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Lyapunov measures on effect algebras

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Abstract. We prove a Lyapunov type theorem for modular measures on lattice ordered effect algebras.

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1. Introduction

The celebrated Lyapunov's theorem says that the range of a non-atomic finite dimensional measure μ on a σ-algebra is convex. In general, this is not true if μ is infinite dimensional. On the other hand, Knowles showed that when μ is properly non-injective with values in a locally convex linear space, then its range is still convex. In [11], De Lucia and Wright, after introducing a notion of a convex set, generalize Knowles' result to the case when μ is group-valued.

In noncommutative measure theory it is known (see [5, Example 3.7]) that there are examples of nonatomic $\mathbb{R}^n$-valued measures on effect algebras which do not have a convex range. Nevertheless, in [5] it is proved (see 3.12) that a Lyapunov type theorem holds for $\mathbb{R}^n$-valued modular measures on lattice ordered effect algebras. Moreover, in [2], the result of [11] has been extended to modular functions on complemented lattices. Then a natural question arises:

Is it possible to extend the result of [11] to modular measures on effect algebras?

In this paper we give an affirmative answer to this question, introducing the notion of a pseudo non-injective measure (see Definition 4.1) in an effect algebra which is equivalent to the notion of properly non-injective measures in the Boolean case.

We recall that effect algebras have been introduced by D.J. Foulis and M.K. Bennett in 1994 (see [7]) for modelling unsharp measurement in a quantum mechanical system. They are a generalization of many structures which arise in quantum physics (see [6]) and in Mathematical Economics (see [14] and [9]), in particular of orthomodular lattices in noncommutative measure theory (e.g. see [12]) and MV-algebras in fuzzy measure theory.

2. Preliminaries

We will fix some notations. First we will give the definition of a D-poset. Examples of D-posets can be found in [10] and [13].

Definition 2.1. Let $(L, \leq)$ be a partial ordered set (a poset for short). A partial binary operation $\ominus$ on L such that $b \ominus a$ is defined iff $a \leq b$ is called a *difference* on $(L, \leq)$ if the following conditions are satisfied for all a, b, $c \in L$:

(1) if $a \leq b$ then $b \ominus a \leq b$ and $b \ominus (b \ominus a) = a$,
(2) if $a \leq b \leq c$ then $c \ominus b \leq c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.

Definition 2.2. Let $(L, \leq, \ominus)$ be a poset with difference and let 0 and 1 be the least and greatest elements in L, respectively. The structure $(L, \leq, \ominus)$ is called a *difference poset* (*D-poset* for short), or a *difference lattice* (*D-lattice* for short) if L is a lattice.

An alternative structure to a D-poset is that of an effect algebra introduced by Foulis and Bennett in [7]. These two structures, D-posets and effect algebras, are equivalent as shown in [13, Theorem 1.3.4].

We recall that a D-lattice is complete (σ-complete) if every set (countable set) has a supremum and an infimum.

If $a \in L$, we set $a^{\perp} = 1 \ominus a$.

We say that a and b are orthogonal if $a \leq b^{\perp}$ and we write $a \perp b$. If $a \perp b$, we set $a \oplus b = (a^{\perp} \ominus b)^{\perp}$. If $a_1, \ldots, a_n \in L$ we define inductively $a_1 \oplus \cdots \oplus a_n = (a_1 \oplus \cdots \oplus a_{n-1}) \oplus a_n$ if the right-hand side exists. The sum is independent of any permutation of the elements. We say that $\{a_1, \ldots, a_n\}$ is orthogonal if $a_1 \oplus \cdots \oplus a_n$ exists. We say that a family $\{a_\alpha\}_{\alpha \in A}$ is *orthogonal* if every finite subfamily is orthogonal. If $\{a_\alpha\}_{\alpha \in A}$ is orthogonal, we define $\bigoplus_{\alpha \in A} a_\alpha := \sup\{\bigoplus_{\alpha \in F} a_\alpha : F \subset A \text{ finite}\}$ if the left-hand side exists.

If $(G, +)$ is an abelian group, a function $\mu : L \to G$ is called *modular* if, for every a, $b \in L$, $\mu(a \vee b) + \mu(a \wedge b) = \mu(a) + \mu(b)$; μ is called a *measure* if, for every a, $b \in L$, with $a \perp b$, $\mu(a \oplus b) = \mu(a) + \mu(b)$. It is easy to see that μ is a measure iff for every a, $b \in L$ with $b \leq a$, $\mu(a \ominus b) = \mu(a) - \mu(b)$.

A measure μ is said to be σ*-additive* if, for every orthogonal sequence in L such that $a = \bigoplus_{n \in \mathbb{N}} a_n$ exists, $\mu(a) = \sum_{n \in \mathbb{N}} \mu(a_n)$. A measure μ is said to be *completely additive* if for every orthogonal family $\{a_\alpha\}_{\alpha \in A}$ in L such that $a = \bigoplus_{\alpha \in A} a_\alpha$ exists, the family $\{\mu(a_\alpha)\}_{\alpha \in A}$ is summable in G and $\mu(a) = \sum_{\alpha \in A} \mu(a_\alpha)$.

Recall that by 3.1 of [17] every modular function $\mu : L \to G$ on any lattice generates a lattice uniformity, $\mathcal{U}(\mu)$, i.e. a uniformity which makes $\wedge$ and $\vee$ uniformly continuous.

We say that $\mathcal{U}(\mu)$ is *exhaustive* if every monotone sequence $\{a_n\}$ is a Cauchy sequence. We say that $\mathcal{U}(\mu)$ is σ*-order* (*order*) *continuous* if every sequence (net)

$\{a_n\}$ which is order converging to a is converging to a. We say that a modular measure is exhaustive, σ-order (order) continuous iff $\mathcal{U}(\mu)$ is so. By 2.2 of [4], a measure is σ-additive iff it is σ-order continuous.

Throughout this article, $(G, +)$ is an abelian topological Hausdorff group which has not $\mathbb{Z}_2$ as a subgroup, L is a σ-complete D-lattice and $\mu : L \to G$ is a σ-additive modular measure.

3. Semi-convexity

We shall call $x \in G$ *infinitely divisible* if for every $n \in \mathbb{N}$ there exists $y \in G$ such that $2^n y = x$. Since $\mathbb{Z}_2$ is not a subgroup of G it is clear that when $2^n y = x$, y is uniquely determined. In what follows we shall denote such a y by $\frac{1}{2^n}x$. If $d = \frac{s}{2^n}$ is a dyadic rational number of the real interval $[0, 1]$ and $x \in G$ is infinitely divisible, we define dx to be sy, where $y = \frac{1}{2^n}x$. By [11] the definition of dx does not depend on the representation of d. Let D be the set of dyadic rationals in $[0, 1]$. For every infinite divisible $x \in G$, let $g_x : D \to G$ be defined by $g_x(d) = dx$ for $d \in D$. If $t \in [0, 1]$ and $\lim_{d\to t} g_x(d)$ exists in G, we define $tx = \lim_{d\to t} g_x(d)$. If $M \subset G$, M is said to be *convex* if for every x, $y \in M$ and $t \in [0, 1]$, tx, $(1-t)y$ exist and $tx + (1-t)y \in M$.

Definition 3.1. A measure μ is said to be *semiconvex* if, for each $b \in L$, there exists $c \in L$ such that $c \leq b$ and $\mu(b) = 2\mu(c)$.

Lemma 3.2. *If μ is semiconvex, then every element of $\mu(L)$ is infinitely divisible.*

Proof: For every $a \in L$ and $n \in \mathbb{N}$, there exists $b \leq a$ such that $\mu(a) = 2^n\mu(b)$. □

Lemma 3.3. *Suppose that μ is semiconvex. Then for every $a \in L$ and $d \in D$, there exists $a_d \leq a$ such that $\mu(a_d) = d\mu(a)$. Moreover, if $d_1 < d_2$, then $a_{d_1} \leq a_{d_2}$.*

Proof: Let $a \in L$.
(i) Claim 1: For every $n \in \mathbb{N}$ there exists an orthogonal family $\Pi_n = \{a_{n,1}, \dots, a_{n,2^n}\}$ in L such that $\bigoplus_{j=1}^{2^n} a_{n,j} = a$ and, for every $i \in \{1, \dots, 2^n\}$ we have:

(a) $2^n\mu(a_{n,i}) = \mu(a)$,
(b) $a_{n,2i-1} \oplus a_{n,2i} = a_{n-1,i}$.

This is trivial for $n = 1$: Since μ is semiconvex, we can choose $a_{1,1} \leq a$ such that $2\mu(a_{1,1}) = \mu(a)$. Let $a_{1,2} := a \ominus a_{1,1}$. Then $a_{1,1} \oplus a_{1,2} = a$ and $2\mu(a_{1,2}) = 2\mu(a) - 2\mu(a_{1,1}) = \mu(a)$.

By induction, suppose that Claim 1 holds for $n \in \mathbb{N}$. Since μ is semiconvex, for every $i \in \{1, \dots, 2^n\}$ we can find $a_{n+1,2i-1}$, $a_{n+1,2i}$ in L such that $a_{n+1,2i-1} \oplus a_{n+1,2i} = a_{n,i}$ and $2\mu(a_{n+1,2i-1}) = 2\mu(a_{n+1,2i}) = \mu(a_{n,i})$.

Set $\Pi_{n+1} = \{a_{n+1,1}, a_{n+1,2}, \dots, a_{n+1,2^{n+1}}\}$. Then Π_{n+1} is orthogonal since $a = \bigoplus_{i=1}^{2^n} a_{n,i} = \bigoplus_{i=1}^{2^n} (a_{n+1,2i-1} \oplus a_{n+1,2i}) = \bigoplus_{i=1}^{2^{n+1}} a_{n+1,i}$ and for every $i \in \{1, \dots 2^{n+1}\}$ we have $2^{n+1}\mu(a_{n+1,i}) = 2^n \mu(a_{n,i}) = \mu(a)$.

(ii) Now we obtain a family $\{b_{n,s} : n \in \mathbb{N}\}$ with $s \in \{0, 1, \dots 2^n\}$ such that:

(1) $b_{n,0} = 0$ and $b_{n,2^n} = a$,
(2) $b_{n,i-1} \leq b_{n,i}$,
(3) $2^n \mu(b_{n,i}) = i\mu(a)$,
(4) if $\frac{r}{2^m} = \frac{s}{2^n}$, then $b_{m,r} = b_{n,s}$.

It is sufficient to set $b_{n,0} = 0$ and, for $i \in \{1, \dots 2^n\}$, $b_{n,i} = \bigoplus_{j \leq i} a_{n,j}$.

(iii) If $d = \frac{r}{2^m}$, set $a_d = b_{m,r}$. Then by (ii), $a_d \leq a$ and $2^m \mu(a_d) = r\mu(a)$, from which $\mu(a_d) = d\mu(a)$. Moreover, by (ii), if $d_1 < d_2$ then $a_{d_1} \leq a_{d_2}$. □

Lemma 3.4. *Suppose that μ is semiconvex. Then for every $a \in L$ and for every 0-neighborhood W in G there exists $m \in \mathbb{N}$ such that for every $p \in D$ with $p \leq \frac{1}{2^m}$, $p\mu(a) \in W$.*

Proof: Let $a \in L$ and W be a 0-neighborhood in G. Since μ is semiconvex, we can construct a decreasing sequence $\{a_n\}$ in L such that $a_n \leq a$ and $2^n \mu(a_n) = \mu(a)$ for every $n \in \mathbb{N}$. Let $b_1 := a \ominus a_1$ and for every $n \geq 2$, let $b_n := a_{n-1} \ominus a_n$. By 3.3 of [1], $\{b_n\}$ is orthogonal and for every $n \in \mathbb{N}$, $2^n \mu(b_n) = 2^n \mu(a_{n-1}) - 2^n \mu(a_n) = 2\mu(a) - \mu(a) = \mu(a)$. Suppose that for every $m \in \mathbb{N}$ there exists c_m such that $\mu(b_m \wedge c_m) \notin W$. Since $\{b_n\}$ is orthogonal, $\{c_m \wedge b_m\}$ is orthogonal, too. Moreover, by 8.1.2 of [16], μ is exhaustive. By 2.4 of [3], μ is exhaustive if and only if $\mu(a_n) \to 0$ for every orthogonal sequence $\{a_n\}$ in L. Therefore, we obtain that $\lim_m \mu(b_m \wedge c_m) = 0$, a contradiction. Hence we can choose $m \in \mathbb{N}$ such that $\mu(b_m \wedge b) \in W$ for every $b \in L$. Set $p = \frac{r}{2^n}$, with $p \leq \frac{1}{2^m}$. By 3.3, we can find $c \leq b_m$ such that $\mu(c) = \frac{r}{2^{n-m}} \mu(b_m)$. Then $p\mu(a) = \frac{r}{2^n} \mu(a) = \frac{r}{2^{n-m}} \mu(b_m) = \mu(c) = \mu(c \wedge b_m) \in W$. □

Lemma 3.5. *Suppose that μ is semiconvex. Then for every $a \in L$ and every $t \in [0, 1]$ there exists $a_t \leq a$ such that $t\mu(a)$ is defined and $t\mu(a) = \mu(a_t)$. Moreover, the map $t \mapsto a_t$ is increasing.*

Proof: We repeat the same argument as in [2]. It follows from 3.3 that there exists a family of elements of L $\{a_d\}_{d \in D}$ such that $\mu(a_d) = d\mu(a)$ for each $d \in D$ and, also, for $d_1 < d_2$, $a_{d_1} \leq a_{d_2} \leq a$. Let $t \in [0, 1] \setminus D$. We define α_t, β_t by $\alpha_t = \bigvee \{a_d : d \in D \text{ and } d < t\}$ and $\beta_t = \bigwedge \{a_d : d \in D \text{ and } d > t\}$. By using the σ-order continuity of μ we find that $\mu(\alpha_t) = \lim_{d \nearrow t} \mu(a_d)$ $\mu(\beta_t) = \lim_{d \searrow t} \mu(a_d)$. Let V be any symmetric 0-neighbourhood in G. It follows from the construction and from 3.4 that we can find $n \in \mathbb{N}$ and $r \in \{0, 1, \dots, 2^n\}$ such that $d = \frac{r}{2^n} < t < \frac{r+1}{2^n} = d'$, $\mu(\beta_t) - \mu(\alpha_{d'}) \in V$, $\mu(\alpha_t) - \mu(a_d) \in V$, and $\frac{1}{2^n}\mu(a) \in V$. Then $(\mu(\beta_t) - \mu(\alpha_t)) \in \mu(a_{d'}) - \mu(a_d) + 2V = \frac{1}{2^n}\mu(a) + 2V \subset 3V$. Since the symmetric neighbourhoods form a base for 0-neighbourhoods, and since the topology is Hausdorff, $\mu(\beta_t) = \mu(\alpha_t)$. Hence we can define a_t, for $t \in [0,1] \setminus D$ to be α_t. Then it is clear that $\mu(a_t) = t\mu(a)$ for each $t \in [0,1]$. $\square$

Lemma 3.6. *Let $t \in [0,1]$ and $\nu_t : L \to G$ be defined as $\nu_t(a) = t\mu(a)$. Then ν_t is a modular measure.*

Proof: Let a, $b \in L$.

First suppose $t = \frac{s}{2^n} \in D$. By 3.3 we can find a_t, $b_t \in L$ with $a_t \leq a$, $b_t \leq b$, $2^n\mu(a_t) = s\mu(a)$ and $2^n\mu(b_t) = s\mu(b)$. Then we have $2^n\mu(a_t \vee b_t) + 2^n\mu(a_t \wedge b_t) = 2^n\mu(a_t) + 2^n\mu(b_t) = s\mu(a) + s\mu(b) = s\mu(a \wedge b) + s\mu(a \vee b)$, from which $\nu_t(a \vee b) + \nu_t(a \wedge b) = \nu_t(a) + \nu_t(b)$.

Now let $t \notin D$ and choose an increasing sequence $\{d_n\}$ in D which converges to t. Then $t\mu(a \vee b) + t\mu(a \wedge b) = \lim_n d_n\mu(a \vee b) + \lim_n d_n\mu(a \wedge b) = t\mu(a) + t\mu(b)$, from which $\nu_t(a \vee b) + \nu_t(a \wedge b) = \nu_t(a) + \nu_t(b)$.

In a similar way we prove that ν_t is a measure. $\square$

4. Lyapunov measures

In this section we set

$$I(\mu) = \{a \in L : \mu([0,a]) = \{0\}\}$$

and

$$N(\mu) = \{(a,b) \in L \times L : \mu \text{ is constant on } [a \wedge b, a \vee b]\}.$$

By 3.1 of [17] and 4.3 of [4] $N(\mu)$ is a congruence relation and the quotient $\hat{L} = L/N(\mu)$ is a D-lattice. Moreover, the function $\hat{\mu} : \hat{L} \to G$ defined as $\hat{\mu}(\hat{a}) = \mu(a)$ for $a \in \hat{a} \in \hat{L}$ is trivially a modular measure.

We say that μ is *closed* if $\hat{L}$ is complete with respect to the uniformity $\mathcal{U}(\hat{\mu})$ generated by $\hat{\mu}$.

Definition 4.1. We say that μ is *pseudo non-injective* if for every $a \in L \setminus I(\mu)$ there exist b, $c \in L \setminus I(\mu)$, $b \perp c$, $b \oplus c \leq a$ and $\mu(b) = \mu(c)$.

Lemma 4.2. (1) *μ is exhaustive.*

(2) *μ is closed iff μ is order continuous and $(\hat{L}, \leq)$ is complete.*

(3) *If G is metrizable, then μ is closed.*

(4) *If μ is order continuous, then μ is completely additive.*

Proof: (1) By 8.1.2 of [16], every σ-order continuous lattice uniformity on a σ-complete lattice is exhaustive.

(2) By (1) and 1.2.6 of [16], the Hausdorff uniformity $\mathcal{U}(\hat{\mu})$ generated by $\hat{\mu}$ on $\hat{L}$ is exhaustive. Then, by 6.3 of [16], $(\hat{L}, \mathcal{U}(\hat{\mu}))$ is complete iff $\mathcal{U}(\hat{\mu})$ is order continuous and $(\hat{L}, \leq)$ is complete. Therefore, if μ is closed, we have that $(\hat{L}, \leq)$ is complete and $\hat{\mu}$ is order continuous, too.

Conversely, if $(\hat{L}, \leq)$ is complete and μ is order continuous, then $\hat{\mu}$ is order continuous by 7.1.9 of [16], and therefore μ is closed.

(3) Since G is metrizable, $\mathcal{U}(\mu)$ is metrizable and, by (1), it is exhaustive. By 8.1.4 of [16] (see also 3.5 and 3.6 of [17]), we get that $(L, \leq)$ is complete and μ is order continuous. By 7.1.9 of [16], $(\hat{L}, \leq)$ is complete, too. Hence μ is closed by (2).

(4) Let $\{a_\alpha\}_{\alpha \in A}$ be an orthogonal family in L such that $a = \sup\{\bigoplus_{\alpha \in F} a_\alpha : F \subset A \text{ finite}\}$ exists in L. For every finite $F \subset A$, let $a_F = \bigoplus_{\alpha \in F} a_\alpha$. Then $\{a_F : F \subset A,\, F \text{ finite}\}$ is an increasing net in L, with $a = \sup_F a_F$. Since μ is order continuous, $\mu(a) = \lim_F \mu(a_F)$. On the other hand $\mu(a_F) = \sum_{\alpha \in F} \mu(a_\alpha)$. Thus $\mu(a) = \sum_{\alpha \in A} \mu(a_\alpha)$. □

Theorem 4.3. *Let L be complete and let μ be completely additive with $I(\mu) = \{0\}$. Then μ is semiconvex if and only if μ is pseudo non-injective.*

PROOF: $\Rightarrow$: Let $a \in L \setminus I(\mu)$.

First, suppose $\mu(a) \neq 0$. Then there exists $b \leq a$ such that $2\mu(b) = \mu(a)$. Put $c := a \ominus b$. Then $b \perp c$, $b \oplus c = a$ and $\mu(b) = \mu(c)$, as $2\mu(c) = 2\mu(a) - 2\mu(b) = \mu(a)$. Moreover, b, $c \notin I(\mu)$, since $\mu(b) = \mu(c) \neq 0$.

Now let $\mu(a) = 0$. As $a \notin I(\mu)$, there exists $d \leq a$ such that $\mu(d) \neq 0$. From above, there exist b, $c \in L \setminus I(\mu)$, $b \perp c$, $b \oplus c \leq d$ and $\mu(b) = \mu(c)$. Obviously, $b \oplus c \leq a$.

$\Leftarrow$: Let $a \neq 0$. We can suppose $\mu(a) \neq 0$.

(i) We will show that $\exists h$, $0 < h \leq a$ such that $\mu(h) = \mu(a)$ and $\mu(k) \neq 0$ for each $0 < k \leq h$.

We can suppose that there exists $b \leq a$, $b \neq 0$ and $\mu(b) = 0$, since otherwise (i) is satisfied with $h = a$.

Recall that in a complete D-lattice, if $\{b_\gamma\}_{\gamma \in \Gamma}$ is an orthogonal family then, for every $\bar{\gamma} \in \Gamma$, the set $\{\gamma \in \Gamma : b_\gamma = b_{\bar{\gamma}}\}$ is finite (see [DP] p.17). Then by Zorn's lemma we can find an orthogonal family $\{a_\alpha\}_{\alpha \in A}$ with the following properties:

(1) For every $\alpha \in A$, $a_\alpha \neq 0$ and $\mu(a_\alpha) = 0$.

(2) For every finite $F \subset A$, $\bigoplus_{\alpha \in F} a_\alpha \leq a$.

(3) If $\{b_\gamma\}_{\gamma \in \Gamma}$ is an orthogonal family in L with (1) and (2), then for each $\bar{\gamma} \in \Gamma$ the set $\{\alpha \in A : a_\alpha = b_{\bar{\gamma}}\} \neq \emptyset$ and $\{\gamma \in \Gamma : b_\gamma = b_{\bar{\gamma}}\} \subset \{\alpha \in A : a_\alpha = b_{\bar{\gamma}}\}$.

Since L is complete, $e = \bigoplus_{\alpha \in A} a_\alpha$ is well-defined. By (2) we get $e \leq a$. Since μ is completely additive we have $\mu(e) = \sum_{\alpha \in A} \mu(a_\alpha) = 0$. Put $h := a \ominus e$. Then $h \leq a$ and $\mu(h) = \mu(a)$.

We will show that, if $0 < b \leq h$, $\mu(b) \neq 0$.

By way of contradiction, assume $b \in L$, $0 < b \leq h$ and $\mu(b) = 0$. Since $b \leq h \leq e^\perp \leq (\bigoplus_{\alpha \in F} a_\alpha)^\perp$ for each finite $F \subset A$, we have, by 4.2 of [7] that every finite subfamily of $\{a_\alpha\}_{\alpha \in A} \cup \{b\}$ is orthogonal. Moreover, if $F \subset A$ is finite, we have $b \bigoplus (\oplus_{\alpha \in F} a_\alpha) \leq h \oplus e = (a \ominus e) \oplus e = a$. Then $\{a_\alpha\}_{\alpha \in A} \cup \{b\}$ gives a contradiction with (3).

Let h be as in (i).

We claim that, if $0 < k \leq h$, then there exist $c, d \in L$ such that $0 < c < d \leq k$ and $2\mu(c) = \mu(d)$.

If $0 < k \leq h$, $\mu(k) \neq 0$ by (i) and, by pseudo non-injectivity, there exist $b_1, b_2 \in L$, $b_1 \perp b_2$, $b_1 \oplus b_2 \leq k$, $b_1 \neq 0$, $b_2 \neq 0$ and $\mu(b_1) = \mu(b_2)$. Then for $c := b_1$ and $d := b_1 \oplus b_2$ we have $0 < c < d \leq k$ as b_1 and b_2 are not zero and $\mu(d) = \mu(b_1) + \mu(b_2) = 2\mu(c)$.

(ii) Zorn's lemma ensures the existence of an orthogonal family $\{d_\alpha\}_{\alpha \in A}$ with the following properties:

(1) for every $\alpha \in A$, $d_\alpha \neq 0$ and there exists c_α such that $0 < c_\alpha < d_\alpha$ and $2\mu(c_\alpha) = \mu(d_\alpha)$;
(2) for every finite $F \subset A$, $\bigoplus_{\alpha \in F} d_\alpha \leq h$;
(3) if $\{c_\gamma : \gamma \in \Gamma\}$ is an orthogonal family in L with properties (1) and (2), then for every $\bar{\gamma} \in \Gamma$ the set $\{\alpha \in A : d_\alpha = c_{\bar{\gamma}}\} \neq \emptyset$ and $\{\gamma \in \Gamma : c_\gamma = c_{\bar{\gamma}}\} \subset \{\alpha \in A : d_\alpha = c_{\bar{\gamma}}\}$.

It is easy to see that the set $\{c_\alpha : \alpha \in A\}$ is orthogonal. Put $d = \bigoplus_{\alpha \in A} d_\alpha$ and $c = \bigoplus_{\alpha \in A} c_\alpha$. We get $c \neq 0$, since $c_\alpha \neq 0$ for every $\alpha \in A$. By (2) $d \leq h$. Moreover, as $\mu(d) = \sum_{\alpha \in A} \mu(d_\alpha) = 2 \sum_{\alpha \in A} \mu(c_\alpha) = 2\mu(c)$ and $c \leq d$, we obtain $c < d$.

(iii) We will show that $d = h$.

Suppose $d < h$. Then $h \ominus d \neq 0$. From above, there exist $c_1, c_2 \in L$ with $0 < c_1 < c_2 \leq h \ominus d$ and $\mu(c_2) = 2\mu(c_1)$.

We will check that $\{d_\alpha\}_{\alpha \in A} \cup \{c_2\}$ has the same properties as $\{d_\alpha\}_{\alpha \in A}$.

Since $c_2 \leq h \ominus d \leq d^\perp \leq (\bigoplus_{\alpha \in F} d_\alpha)^\perp$ for every finite $F \subset A$, from 4.2 of [7] it follows that every finite subfamily of $\{d_\alpha\}_{\alpha \in A} \cup \{c_2\}$ is orthogonal and so, the family is orthogonal. Moreover, if $F \subset A$ is finite, then $c_2 \oplus (\bigoplus_{\alpha \in F} d_\alpha) \leq (h \ominus d) \oplus d = h$. Obviously, c_2 verifies (1). Then $\{d_\alpha\}_{\alpha \in A} \cup \{c_2\}$ contradicts property (3). Hence $d = h$.

It follows that $\mu(a) = \mu(h) = \mu(d) = 2\mu(c)$. Therefore μ is semiconvex. □

Theorem 4.4. *Let μ be closed and pseudo non-injective. Then $\mu(L)$ is convex.*

PROOF: It is clear that we can replace L by $L/N(\mu)$ and μ by $\hat{\mu}$. Then by 4.2 we can suppose L complete, μ completely additive and $I(\mu) = \{0\}$. Hence by 4.3 μ is semiconvex.

Let $b, c \in L$ and $t \in [0,1]$.

First, suppose $b \wedge c = 0$.

By 3.3 there exist $d, e \in L$ such that $d \leq b$, $e \leq c$, $\mu(d) = t\mu(b)$ and $\mu(e) = (1-t)\mu(c)$. Since $b \wedge c = 0$, we have $d \wedge e = 0$. It follows that $t\mu(b) + (1-t)\mu(c) = \mu(d) + \mu(e) = \mu(d \vee e) + \mu(d \wedge e) = \mu(d \vee e)$.

Now let $b, c \in L$. Put $b_1 := b \ominus (b \wedge c)$ and $c_1 = c \ominus (b \wedge c)$. By 1.8.5 of [13] we have $b_1 \wedge c_1 = 0$. Then, from above, there exist $b_2, c_2 \in L$ with $b_2 \leq b_1$, $c_2 \leq c_1$ and $t\mu(b_1) + (1-t)\mu(c_1) = \mu(b_2 \vee c_2)$.

Since $b = (b\wedge c)\oplus b_1$ and $c = (b\wedge c)\oplus c_1$, by 3.6 we obtain $t\mu(b) = t\mu(b_1)+t\mu(b\wedge c)$ and $(1-t)\mu(c) = (1-t)\mu(b\wedge c)+(1-t)\mu(c_1)$. It follows that $t\mu(b)+(1-t)\mu(c) = \mu(b\wedge c) + t\mu(b_1) + (1-t)\mu(c_1) = \mu(b\wedge c) + \mu(b_2 \vee c_2)$.

We claim that $b\wedge c \perp b_2 \vee c_2$. By 1.8.4 of [13] applied with $c = a\wedge b$, we obtain $b_1 \vee c_1 = (b \ominus (b\wedge c)) \vee (c \ominus (b\wedge c)) = (b\vee c) \ominus (b\wedge c)$, hence $b_2 \vee c_2 \leq b_1 \vee c_1 \leq 1 \ominus (b\wedge c) = (b\wedge c)^{\perp}$.

It follows that $\mu(b\wedge c) + \mu(b_2 \vee c_2) = \mu((b\wedge c) \oplus (b_2 \vee c_2))$ and, therefore, $t\mu(b) + (1-t)\mu(c) \in \mu(L)$. □

Corollary 4.5. *Let μ be closed. Then μ is pseudo non-injective iff for every $a \in L$, $\mu([0,a])$ is convex.*

Proof: $\Leftarrow$: From the assumptions we get that μ is semiconvex. Hence, $\hat{\mu}$ is semiconvex, too. Moreover, since μ is closed, by 4.2 we have that $L/N(\mu)$ is complete and $\hat{\mu}$ is completely additive. Since $I(\hat{\mu}) = \{\hat{0}\}$, by 4.3 we have that $\hat{\mu}$ is pseudo non-injective. We see that μ is pseudo non-injective, too. Let $a \in L \setminus I(\mu)$ and choose $b \leq a$ such that $\mu(b) \neq 0$. Since $\hat{\mu}$ is pseudo non-injective, there exist $\hat{c}$, $\hat{d}$; $\hat{c}, \hat{d} \neq \hat{0}$, $\hat{c} \perp \hat{d}$, $\hat{c} \oplus \hat{d} \leq \hat{b}$ and $\hat{\mu}(\hat{c}) = \hat{\mu}(\hat{d})$. Then there exist c, $d \in L \setminus I(\mu)$, $c \perp d$, $c \oplus d \leq b \leq a$ and $\mu(b) = \mu(c)$.

$\Rightarrow$: As in 4.4 we can suppose $L = L/N(\mu)$. Let $a \in L$ and denote by μ_a the restriction of μ to $[0,a]$. Observe that $[0,a]$ is a complete D-lattice and μ_a is a σ-order continuous pseudo non-injective modular measure, since $\mathcal{U}(\mu_a)$ coincides with the restriction of $\mathcal{U}(\mu)$ to $[0,a]$ and $N(\mu_a) = N(\mu) \cap ([0,a] \times [0,a])$. Hence by 4.4 we have that $\mu([0,a])$ is convex. □

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