow}P\Sigma^*$		

b) Now we introduce convergence condition (Σ').

(Σ') There is a monotone unbounded sequence of integers $\{k_n\}_{n=0}^{\infty}$ such that $\sum_{n=0}^{\infty} k_n|f_n(x) - f(x)| < \infty$.

Notice that f_n QN-converge to f if and only if there is a monotone unbounded sequence of integers $\{k_n\}_{n=0}^{\infty}$ such that $\lim_{n\to\infty} k_n|f_n(x)-f(x)| = 0$ for all x. It is easy to see that f_n Σ'-converge to f if and only if there is an increasing sequence of integers $\{n_i\}_{i=0}^{\infty}$ such that the sequence $\{\sum_{n=n_i}^{n_{i+1}-1} |f_n(x)-f(x)|\}_{i=0}^{\infty}$ Σ^*-converge to 0. Clearly $(\Sigma^*) \to (\Sigma') \to (\Sigma)$ & (QN). Replacing Σ^* or Σ by Σ' in some of the above cases we obtain further 5×4 possibilities $\alpha\Sigma\Sigma'$, $\alpha\Sigma'\Sigma^*$, αPΣ', w$\alpha\Sigma\Sigma'$, w$\alpha\Sigma'\Sigma^*$ for $\alpha = \mathcal{F}$, $\overline{\mathcal{F}}$, $\mathcal{F}^{\downarrow}$, $\overline{\mathcal{F}}^{\downarrow}$. But we can easily see that wαP$\Sigma' \equiv$ wαPQN, w$\alpha\Sigma\Sigma' \equiv$ w$\alpha\Sigma$QN, and every space is a w$\alpha\Sigma'\Sigma^*$-space. So we can consider the additional 8 properties of spaces:

$\mathcal{F}\Sigma\Sigma'$ $\quad\overline{\mathcal{F}}\Sigma\Sigma'$ $\quad\mathcal{F}^{\downarrow}\Sigma\Sigma'$ $\quad\overline{\mathcal{F}}^{\downarrow}\Sigma\Sigma'$ $\quad\mathcal{F}\Sigma'\Sigma^*$ $\quad\overline{\mathcal{F}}\Sigma'\Sigma^*$ $\quad\mathcal{F}^{\downarrow}\Sigma'\Sigma^*$ $\quad\overline{\mathcal{F}}^{\downarrow}\Sigma'\Sigma^*$

To simplify the notation we omit the letter P since it never occurs at the end of any prefix ($\alpha\gamma$-space and w$\alpha\gamma$-space mean αPγ-space and wαPγ-space, respectively).

The result of Section 1 is Diagram 1 and the next equalities describe the known relationship between the classes of spaces with particular properties:

$\mathcal{F}^{\downarrow}\mathcal{QN} = w\mathcal{F}^{\downarrow}\mathcal{QN} = w\mathcal{F}^{\downarrow}\Sigma^* = w\mathcal{F}^{\downarrow}\Sigma$ (Lemma 1.1(i)),
$\overline{\mathcal{F}}^{\downarrow}\mathcal{QN} = w\overline{\mathcal{F}}^{\downarrow}\mathcal{QN} = w\overline{\mathcal{F}}^{\downarrow}\Sigma^* = w\overline{\mathcal{F}}^{\downarrow}\Sigma = \mathcal{F}\Sigma\Sigma' = \overline{\mathcal{F}}\Sigma\Sigma' = \overline{\mathcal{F}}^{\downarrow}\Sigma\Sigma'$ (Lemma 1.2 and Lemma 1.4(1)),
$\mathcal{F}\Sigma'\Sigma^* = \overline{\mathcal{F}}\Sigma'\Sigma^*$ (Lemma 1.4(2)),
$\mathcal{F}\Sigma\Sigma^* = \overline{\mathcal{F}}\Sigma\Sigma^*$ (Lemma 1.4(3)),
$w\mathcal{F}\mathcal{QN} = w\mathcal{F}\Sigma^*$,
$w\overline{\mathcal{F}}\mathcal{QN} = w\overline{\mathcal{F}}\Sigma^* = w\overline{\mathcal{F}}\Sigma$ ([3]),
$w\mathcal{F}\Sigma\mathcal{QN} = w\mathcal{F}\Sigma\Sigma^*$,
$w\overline{\mathcal{F}}\Sigma\mathcal{QN} = w\overline{\mathcal{F}}\Sigma\Sigma^*$,
$\mathcal{F}\Sigma\Sigma^* = \overline{\mathcal{F}}^{\downarrow}\mathcal{QN} \cap \mathcal{F}\Sigma'\Sigma^*$ (Lemma 1.6),
$\overline{\mathcal{F}}^{\downarrow}\Sigma\Sigma^* = \overline{\mathcal{F}}^{\downarrow}\mathcal{QN} \cap \overline{\mathcal{F}}^{\downarrow}\Sigma'\Sigma^*$ (Lemma 1.6),
$\mathcal{F}^{\downarrow}\Sigma\Sigma^* = \mathcal{F}^{\downarrow}\Sigma\Sigma' \cap \mathcal{F}^{\downarrow}\Sigma'\Sigma^*$,
$w\mathcal{F}\mathcal{QN} = w\mathcal{F}\Sigma \cap w\mathcal{F}\Sigma\mathcal{QN}$.

Section 2 deals with the case $\mathcal{F} = \mathcal{M}$, the class of Borel measurable functions. The result of Section 3 is a picture of relationship between the properties from Definition 0.1 for all classes $\mathcal{F}$ with $\mathcal{C} \subseteq \mathcal{F} \subseteq \mathcal{M}$ where $\mathcal{C}$ is the class of continuous real-valued functions and in Section 4 we give some cardinal characterizations for all these properties.

We will use these standard families of real-valued functions defined on X: $\mathcal{C}(X) = \{f \in {}^X\mathbb{R} : f \text{ is continuous}\}$, $\mathcal{M}_1(X) = \{f \in {}^X\mathbb{R} : f \text{ is } F_\sigma \text{ measurable}\}$, $\mathcal{M}(X) = \{f \in {}^X\mathbb{R} : f \text{ is Borel measurable}\}$.

1. Classification in general

Every monotone Σ-convergent sequence of functions is QN-convergent ([3, Lemma 1.2]). The next lemma generalizes this fact and gives a comparison of Σ^*-convergence and Σ'-convergence for monotone series. We set $K(n) = \sum_{i=0}^{n} k_i$. The assumption that $\{k_n\}_{n=0}^{\infty}$ is monotone is not used here.

Lemma 1.1.

(i) *If $0 \le f_{n+1}(x) \le f_n(x)$ and $\sum_{k=0}^{\infty} k_n f_n(x) < \infty$ for $x \in X$, then for every $x \in X$, $f_n(x) < 1/K(n)$ for all but finitely many $n \in \omega$.*

(ii) *If $\sum_{n=0}^{\infty} 1/K(n) = \infty$ and $\sum_{n=0}^{\infty} \varepsilon_n < \infty$, then there is a sequence of positive reals $\{\xi_n\}_{n=0}^{\infty}$ such that $\xi_{n+1} \le \xi_n$ for all $n \in \omega$, $\sum_{n=0}^{\infty} k_n\xi_n < \infty$, and $\xi_n \ge \varepsilon_n$ for infinitely many $n \in \omega$.*

Proof: (i) If the conclusion is not correct then we get an infinite increasing sequence of integers $\{n_k\}_{k=0}^{\infty}$ such that for some $x \in X$ we have $f_{n_k}(x) \ge 1/K(n_k)$ for all $k \in \omega$. Define $\xi \in {}^\omega\mathbb{R}$ by $\xi(n) = 1/K(n_k)$ for $n_{k-1} < n \le n_k$ and $\xi(n) = 1/K(n_0)$ for $n \le n_0$. Clearly, $\xi(n) = 1/K(n_k) \le f_{n_k}(x) \le f_n(x)$ for all k and $n_{k-1} < n \le n_k$, and so $\sum_{n=0}^{\infty} k_n\xi(n) \le \sum_{n=0}^{\infty} k_n f_n(x) < \infty$. On the other hand (we can assume that $K(n_k) - K(n_{k-1}) \ge K(n_k)/2$)

$$\sum_{n>n_0} k_n\xi_n = \sum_{k=1}^{\infty} \sum_{n=n_{k-1}+1}^{n_k} k_n/K(n_k) = \sum_{k=1}^{\infty} (K(n_k) - K(n_{k-1}))/K(n_k) = \infty$$

because $\lim_{n\to\infty} K(n) = \infty$. This is a contradiction.

(ii) Then $\liminf_{n\to\infty} K(n)\varepsilon_n = 0$ and there is an increasing sequence of integers $\{n_k\}_{k=0}^{\infty}$ such that $\sum_{k=0}^{\infty} K(n_k)\varepsilon_{n_k} < \infty$ and $\varepsilon_{n_{k+1}} \le \varepsilon_{n_k}$. Define $\xi_n = \varepsilon_{n_k}$ for $n_{k-1} < n \le n_k$ and $\xi_n = \varepsilon_{n_0}$ for $n \le n_0$. Then

$$\sum_{n>n_0} k_n\xi_n = \sum_{k=1}^{\infty}\sum_{n=n_{k-1}+1}^{n_k} k_n\varepsilon_{n_k} \le \sum_{k=1}^{\infty} K(n_k)\varepsilon_{n_k} < \infty.$$

Clearly, $\xi_{n_k} \ge \varepsilon_{n_k}$ for every $k \in \omega$. □

Let $X = \{x \in {}^{\omega}\omega : \sum_{n=0}^{\infty} \log(n)/x(n) < \infty \;\&\; (\forall n)\; x(n) < x(n+1)\}$. By Lemma 1.1(ii) X is not an $\mathcal{F}^{\downarrow}\Sigma'\Sigma^*$-space whenever $\mathcal{F}$ contains the functions $f_n : X \to \mathbb{R}$ defined by $f_n(x) = 1/x(n)$.

Lemma 1.2. *$\mathcal{F}\Sigma\Sigma' = \overline{\mathcal{F}}^{\downarrow}\mathcal{QN}$.*

Proof: $\mathcal{F}\Sigma\Sigma' \subseteq \overline{\mathcal{F}}^{\downarrow}\mathcal{QN}$. Let $f_{n+1} \le f_n$, $\lim_{n\to\infty} f_n(x) = f(x)$ for $x \in X$, and let X be an $\mathcal{F}\Sigma\Sigma'$-space. Since $\sum_{n=0}^{\infty}(f_n(x) - f_{n+1}(x)) = f_0(x) - f(x) < \infty$, there is a monotone sequence of integers $\{k_n\}_{n=0}^{\infty}$ such that $\sum_{n=0}^{\infty} k_n(f_n(x) - f_{n+1}(x)) < \infty$. Let $\{n_i\}_{i=0}^{\infty}$ be the increasing enumeration of all $n > 0$ such that $k_n - k_{n-1} > 0$. Then

$$\sum_{i=0}^{\infty}(f_{n_i}(x) - f(x)) = \sum_{i=0}^{\infty}\sum_{n=n_i}^{\infty}(f_n(x) - f_{n+1}(x)) \le \sum_{n=n_0}^{\infty} k_n(f_n(x) - f_{n+1}(x)) < \infty.$$

Since $\mathrm{w}\overline{\mathcal{F}}^{\downarrow}\Sigma = \overline{\mathcal{F}}^{\downarrow}\mathrm{QN}$ X is an $\overline{\mathcal{F}}^{\downarrow}\mathrm{QN}$-space.

$\overline{\mathcal{F}}^{\downarrow}\mathcal{QN} \subseteq \mathcal{F}\Sigma\Sigma'$. Let $\sum_{n=0}^{\infty} |f_n(x)| < \infty$ for $x \in X$, and let X be an $\overline{\mathcal{F}}^{\downarrow}\mathrm{QN}$-space. Set $g(x) = \sum_{m=0}^{\infty} |f_m(x)|$, $g_n(x) = \sum_{m=0}^{n-1} |f_m(x)|$. There is an increasing sequence of integers $\{n_i\}_{i=0}^{\infty}$ such that $(\forall x \in X)(\forall^{\infty} i)\; g(x) - g_{n_i}(x) \le 2^{-i}$. Let $k_n = |\{i : n_i \le n\}|$. Then $\sum_{n=0}^{\infty} k_n|f_n(x)| = \sum_{i=0}^{\infty}(g(x) - g_{n_i}(x)) < \infty$. □

Lemma 1.3. *$\overline{\mathcal{F}}^{\downarrow}\mathcal{QN} \subseteq \overline{\mathcal{F}}\Sigma\mathcal{QN}$.*

Proof: See [3], Lemma 1.6. □

Lemma 1.4.

(1) $\mathcal{F}\Sigma\Sigma' = \overline{\mathcal{F}}\Sigma\Sigma' = \overline{\mathcal{F}}^{\downarrow}\Sigma\Sigma' \subseteq \mathcal{F}^{\downarrow}\Sigma\Sigma'$.
(2) $\mathcal{F}\Sigma'\Sigma^* = \overline{\mathcal{F}}\Sigma'\Sigma^* \subseteq \overline{\mathcal{F}}^{\downarrow}\Sigma'\Sigma^* \subseteq \mathcal{F}^{\downarrow}\Sigma'\Sigma^*$.
(3) $\mathcal{F}\Sigma\Sigma^* = \overline{\mathcal{F}}\Sigma\Sigma^* \subseteq \overline{\mathcal{F}}^{\downarrow}\Sigma\Sigma^* \subseteq \mathcal{F}^{\downarrow}\Sigma\Sigma^*$.

Proof: (1) The inclusion $\mathcal{F}\Sigma\Sigma' \subseteq \overline{\mathcal{F}}\Sigma\Sigma'$. Let $\sum_{n=0}^{\infty} |f_n(x) - f(x)| < \infty$ for all $x \in X$ and let X be an $\mathcal{F}\Sigma\Sigma'$-space. By Lemma 1.2 and Lemma 1.3 there is a sequence of positive reals $\{\varepsilon_n\}_{n=0}^{\infty}$ tending to 0 such that $(\forall x \in X)$ $(\forall^{\infty} n)\; |f_n(x) - f(x)| < \varepsilon_n$. Let $\varphi \in {}^{\omega}\omega$ be such that $\sum_{n=0}^{\infty} \varepsilon_{\varphi(n)} < \infty$. Then $\sum_{n=0}^{\infty} |f_n(x) - f_{\varphi(n)}(x)| \le \sum_{n=0}^{\infty} |f_n(x) - f(x)| + \sum_{n=0}^{\infty} |f_{\varphi(n)}(x) - f(x)| < \infty$.

Since X is an $\mathcal{F}\Sigma\Sigma'$-space, there is a sequence of integers $\{k'_n\}_{n=0}^{\infty}$ such that $\sum_{n=0}^{\infty} k'_n|f_n(x) - f_{\varphi(n)}(x)| < \infty$. There is a monotone unbounded sequence of integers $\{k''_n\}_{n=0}^{\infty}$ such that $\sum_{n=0}^{\infty} k''_n\varepsilon_{\varphi(n)} < \infty$. Set $k_n = \min\{k'_n, k''_n\}$. Then $\sum_{n=0}^{\infty} k_n|f_n(x)-f(x)| \le \sum_{n=0}^{\infty} k''_n|f_{\varphi(n)}(x)-f(x)|+\sum_{n=0}^{\infty} k'_n|f_n(x)-f_{\varphi(n)}(x)| < \infty$ and X is an $\overline{\mathcal{F}}\Sigma\Sigma'$-space.

The inclusion $\overline{\mathcal{F}}^{\downarrow}\Sigma\Sigma' \subseteq \mathcal{F}\Sigma\Sigma'$. Let $\sum_{n=0}^{\infty}|f_n(x)| < \infty$ for $x \in X$ and let X be an $\overline{\mathcal{F}}^{\downarrow}\Sigma\Sigma'$-space. Let us set

$$g(x) = \sum_{m=0}^{\infty} |f_m(x)|/(m+1), \qquad g_n(x) = \sum_{m=0}^{n-1} |f_m(x)|/(m+1).$$

Clearly $g_n \le g_{n+1}$ and we have

$$\sum_{n=0}^{\infty}(g(x) - g_n(x)) = \sum_{n=0}^{\infty}\sum_{m=n}^{\infty} |f_m(x)|/(m+1) = \sum_{m=0}^{\infty} |f_m(x)| < \infty.$$

Therefore there is a sequence of integers $\{k'_n\}_{n=0}^{\infty}$ such that $\sum_{n=0}^{\infty} k'_n(g(x) - g_n(x)) < \infty$ for $x \in X$. Let $\{k_n\}_{n=0}^{\infty}$ be another monotone unbounded sequence of integers such that $k_n \le \sum_{m=0}^{n} k'_m/(n+1)$. Then for all $x \in X$ we have

$$\sum_{n=0}^{\infty} k_n|f_n(x)| \le \sum_{n=0}^{\infty}\sum_{m=0}^{n} k'_m|f_n(x)|/(n+1) = \sum_{m=0}^{\infty} k'_m(g(x) - g_m(x)) < \infty.$$

Hence X is an $\mathcal{F}\Sigma\Sigma'$-space.

(2) The inclusion $\mathcal{F}\Sigma'\Sigma^* \subseteq \overline{\mathcal{F}}\Sigma'\Sigma^*$. Let $\sum_{n=0}^{\infty} k_n|f_n(x) - f(x)| < \infty$ for $x \in X$ and let X be an $\mathcal{F}\Sigma'\Sigma^*$-space. Let $\varphi \in {}^{\omega}\omega$ be increasing such that $\sum_{n=0}^{\infty} 1/k_{\varphi(n)} < \infty$. Since

$$\sum_{n=0}^{\infty} k_n|f_n(x)-f_{\varphi(n)}(x)| \le \sum_{n=0}^{\infty} k_n|f_n(x)-f(x)|+\sum_{n=0}^{\infty} k_{\varphi(n)}|f_{\varphi(n)}(x)-f(x)| < \infty,$$

there is a sequence of positive reals $\{\varepsilon_n\}_{n=0}^{\infty}$ such that $\sum_{n=0}^{\infty}\varepsilon_n < \infty$ and $(\forall x \in X)(\forall^{\infty} n)\ |f_n(x) - f_{\varphi(n)}(x)| < \varepsilon_n$. Now for all $x \in X$ for all but finitely many $n \in \omega$ we have $|f_n(x) - f(x)| \le |f_n(x) - f_{\varphi(n)}(x)| + |f_{\varphi(n)}(x) - f(x)| < \varepsilon_n + 1/k_{\varphi(n)}$.

(3) These inclusions are proved in [3, Lemma 1.7] and they are trivial consequences of inclusions (1) and (2) because $\alpha\Sigma\Sigma^* = \alpha\Sigma\Sigma' \cap \alpha\Sigma'\Sigma^*$ for $\alpha = \mathcal{F}$, $\mathcal{F}^{\downarrow}$, $\overline{\mathcal{F}}$, $\overline{\mathcal{F}}^{\downarrow}$. □

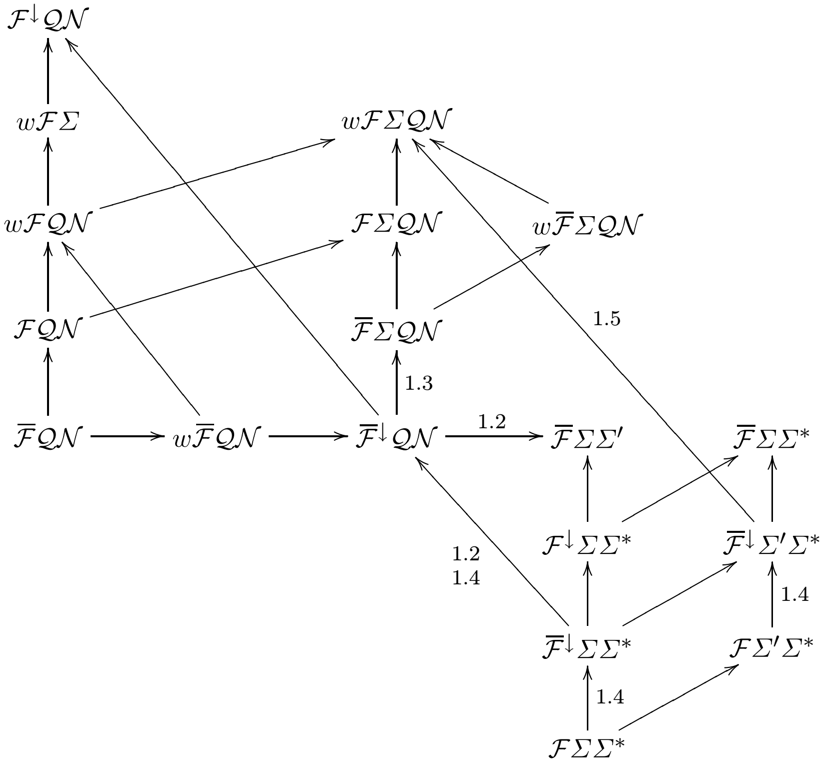

Diagram 1. This diagram shows all inclusions between the classes of spaces with introduced properties. The numbers at the arrows refer for the proofs of the inclusions. The other inclusions are easy consequences of definitions.

Lemma 1.5. *$\overline{\mathcal{F}}^{\downarrow}\Sigma'\Sigma^* \subseteq w\mathcal{F}\Sigma\mathcal{QN}$ and if a class of functions $\mathcal{G}$ is closed on limits of sequences of functions from $\mathcal{F}$, then $\mathcal{G}^{\downarrow}\Sigma'\Sigma^* \subseteq w\overline{\mathcal{F}}\Sigma\mathcal{QN}$.*

Proof: We prove the first part only, the second part is the same. Let X be an $\overline{\mathcal{F}}^{\downarrow}\Sigma'\Sigma^*$-space and let $\sum_{n=0}^{\infty}|f_n(x)| < \infty$ for $x \in X$. Let us choose a monotone unbounded sequence of integers $\{k_n\}_{n=0}^{\infty}$, and let us set $K(n) = \sum_{i=0}^{n} k_i$, so that $\sum_{n=0}^{\infty} 1/K(n) = \infty$ (e.g. $k_n = \log n$, $K(n) \leq n \log n$). Set $g(x) = \sum_{m=1}^{\infty}|f_m(x)|/K(m)$ and $g_n(x) = \sum_{m=1}^{n-1}|f_m(x)|/K(m)$. Then

$$\sum_{n=0}^{\infty} k_n(g(x) - g_n(x)) = \sum_{m=0}^{\infty}\sum_{n=0}^{m} k_n|f_m(x)|/K(m) = \sum_{m=0}^{\infty}|f_m(x)| < \infty.$$

Therefore there is a sequence of positive reals $\{\varepsilon_n\}_{n=0}^{\infty}$ such that $\sum_{n=0}^{\infty} \varepsilon_n < \infty$ and $(\forall x \in X)(\forall^\infty n)\ |f_n(x)|/K_n \leq g(x) - g_n(x) < \varepsilon_n$. There is an increasing sequence of integers $\{n_k\}_{k=0}^{\infty}$ such that $\lim_{k\to\infty} K(n_k)\varepsilon_{n_k} = 0$, and hence the sequence $\{K(n_k)\varepsilon_{n_k}\}_{k=0}^{\infty}$ witnesses the QN-convergence of $\{f_{n_k}\}_{k=0}^{\infty}$ to 0. □

Lemma 1.6.

(1) $\mathcal{F}\Sigma\Sigma^* = \overline{\mathcal{F}}^{\downarrow}\mathcal{QN} \cap \mathcal{F}\Sigma'\Sigma^*$.
(2) $\overline{\mathcal{F}}^{\downarrow}\Sigma\Sigma^* = \overline{\mathcal{F}}^{\downarrow}\mathcal{QN} \cap \overline{\mathcal{F}}^{\downarrow}\Sigma'\Sigma^*$.
(3) $\mathcal{F}^{\downarrow}\Sigma\Sigma^* = \mathcal{F}^{\downarrow}\Sigma\Sigma' \cap \mathcal{F}^{\downarrow}\Sigma'\Sigma^*$.

Proof: (1) and (2) are easy consequences of Lemma 1.2 and (3) is trivial. □

2. Borel measurable functions

For the class $\mathcal{F} = \mathcal{M}$, the class of Borel measurable real-valued functions, all the defined properties are hereditary for Borel subsets.

Theorem 2.1.

(1) $\overline{\mathcal{M}}\mathcal{QN} = \mathcal{M}^{\downarrow}\mathcal{QN}$,
(2) $\mathcal{M}^{\downarrow}\Sigma\Sigma^* = \overline{\mathcal{M}}^{\downarrow}\Sigma\Sigma^*$,
(3) $\mathcal{M}\Sigma'\Sigma^* = \mathcal{M}^{\downarrow}\Sigma'\Sigma^*$,
(4) $\mathcal{M}\Sigma\mathcal{QN} = \overline{\mathcal{M}}\Sigma\mathcal{QN}$,
(5) $w\mathcal{M}\Sigma\mathcal{QN} = w\overline{\mathcal{M}}\Sigma\mathcal{QN}$,
(6) $\overline{\mathcal{M}}^{\downarrow}\mathcal{QN} = \mathcal{M}^{\downarrow}\Sigma\Sigma'$,
(7) $\mathcal{M}\Sigma\Sigma^* = \mathcal{M}^{\downarrow}\Sigma\Sigma^*$.

Proof: The equalities (1)–(5) are easy and (6) is consequence of Lemma 1.2 since in Lemma 1.4(1) all families are equal. We prove (7). It is enough to prove the inclusion $\mathcal{M}^{\downarrow}\Sigma\Sigma^* \subseteq \mathcal{M}\Sigma\Sigma^*$. Let $X \in \mathcal{M}^{\downarrow}\Sigma\Sigma^*$, $f_n > 0$, $n \in \omega$ be Borel functions on X and let $\sum_{n=0}^{\infty} f_n(x) < \infty$. The function $\varphi : X \to {}^{\omega}\omega$ defined by $\varphi(x)(n) = \min\{m \in \omega : f_{n+1}(x)/m \leq f_n(x)\}$ is Borel. As X is an $\mathcal{M}$QN-space the image $\varphi(X)$ is a bounded subset of ${}^{\omega}\omega$ in the eventual ordering (see [3, Theorem 2.1]). Let $\alpha \in {}^{\omega}\omega$ eventually dominates $\varphi(X)$. Let us set $m_n = \prod_{i<n} \alpha(n)$. Then $f_{n+1}(x)/m_{n+1} = f_{n+1}(x)/(\alpha(n)m_n) \leq f_n(x)/m_n$ for all but finitely many n. The set $X_k = \{x \in X : (\forall n \geq k)\ f_{n+1}(x)/m_{n+1} \leq f_n/m_n\}$ is Borel for all k and $X = \bigcup_{k=0}^{\infty} X_k$. As X_k is an $\mathcal{M}^{\downarrow}\Sigma\Sigma^*$-space the monotone series $\sum_{n=k}^{\infty}\sum_{i=1}^{m_n} f_n(x)/m_n$ Σ^*-converge on X_k. Therefore the sequence $\{f_n\}_{n=0}^{\infty}$ Σ^*-converge on X_k for all k and hence also on their union X.

□

By Theorem 2.1 in the case $\mathcal{F} = \mathcal{M}$ every class of Diagram 1 is equal to one of the six classes of Diagram 2.

In the case $\mathcal{F} = \mathcal{C}$, the class of continuous real-valued functions, the next definition introduces a shorter notation. The first seven items of this definition were introduced in [3, Definition 2.2].

$$\begin{tikzcd}
M\Sigma'\Sigma^* \arrow[r] & \mathcal{M}^{\downarrow}\Sigma'\Sigma^* \arrow[r] & w\mathcal{M}\Sigma\mathcal{QN} \\
\mathcal{M}\Sigma\Sigma^* \arrow[u] \arrow[r] & \mathcal{MQN} \arrow[r] & \mathcal{M}\Sigma\mathcal{QN}
\end{tikzcd}$$

Diagram 2.

Definition 2.2.

(1) *An* mQN-*space* ($\overline{\mathrm{m}}$QN-*space*) *is a* $\mathcal{C}^{\downarrow}$QN-*space* ($\bar{\mathcal{C}}^{\downarrow}$QN-*space*).
(2) *A* QN-*space* (wQN-*space*) *is a* $\mathcal{C}$QN-*space* (w$\mathcal{C}$QN-*space*).
(3) *A* ΣQN-*space* (wΣQN-*space*) *is a* $\mathcal{C}\Sigma$QN-*space* (w$\mathcal{C}\Sigma$QN-*space*).
(4) *A* $\overline{\mathrm{QN}}$-*space is a* $\bar{\mathcal{C}}$QN-*space.*
(5) *A* $\overline{\Sigma}$QN-*space* (w$\overline{\Sigma}$QN-*space*) *is a* $\bar{\mathcal{C}}\Sigma$QN-*space* (w$\bar{\mathcal{C}}\Sigma$QN-*space*).
(6) *A* Σ-*space is a* w$\mathcal{C}\Sigma$-*space.*
(7) *A* $\Sigma\Sigma^*$-*space* (m$\Sigma\Sigma^*$-*space,* $\overline{\mathrm{m}}\Sigma\Sigma^*$-*space*) *is a* $\mathcal{C}\Sigma\Sigma^*$-*space* ($\mathcal{C}^{\downarrow}\Sigma\Sigma^*$-*space,* $\bar{\mathcal{C}}^{\downarrow}\Sigma\Sigma^*$-*space*).
(8) *A* $\Sigma'\Sigma^*$-*space* (m$\Sigma'\Sigma^*$-*space,* $\overline{\mathrm{m}}\Sigma'\Sigma^*$-*space*) *is a* $\mathcal{C}\Sigma'\Sigma^*$-*space* ($\mathcal{C}^{\downarrow}\Sigma'\Sigma^*$-*space,* $\bar{\mathcal{C}}^{\downarrow}\Sigma'\Sigma^*$-*space*).
(9) *A* $\Sigma\Sigma'$-*space* (m$\Sigma\Sigma'$-*space,* $\overline{\mathrm{m}}\Sigma\Sigma'$-*space*) *is a* $\mathcal{C}\Sigma\Sigma'$-*space* ($\mathcal{C}^{\downarrow}\Sigma\Sigma'$-*space,* $\bar{\mathcal{C}}^{\downarrow}\Sigma\Sigma'$-*space*).

In [3] the following theorem is proved for the first seven items in Definition 2.2 but the same arguments work also for the additional two.

Theorem 2.3. *Every of the properties in Definition* 2.2 *is* σ*-additive, and for perfectly normal spaces, it is hereditary for* F_σ *subsets, and it is preserved by* $\mathcal{D}_1$ *images.* □

3. Classification with use of topology

Theorem 3.1. *Let* X *be a perfectly normal space.*

(1) X *is an* m$\Sigma\Sigma^*$-*space if and only if* X *is an* $\overline{\mathrm{m}}\Sigma\Sigma^*$-*space.*
(2) X *is an* m$\Sigma'\Sigma^*$-*space if and only if* X *is an* $\overline{\mathrm{m}}\Sigma'\Sigma^*$-*space.*
(3) X *is an* m$\Sigma\Sigma'$-*space if and only if* X *is an* $\overline{\mathrm{m}}\Sigma\Sigma'$-*space.*
(4) X *is a* $\overline{\Sigma}$QN-*space if and only if* X *is a* w$\overline{\Sigma}$QN-*space and a* ΣQN-*space.*

Proof: The assertions (1) and (4) are proved in [3, Theorem 2.4]. The proof of all cases is based on Theorem 2.3. We prove only assertion (3), the other are similar. Let X be an m$\Sigma\Sigma'$-space and let $f_{n+1} \leq f_n$ for $n \in \omega$ be continuous functions, $f(x) = \lim_{n\to\infty} f_n(x)$ for $x \in X$ and let $\sum_{n=0}^{\infty} (f_n(x) - f(x)) < \infty$ for $x \in X$. By Lemma 1.1(i) these functions quasi-normally converge and so $X = \bigcup_{k=0}^{\infty} F_k$ with F_k closed such that the convergence is uniform on each set F_k. Therefore $(f_n - f)\restriction F_k$ are continuous and as all closed subsets of X are m$\Sigma\Sigma'$-spaces the sequence of functions $\{f_n - f\}_{n=0}^{\infty}$ Σ'-converges on each set F_k.

Each Σ'-convergence is witnessed by a monotone unbounded sequence of integers. As for every countable family of monotone unbounded sequences of integers there is a monotone unbounded sequence of integers majorized by all sequences from the family we easily obtain Σ'-convergence of the sequence of functions on the whole space X. □

Theorem 3.2. *Let $\mathcal{F}$ be any class of Borel functions containing all continuous functions. The following equivalences between the properties hold true for perfectly normal spaces:*

(i) $\overline{\mathrm{QN}} \equiv \overline{\mathcal{F}}\mathrm{QN} \equiv \mathrm{w}\overline{\mathcal{F}}\mathrm{QN} \equiv \overline{\mathcal{F}}^{\downarrow}\mathrm{QN}$,

(ii) $\Sigma\Sigma^* \equiv \mathcal{F}\Sigma\Sigma^* \equiv \mathcal{F}^{\downarrow}\Sigma\Sigma^* \equiv \overline{\mathcal{F}}\Sigma\Sigma^* \equiv \overline{\mathcal{F}}^{\downarrow}\Sigma\Sigma^*$,

(iii) $\mathcal{M}\Sigma\mathrm{QN} \equiv \mathcal{M}_1\Sigma\mathrm{QN}$,

(iv) $\overline{\mathrm{QN}} \equiv \mathcal{F}\Sigma\Sigma' \equiv \mathcal{F}^{\downarrow}\Sigma\Sigma' \equiv \overline{\mathcal{F}}\Sigma\Sigma' \equiv \overline{\mathcal{F}}^{\downarrow}\Sigma\Sigma'$.

Proof: Assertions (i)–(iii) are proved in [3, Theorem 6.2]. We prove (iv). By (i) and Lemma 1.2 $\Sigma\Sigma' \equiv \overline{\mathrm{m}}\mathrm{QN} \equiv \overline{\mathcal{F}}^{\downarrow}\mathrm{QN} \equiv \mathcal{F}\Sigma\Sigma'$, by Lemma 1.4(1) and Theorem 3.1(3) $\mathcal{F}\Sigma\Sigma' \equiv \overline{\mathcal{F}}\Sigma\Sigma' \equiv \overline{\mathcal{F}}^{\downarrow}\Sigma\Sigma' \to \mathcal{F}^{\downarrow}\Sigma\Sigma' \to \mathrm{m}\Sigma\Sigma' \equiv \overline{\mathrm{m}}\Sigma\Sigma' \equiv \Sigma\Sigma'$. □

Let us recall that a space X is *nestled* (see [3]) if for every sequence of strictly positive functions $f_n \in \mathcal{C}(X)$ there is a sequence of integers k_n such that $(\forall x \in X)$ $(\forall^\infty n)(\forall i \leq n)\ f_i(x) > 1/k_n$. We will need the following characterization.

Lemma 3.3. *Let X be arbitrary space. The following conditions are equivalent.*

(1) *X is a nestled space.*

(2) *Every continuous image of X into ${}^\omega\mathbb{R}$ is eventually bounded.*

(3) *For every sequence of strictly positive functions $f_n \in \mathcal{C}(X)$ for $n \in \omega$ there are integers m_n such that $(\forall x \in X)(\forall^\infty n)\ f_{n+1}(x)/m_{n+1} < f_n(x)/m_n$.*

Proof: See [3, Lemma 3.1]. □

Theorem 3.4. *Let X be a perfectly normal space.*

(1) *X is a $\Sigma\Sigma^*$-space if and only if X is a $\overline{\mathrm{QN}}$-space and an $\mathrm{m}\Sigma'\Sigma^*$-space.*

(2) *X is an $\mathrm{m}\Sigma\Sigma^*$-space if and only if X is a $\Sigma\Sigma^*$-space.*

(3) *If X is a nestled space and an $\mathrm{m}\Sigma'\Sigma^*$-space, then X is a $\Sigma'\Sigma^*$-space.*

Proof: (1) By Theorem 3.2 $\Sigma\Sigma^* \equiv \overline{\mathrm{m}}\Sigma\Sigma^*$ and $\overline{\mathrm{QN}} \equiv \overline{\mathrm{m}}\mathrm{QN}$. By Lemma 3.1(2) $\overline{\mathrm{m}}\Sigma'\Sigma^* \equiv \mathrm{m}\Sigma'\Sigma^*$. Lemma 1.6(2) in this case says $\overline{\mathrm{m}}\Sigma\Sigma^* \equiv \overline{\mathrm{m}}\mathrm{QN}\ \&\ \overline{\mathrm{m}}\Sigma'\Sigma^*$. Condition (2) is proved in [3, Theorem 6.1] and it can be obtain from the next part by taking $k_n = 1$.

(3) Let X be a nestled $\mathrm{m}\Sigma'\Sigma^*$-space. Let $\sum_{n=0}^{\infty} k_n |f_n(x)| < \infty$ for $x \in X$ where $\{k_n\}_{n=0}^{\infty}$ is monotone unbounded sequence of integers. We want to obtain the Σ^*-convergence. Without loss of generality we can assume that $f_n(x) > 0$ (otherwise take $f'_n(x) = \max\{f_n(x), 2^{-n}\}$ and $k'_n = \min\{k_n, n\}$). By Lemma 3.3 there are integers m_n such that $(\forall x \in X)(\forall^\infty n)\ f_{n+1}(x)/m_{n+1} < f_n(x)/m_n$. The sets

$$F_k = \{x \in X : (\forall n \geq k)\ f_{n+1}(x)/m_{n+1} < f_n(x)/m_n\}$$

are closed, by Theorem 2.3 they are $\mathrm{m}\Sigma\Sigma^*$-spaces (resp. $\mathrm{m}\Sigma'\Sigma^*$-spaces), and $X = \bigcup_{k=0}^{\infty} F_k$. The series $\sum_{n=k}^{\infty}\sum_{i=1}^{m_n} k_n f_n/m_n$ converges on F_k and hence the series $\sum_{n=k}^{\infty}\sum_{i=1}^{m_n} f_n/m_n$ Σ^*-converges on F_k. It follows that the series $\sum_{n=0}^{\infty} f_n(x)$ Σ^*-converges on F_k for every $k \in \omega$ and hence the series Σ^*-converges on the whole X (since every countable sequence of convergent series can be majorized by a single convergent series). $\square$

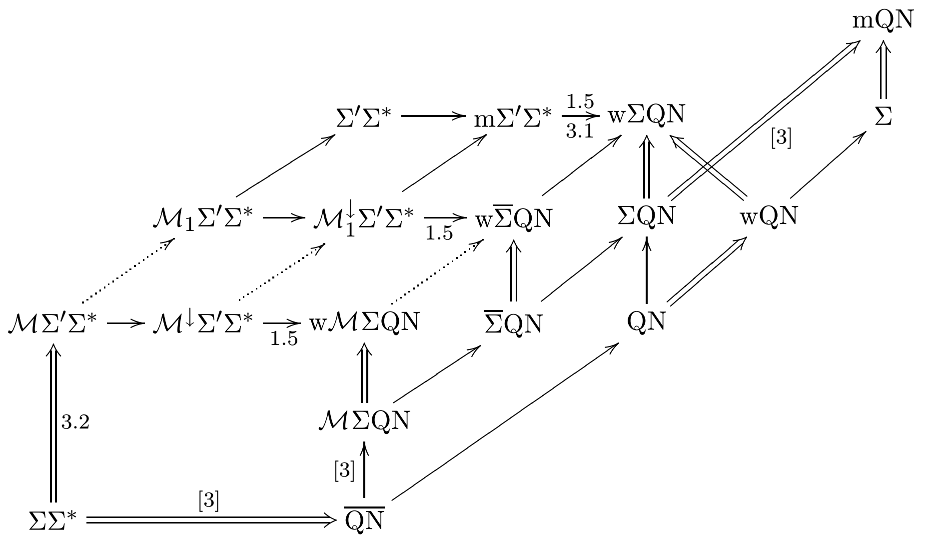

Diagram 3. The implications in the diagram hold true for perfectly normal spaces. Thick arrows are for the implications we know they are proper. Dotted arrows stand for infinite hierarchies of properties corresponding to the families of functions $\mathcal{F}$ with $\mathcal{C}\subseteq\mathcal{F}\subseteq\mathcal{M}$. Some equivalences: $\mathrm{wQN}\equiv\Sigma\&\mathrm{w}\Sigma\mathrm{QN}$, $\overline{\Sigma}\mathrm{QN}\equiv\Sigma\mathrm{QN}\&\mathrm{w}\overline{\Sigma}\mathrm{QN}$ (Theorem 3.1), $\Sigma\Sigma^*\equiv\overline{\mathrm{QN}}\&\mathrm{m}\Sigma'\Sigma^*$ (Theorem 3.4).

With connection to Diagram 3 let us remark that every $\mathfrak{b}$-Sierpiński set is a $\overline{\mathrm{QN}}$-set, γ-set is a wQN-set, σ-compact space is an mQN-space, separable metric Σ-space or $\mathrm{w}\Sigma\mathrm{QN}$-space is perfectly meager, and all properties below $\Sigma\mathrm{QN}$ are hereditary for subspaces of perfectly normal spaces (see [2] and [3]).

4. Cardinal invariants

We shall summarize characterizations of minimal cardinalities of spaces which do not have a property of a Diagram 3 and show that each of them is equal to one of five cardinals. It is a simple observation that $\mathrm{non}(\overline{\mathrm{QN}}) = \mathrm{non}(\mathrm{mQN}) = \mathfrak{b}$ and $\mathrm{non}(\Sigma\Sigma^*) = \mathrm{add}(\mathcal{N})$. In [3, Theorem 7.11] it is proved that $\mathrm{non}(\mathrm{w}\mathcal{M}\Sigma\mathrm{QN}) = \mathrm{non}(\mathrm{w}\Sigma\mathrm{QN}) = \mu$, where

$$\mu = \min\{|\mathcal{F}| : \mathcal{F} \subseteq {}^{\omega}\omega \;\&\; (\forall h \in {}^{\omega}\omega)(\exists g \in \mathcal{F})(\exists^{\infty} n)\; h(n) \in \{g(m) : m < n\}\}.$$

We prove $\mathrm{non}(\mathcal{M}\Sigma'\Sigma^*) = \mathrm{non}(\Sigma'\Sigma^*) = \mathfrak{k}$ and $\mathrm{non}(\mathcal{M}^{\downarrow}\Sigma'\Sigma^*) = \mathrm{non}(\mathrm{m}\Sigma'\Sigma^*) = \kappa$ ($\mathfrak{k}$ and κ are defined below).

Let us recall that for any function $h \in {}^\omega\omega$ with $\lim_{n\to\infty} h(n) = \infty$, $\mathfrak{k}$ is the following cardinal invariant.

$$\begin{aligned}\mathfrak{k} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq {}^\omega\omega \text{ is bounded and}\\ (\forall\varphi \text{ with } |\varphi(n)| \le n)(\exists g \in \mathcal{F})(\exists^\infty n)\ g(n) \notin \varphi(n)\}\\ = \min\{|\mathcal{F}| : \mathcal{F} \subseteq {}^\omega\omega \text{ is bounded and}\\ (\forall\varphi \text{ with } |\varphi(n)| \le h(n))(\exists g \in \mathcal{F})(\exists^\infty n)\ g(n) \notin \varphi(n)\}.\end{aligned}$$

Theorem 4.1. $\mathrm{non}(\mathcal{M}\Sigma'\Sigma^*) = \mathrm{non}(\Sigma'\Sigma^*) = \mathfrak{k}$.

PROOF: The proof is a literal transcription of Bartoszyński's characterization [1] of convergence of series via localizations of functions.

$\mathfrak{k} \le \mathrm{non}(\mathcal{M}\Sigma'\Sigma^*)$. Let $|X| < \mathfrak{k}$ and let $f_n : X \to \mathbb{R}$ be Borel functions such that $\sum_{n=0}^\infty k_n f_n(x) < \infty$ for all $x \in X$ for some monotone unbounded sequence $\{k_n\}_{n=0}^\infty$. Without loss of generality we can assume that $2^{-n} \le f_n(x) < 1$, otherwise take $f'_n = \max\{2^{-n}, \min\{f_n, 1/2\}\}$ and $k'_n = \min\{n, k_n\}$. Let us define $\varphi_x \in {}^\omega\omega$ for $x \in X$ so that $1/(\varphi_x(n) + 1) \le f_n(x) < 1/\varphi_x(n)$. Clearly, $0 < \varphi_x(n) < 2^n$ and $\sum_{n=0}^\infty k_n f_n(x) \le \sum_{n=0}^\infty k_n/\varphi_x(n) < \infty$. Let $X_m = \{x \in X : \sum_{n=0}^\infty k_n/\varphi_x(n) < m\}$. To prove the Σ^*-convergence on the set X it is enough to prove the Σ^*-convergence on each set X_m and since we can pass to coefficients $k'_n = k_n/m$, without loss of generality we can assume that $X = X_1$, i.e. $\sum_{n=0}^\infty k_n/\varphi_x(n) < 1$ for $x \in X$. Let $h \in {}^\omega\omega$ be a strictly increasing function such that $k_{h(m)} > 2^m$. Then $\sum_{n\ge h(m)} 1/\varphi_x(n) < 2^{-m}$ for all m. Let $g_x(n) = \varphi_x \restriction h(n+1) \in H(n) = \prod_{i<h(n+1)} 2^i$. There is φ with $|\varphi(m)| \le m$ such that $(\forall x \in X)(\forall^\infty m)\ g_x(m) \in \varphi(m)$. Let us define

$$\varepsilon_n = \max\left\{\frac{1}{s(n)} : s \in \varphi(m)\ \&\ \sum_{i=h(m)}^{h(m+1)-1} \frac{1}{s(i)} < 2^{-m}\right\},$$

for $h(m) \le n < h(m+1)$. Then $\sum_{n=0}^\infty \varepsilon_n < \infty$ and for every $x \in X$ for all but finitely many n, $f_n(x) \le 1/\varphi_x(n) \le \varepsilon_n$.

$\mathrm{non}(\Sigma'\Sigma^*) \le \mathfrak{k}$. Let $\mathcal{F} \subseteq {}^\omega\omega$ be a bounded family of functions and let $h \in {}^\omega\omega$ be a strictly increasing function bounding $\mathcal{F}$. We set $k_n = i$ for $h(i) \le n < h(i+1)$. For $g \in \mathcal{F}$ let $x_g \in {}^\omega\omega$ be defined by

$$x_g(n) = \begin{cases} \min\{i : n = g(i)\}, & \text{if } n \in \mathrm{rng}(g),\\ n, & \text{if } n \notin \mathrm{rng}(g).\end{cases}$$

Since $|\mathcal{F}| < \mathrm{non}(\Sigma'\Sigma^*)$, the set $X = \{x_g : g \in \mathcal{F}\}$ is a $\Sigma'\Sigma^*$-space. Let $f_n \in \mathcal{C}(X)$ be defined by $f_n(x) = 2^{-x(n)}$. Let $a_g = \{i : (\forall k < i)\ g(i) \ne g(k)\}$. Then

$\sum_{n=0}^{\infty} k_n f_n(x) \leq \sum_{n=0}^{\infty} n2^{-n} + \sum_{i\in a_g} k_{g(i)}2^{-i} < \infty$ because $g(i) \leq h(i)$ and hence $k_{g(i)} \leq i$ for all but finitely many i. There exists a convergent series of positive reals $\sum_{n=0}^{\infty} \varepsilon_n < 1$ such that for every $x \in X$, $f_n(x) \leq \varepsilon_n$ for all but finitely many n. Let $\varphi(k) = \{n : 2^{-k} \leq \varepsilon_n\}$. Clearly $|\varphi(k)| < 2^k$. For $g \in \mathcal{F}$ there is n_0 such that $2^{-x_g(n)} = f_n(x_g) \leq \varepsilon_n$ for all $n \geq n_0$. Then for all but finitely many $k \in \omega$ either $g(k) \geq n_0$, and then $2^{-k} \leq 2^{-x_g(g(k))} \leq \varepsilon_{g(k)}$, or $g(k) < n_0$ and $2^{-k} \leq \min\{\varepsilon_n : n < n_0\}$, and hence $g(k) \in \varphi(k)$. $\square$

Every perfectly normal ΣQN-space is a hereditary mQN-space since it is a σ-space (see [3, Theorem 5.7]). A wΣQN-space need not be a σ-space because the property wQN is not hereditary.

Corollary 4.2. *The implications $\mathcal{M}\Sigma'\Sigma^* \to$ mQN, $\mathcal{M}\Sigma'\Sigma^* \to \sigma$, $\overline{\mathrm{QN}} \to \mathrm{m}\Sigma'\Sigma^*$ are not provable.*

PROOF: (a) If $\mathfrak{t} = \mathfrak{b}$, then there exists a wQN-set $X \subseteq \mathbb{R}$ of size $\mathfrak{b}$ which is $\mathfrak{b}$-concentrated on a countable subset A (see [3] and [9]). X is an mQN-set but as mQN-subsets of X which are disjoint from A have size less than $\mathfrak{b}$ ([3, Theorem 4.1]) the set X is not a hereditary mQN-set. Therefore X is not a σ-set ([3, Theorem 3.12]). M. Kada and S. Kamo [6] have proved the consistency of $\omega_1 = \mathfrak{b} < \mathfrak{k}$. In the model let X_1 be a set of reals of size $\mathfrak{b}$ which is not an mQN-set. Since $\mathfrak{t} = \mathfrak{b} = \omega_1$ there is an mQN-set X_2 of size $\mathfrak{b}$ which is not a σ-set. Since $\mathfrak{b} < \mathfrak{k}$, X_1, X_2 are $\mathcal{M}\Sigma'\Sigma^*$-sets.

(b) By Theorem 3.4(1) $\mathrm{non}(\Sigma\Sigma^*) = \min\{\mathrm{non}(\overline{\mathrm{QN}}), \mathrm{non}(\mathrm{m}\Sigma'\Sigma^*)\}$. By consistency of $\mathrm{add}(\mathcal{N}) < \mathfrak{b}$ and equalities $\mathrm{non}(\Sigma\Sigma^*) = \mathrm{add}(\mathcal{N})$, $\mathrm{non}(\overline{\mathrm{QN}}) = \mathfrak{b}$, also $\mathrm{non}(\mathrm{m}\Sigma'\Sigma^*) < \mathrm{non}(\overline{\mathrm{QN}})$ is consistent. $\square$

Let us denote

$$B = \{f \in {}^\omega\omega : \lim_{n\to\infty} f(n) = \infty\},$$

$$X_f = \{x \in {}^\omega\omega : (\forall n \in \omega)\ x(n) \leq x(n+1)\ \&\ \textstyle\sum_{n=0}^{\infty} f(n)2^{-x(n)} < \infty\}, \quad f \in B,$$

$$\mathcal{K} = \{g \in {}^\omega\omega : \textstyle\sum_{n=0}^{\infty} 2^{-g(n)} < \infty\},$$

$$\kappa = \min\{|X| : (\exists f \in B)\ X \subseteq X_f\ \&\ (\forall g \in \mathcal{K})(\exists x \in X)(\exists^\infty n)\ x(n) < g(n)\}.$$

Theorem 4.3. $\mathrm{non}(\mathcal{M}^\downarrow\Sigma'\Sigma^*) = \mathrm{non}(m\Sigma'\Sigma^*) = \kappa$.

PROOF: Let $f \in B$ and $X \subseteq X_f$ be such that X is a witness for $\kappa = |X|$. The functions $f_n : X \to \mathbb{R}$ defined by $f_n(x) = 2^{-x(n)}$ are continuous, $f_{n+1} \leq f_n$, and by the choice of X the sequence $\{f_n\}_{n=0}^{\infty}$ Σ'-converge to 0 but does not Σ^*-converge. Therefore $\mathrm{non}(\mathcal{M}^\downarrow\Sigma'\Sigma^*) \leq \mathrm{non}(m\Sigma'\Sigma^*) \leq \kappa$. Conversely, let X be a space of minimal size which is not an $\mathcal{M}^\downarrow\Sigma'\Sigma^*$-space and let a monotone sequence of Borel functions $\{f_n\}_{n=0}^{\infty}$ and a sequence of integers $\{k_n\}_{n=0}^{\infty}$ witness this fact. For $x \in X$ let $\overline{x} \in {}^\omega\omega$ be such that $2^{-\overline{x}(n)} \leq f_n(x) < 2^{-\overline{x}(n)+1}$, $\overline{X} = \{\overline{x} : x \in X\}$, and let $f \in B$ be defined by $f(n) = k_n$. Then $\overline{X} \subseteq X_f$, $|\overline{X}| \leq |X|$, and $\overline{X}$ witnesses the inequality $\kappa \leq |\overline{X}|$. $\square$

For $\mathrm{non}(m\Sigma'\Sigma^*)$ we have yet another upper bound. By induction let us define a sequence of integers $d_0 = 1$, $d_{n+1} \geq d_n 2^{d_n}$ and let us denote

$$Y = \{x \in {}^\omega\omega : (\forall n)\ x(n) < 2^{d_n}\},$$

$$\mathcal{L} = \{g \in {}^\omega([\omega]^{<\omega}) : (\forall k)(\forall^\infty n)\ |g(n)| \geq 2^{d_n(1-1/k)}\ \&\ (\forall n)\ g(n) \subseteq 2^{d_n}\},$$

$$\nu = \min\{|X| : X \subseteq Y\ \&\ (\forall g \in \mathcal{L})(\exists x \in X)(\exists^\infty n)\ x(n) \in g(n)\}.$$

Theorem 4.4. $\mathrm{non}(m\Sigma'\Sigma^*) \leq \nu$.

Proof: Let $a_0 = 0$, $a_{n+1} = a_n + 2^{d_n}$, $I_n = [a_n, a_{n+1})$. For $k < 2^{d_n}$ we define $s_{n,k} \in {}^{I_n}\mathbb{R}$ by

$$s_{n,k}(m) = \begin{cases} \dfrac{1}{d_n(k+1)}, & \text{for } a_n \leq m \leq a_n + k, \\ \dfrac{1}{d_{n+1}}, & \text{for } a_n + k < m < a_{n+1}. \end{cases}$$

Since $d_n(k+1) \leq d_{n+1}$ for $k < 2^{d_n}$ we have $s_{n,k}(m) \geq s_{n,k}(m')$ for $m \leq m'$ in I_n. Therefore the sequence of functions $f_m : Y \to \mathbb{R}$ defined by

$$f_m(x) = s_{n,x(n)}(m), \qquad \text{for } m \in I_n,$$

is a monotone sequence of continuous functions. Since

$$\sum_{i \in I_n} s_{n,k}(i) \leq (k+1)\frac{1}{d_n(k+1)} + (2^{d_n} - k - 1)\frac{1}{d_{n+1}} \leq \frac{2}{d_n},$$

we have $\sum_{m \in I_n} f_m(x) \leq 2/d_n$, and the series $\sum_{n=0}^\infty 2/d_n < \infty$ witnesses the Σ'-convergence of f_m.

Let $X \subseteq Y$, $|X| < \mathrm{non}(m\Sigma'\Sigma^*)$. Since X is an $\mathrm{m}\Sigma'\Sigma^*$-space, there exists a convergent series of positive reals $\sum_{m=0}^\infty \varepsilon_m$ such that $(\forall x \in X)(\forall^\infty m)$ $f_m(x) \leq \varepsilon_m$. We can assume that $\varepsilon_{m+1} \leq \varepsilon_m$ for all $m \in \omega$. Let us set

$$g(n) = \{k < 2^{d_n} : s_{n,k}(d_n + k) > \varepsilon_{d_n+k}\}.$$

Since $s_{n,x(n)}(d_n + x(n)) = f_{d_n+x(n)}(x) \leq \varepsilon_{d_n+x(n)}$, $x(n) \notin g(n)$ for all but finitely many $n \in \omega$. We prove that $g \in \mathcal{L}$. Assume that for some n and for some $k > 0$, $|g(n)| < 2^{d_n(1-1/k)}$ and let l_i, $i = 1, 2, \dots, 2^{d_n} - 2^{d_n(1-1/k)}$ be an increasing sequence of integers such that $s_{n,l_i}(d_n + l_i) \leq \varepsilon_{d_n+l_i}$. We set $l_0 = -1$ and notice that $(i+1)/j \geq 1/(j-i)$ for $j > i$.

$$\sum_{m \in I_n} \varepsilon_m \geq \sum_i \frac{l_i - l_{i-1}}{d_n(l_i+1)} + \frac{2^{d_n} - l_{\max} - 1}{d_{n+1}} \geq \frac{1}{d_n}\sum_i \frac{1}{l_{i-1}+2}$$

$$\geq \frac{1}{d_n}\sum_{i=2^{d_n(1-1/k)}+1}^{2^{d_n}} \frac{1}{i} \geq \frac{1}{d_n}\sum_{j=d_n(1-1/k)}^{d_n-1} \frac{1}{2} = \frac{1}{2k}.$$

This is possible only for finitely many n since $\sum_{m=0}^\infty \varepsilon_m < \infty$. Therefore $|g(n)| \geq 2^{d_n(1-1/k)}$ for all but finitely many $n \in \omega$. □

Corollary 4.5. $\mathfrak{k} \leq \kappa \leq \min\{\mu, \nu\}$. □

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