Emmanuel Vrontakis

On the boundary of 2-dimensional ideal polyhedra

Commentationes Mathematicae Universitatis Carolinae, Vol. 47 (2006), No. 2, 359--367

Persistent URL: `http://dml.cz/dmlcz/119597`

On the boundary of 2-dimensional ideal polyhedra

Emmanuel Vrontakis

Abstract. It is proved that for every two points in the visual boundary of the universal covering of a 2-dimensional ideal polyhedron, there is an infinity of paths joining them.

Keywords: CAT(−1) spaces, ideal polyhedron, visual boundary

Classification: 57M20 (53C23)

1. Introduction and definitions

The study of the homeomorphic type of the visual boundary of a CAT(−1) space is an interesting problem, see for example [1], [12], [13]. The goal of the present work is to investigate some properties of the visual boundary of an ideal polyhedron of dimension 2.

Firstly, we recall some standard definitions (for more details see for example [14]).

A *path* in a topological space X is a continuous map $f : I \to X$, where I is an interval in $\mathbb{R}$. If the interval is compact, i.e. $I = [a, b]$ and if $x = f(a)$ and $y = f(b)$, then we say that f joins x and y; in this case the path is also called an *arc*. We denote the *distance* between two points x, y in a metric space X by $d_X(x, y)$. Let X and X' be two metric spaces. A map $f : X \to X'$ is called an *isometry* if $d_X(x, y) = d_{X'}(f(x), f(y))$ for every $x, y \in X$. A *geodesic* in X is a path $g : I \to X$ which is an isometry. If $I = [0, +\infty)$ we say that g is a *geodesic ray*, if $I = [a, b]$ we say that g is a *geodesic segment* and if $I = (-\infty, +\infty)$, we say that g is a geodesic line. A *local geodesic* in X is a path $g : I \to X$ such that for each $t \in I$, there exists an interval $I(t) \subset I$, which is a neighbourhood of t in I, such that the restriction of g to $I(t)$ is a geodesic. A *closed geodesic* in X is a periodic map $g : (-\infty, +\infty) \to X$ which is a local geodesic. A metric space X is said to be *geodesic* if for all x and y in X, there is a geodesic segment $g : [a, b] \to X$ with $g(a) = x$ and $g(b) = y$.

Now, we consider a family $\mathcal{T}$ of ideal triangles of the hyperbolic plane H^2 glued by isometries along their sides. We denote by X the resulting space. An *edge* e of X (i.e. a 1-dimensional (open) simplex of X) is the image in X of a side of an

The author was supported by a graduate fellowship from the *Greek State Scholarships Foundation* (I.K.Y.).

ideal triangle $T \in \mathcal{T}$. An *ideal vertex* of X is the image in X of an ideal vertex of some $T \in \mathcal{T}$. An edge e of X is said to have *index* k, $k > 1$, if e is obtained by the identification of k sides of the ideal triangles of $\mathcal{T}$. If an edge e is of index $k > 2$, then e is called *singular*.

We assume that X satisfies the following:

(a) *It is locally finite, i.e. every point belongs to a finite number of triangles.*

(b) *Every edge is adjacent to, at least, two triangles.*

X becomes a metric space; its metric $d_X(\cdot,\cdot)$ is induced from the ideal triangles as follows: the length $l_X(f)$ of a path $f : [a,b] \to X$ is defined as the sum of the lengths of the components of the intersection of f with the triangles of X. For x, y in X, we define $d_X(x,y)$ to be the infimum of the set of lengths of all paths joining these points. Actually, $d_X(x,y)$ is a pseudo-metric but the local finiteness property implies that $d_X(x,y)$ is a metric (see Proposition 1.2 of [4]).

We impose further in the assumptions of X that:

(c) *The metric $d_X(\cdot,\cdot)$ is complete.*

Therefore, we deduce that X is geodesic since it is locally compact and complete (Cohn-Vossen, see p. 4 of [10]).

We remark here that the completeness of $d_X(\cdot,\cdot)$ puts some restrictions on the gluing isometries between the sides of the ideal triangles of $\mathcal{T}$. If X consists of a finite number of ideal triangles then we can describe the completeness in terms of a geometrical property at the ideal vertices of X. With each ideal vertex v of X we can associate a natural foliation of a subset of X (a "neighborhood" of the ideal vertex), which is well defined up to restriction to a smaller neighborhood. The definition is as follows. Consider an ideal triangle in X, having v as one of its ideal vertices. Consider a foliation of a horoball neighborhood of v in the ideal triangle, whose leaves are pieces of horocycles which are centered at the ideal vertex. Considering now the various ideal triangles in X abutting on v, we can glue together these foliations and define a foliation of a "horoball neighborhood" of the ideal vertex v in X. (Note that the foliation is singular, with singular locus contained in the singular edges of X). Then, X is complete as a metric space if and only if, for each ideal vertex v of X and for each foliation of a horoball neighborhood of v, which is obtained in this manner, the restriction of the foliation to a smaller neighborhood of v is a product foliation on a space of the form $L \times [0,\infty)$, the leaves of which are the fibers $L \times \{x\}$, where L is homeomorphic to a closed graph (which in fact is the link of v). Proposition 3.4.18 of [15] concerns the case where X is a hyperbolic cusped surface, but the case of an ideal polyhedron follows with the same discussion.

Now, for each $x \in X$, there is a neighborhood U of x such that either U is isometric to an open ball in H^2 or to a finite number of half-discs in H^2 with equal diameters, which are glued along their diameters. It follows, from Corollary 5 of [9], p. 192, that X is of *curvature less than or equal to* -1 i.e. X satisfies locally the CAT(-1)-*inequality* (for more details on spaces of curvature less than or equal to -1 as well as on CAT(-1)-inequality, see for example [2], [9], [14]).

The space X defined above, equipped with a triangulation $\mathcal{T}$ of hyperbolic ideal triangles, will be called *an ideal polyhedron of dimension* 2. Such a triangulation $\mathcal{T}$ will be referred to as *ideal triangulation*. The universal covering $\widetilde{X}$ obviously inherits the structure of an ideal polyhedron because the triangulation $\mathcal{T}$ is lifted to an ideal triangulation $\widetilde{\mathcal{T}}$ of $\widetilde{X}$.

The geodesic metric space $\widetilde{X}$ is locally compact, complete and CAT(-1). Therefore the *visual boundary* $\partial\widetilde{X}$ of $\widetilde{X}$ is defined by means of the geodesic rays emanating from a fixed point of $\widetilde{X}$ (see for example [2], [13]).

Several properties of ideal polyhedra have been studied in [4], [5], [7], [8].

Throughout this paper, we assume that the triangulation $\mathcal{T}$ of X contains a finite number of ideal triangles, i.e. X is a finite ideal polyhedron, and that X contains at least one singular edge. The main result of this work is the following:

For every $\eta, \xi \in \partial\widetilde{X}$, there exists an infinite (*uncountable*) *number of distinct subspaces C_i of $\partial\widetilde{X}$, which are homeomorphic to S^1 and which contain η and ξ.*

2. Proof of the main theorem

Let X be a finite ideal polyhedron and let $\widetilde{X}$ be its universal covering. We assume that X and $\widetilde{X}$ are equipped with ideal triangulations $\mathcal{T}$ and $\widetilde{\mathcal{T}}$ respectively, which are considered fixed in the sequel.

A subpolyhedron of $\widetilde{X}$ is, by definition, a connected subset P of $\widetilde{X}$ equipped with an ideal triangulation $\widetilde{\mathcal{T}}_P$ such that each triangle of $\widetilde{\mathcal{T}}_P$ belongs to $\widetilde{\mathcal{T}}$. The metric of P is induced from the metric of $\widetilde{X}$. Denote by $H^+ = \{(x, y) \in H^2 : x \geq 0\}$.

Lemma 1. *Let P be an ideal polyhedron and let $h : P \to \widetilde{X}$ be a local isometry. Then h is an isometry and P is simply connected.*

Proof: In order to prove that h is an isometry we proceed as follows: If P is not simply connected, we consider the universal covering space $\widetilde{P}$ of P. If $p_1 : \widetilde{P} \to P$ is the covering projection, we will firstly show that the mapping $h \circ p_1 : \widetilde{P} \to \widetilde{X}$ is an isometry, i.e. $d_{\widetilde{P}}(\tilde{x}, \tilde{y}) = d_{\widetilde{X}}(h \circ p_1(\tilde{x}), h \circ p_1(\tilde{y}))$. Every two points in $\widetilde{P}$ and $\widetilde{X}$ can be joined by a unique geodesic segment in $\widetilde{P}$ and $\widetilde{X}$, respectively. Therefore, in order to prove that $h \circ p_1$ is an isometry it suffices to show that the geodesics of $\widetilde{P}$ are mapped to geodesics of $\widetilde{X}$. But this last assertion is true because h and p_1 are local isometries and since every local geodesic of $\widetilde{X}$ is a geodesic of $\widetilde{X}$ (see for example Proposition 1.4. of [6]). As a result, $h \circ p_1$ is an isometry and hence p_1 must be an isometry. Therefore P is simply connected and h is an isometry too. $\square$

When P is a subpolyhedron of $\widetilde{X}$ then Lemma 1 shows that the natural embedding $h : P \to \widetilde{X}$ is an isometry. Hence, due to the Theorem of Gromov [11], the space ∂P, is embedded in $\partial\widetilde{X}$.

Now, let $\eta, \xi \in \partial\widetilde{X}$ and $\widetilde{g} : (-\infty, +\infty) \to \widetilde{X}$ be a geodesic, with $\widetilde{g}(-\infty) = \eta$ and $\widetilde{g}(+\infty) = \xi$, and denote by $\mathrm{Im}\,\widetilde{g}$ the image of $\widetilde{g}$. We have the following elementary lemma:

Lemma 2. (a) *If e is an edge of $\widetilde{X}$ and if $e \neq \mathrm{Im}\,\widetilde{g}$, then $e \cap \mathrm{Im}\,\widetilde{g}$ is a singleton, provided that $e \cap \mathrm{Im}\,\widetilde{g} \neq \emptyset$.*

(b) *There are no $t_1, t_2, t_3 \in (-\infty, +\infty)$, with $t_1 < t_2 < t_3$, such that: $\widetilde{g}(t_1), \widetilde{g}(t_3) \in T_j$ and $\widetilde{g}(t_2) \in T_i$, where T_i, $T_j \in \mathcal{T}'$ and $T_i \neq T_j$.*

Proof: The proof of (a) and (b) follows immediately from the fact that e and T_j, are convex subsets of $\widetilde{X}$ and $\widetilde{X}$ is a CAT(-1) space. Therefore two points either in e or in T_j can be joined by a unique geodesic segment which belongs to e or to T_j, respectively. $\square$

Let $\widetilde{g} : (-\infty, +\infty) \to \widetilde{X}$ be a geodesic such that $\mathrm{Im}\,\widetilde{g}$ does not coincide with an edge e of $\widetilde{X}$. Let $S = \{T \in \mathcal{T}' : T \cap \mathrm{Im}\,\widetilde{g} \neq \emptyset\}$, be the collection of triangles intersected by $\mathrm{Im}\,\widetilde{g}$. We order the elements of S as follows: consider a partition of $(-\infty, +\infty)$, as a union of closed intervals $\bigcup I_i$, $i \in I$, such that: for every $i \in I$, $\widetilde{g}(I_i)$ belongs to a single T which we label by T_i, and the intersection $I_i \cap I_{i+1}$ contains a single point t_i, while the triangles T_i, T_{i+1} have only one side adjacent that contains $\widetilde{g}(t_i)$. We call $\{T_i\}_{i\in I}$ *an ordered sequence of triangles which are intersected by* $\widetilde{g}$. Therefore $\mathrm{Im}\,\widetilde{g}$ can intersect two successive triangles T_i, T_{i+1}, $i \in I$, in the following way: T_i, T_{i+1} have a common side, which coincides with an edge e of $\widetilde{X}$ and $\widetilde{g}$ intersects e transversely, passing from the interior of T_i to the interior of T_{i+1}.

The number of triangles $\{T_i\}_{i\in I}$, may be infinite or finite and by the previous lemma we deduce that $T_i \neq T_j$ if $i \neq j$.

We need the following lemmata:

Lemma 3. *If e is an edge of $\widetilde{X}$ of index k, $k > 1$, there are exactly k ideal triangles of $\mathcal{T}'$ which have e as a common side.*

Proof: If the lemma is not true, then there is a triangle $T_i \in \mathcal{T}'$ which has two different sides identified to the edge e of $\widetilde{X}$. Therefore we can find a simple closed loop α in T_i which intersects e once. But α must be null-homotopic in $\widetilde{X}$. By Corollary 2.4 of [8], α must belong in the interior of T_i, which gives a contradiction. $\square$

Lemma 4. *Let R be a subpolyhedron of $\widetilde{X}$ which is isometric to H^2 and which contains at least one singular edge $\widetilde{e}$ of $\widetilde{X}$. Let $p_{|R}(R) = Y$, where $p_{|R}$ denotes the restriction of the covering map $p : \widetilde{X} \to X$ to R. If $z, w \in \partial R$ and* $\mathrm{arc}[zw]$

denotes an arc if ∂R joining z and w, then there exists a countable number of disjoint subarcs $\mathrm{arc}[z_i w_i]$ *of* $\mathrm{arc}[zw]$ *such that:* $p_{|R}((z_i, w_i)) = p(\widetilde{e})$ *for every* i*, where by* (z_i, w_i) *we denote the geodesic of* R *which joins* z_i *and* w_i*.*

Proof: It is well known that $X = \widetilde{X}/\operatorname{Isom}(\widetilde{X})$, where $\operatorname{Isom}(\widetilde{X})$ denotes the discrete group of isometries of $\widetilde{X}$. Let G be the subgroup of $\operatorname{Isom}(\widetilde{X})$ whose elements leave R invariant, hence for every $\phi \in G$ we have $\phi(R) = R$. Therefore, $Y = R/G$. The mapping $p_{|R}$ is a local isometry, therefore every local geodesic g of Y has a lifting $\widetilde{g}$ in R which is a local geodesic in R and hence in $\widetilde{X}$. By Proposition 1.4 of [6] $\widetilde{g}$ is a geodesic of $\widetilde{X}$. We consider an edge $\widetilde{e}$ of R such that $p(\widetilde{e}) = e$ is a singular edge of X. Let $\mathrm{arc}[zw]$ be an arc of $\partial R \approx S^1$ and let $y \in \operatorname{Int}(\mathrm{arc}[zw])$. Let $\widetilde{g} : (-\infty, +\infty) \to R$ be a geodesic with $\widetilde{g}(+\infty) = y$ and which intersects $\widetilde{e}$. Y is a finite ideal polyhedron. Therefore, from Theorem 2 of [7], the set of closed geodesics of Y is dense in the set of local geodesics of Y with the compact-open topology. So, we deduce that there exists a geodesic $\widetilde{h} : (-\infty, +\infty) \to R$ which is arbitrarily close to $\widetilde{g}$ and whose image by $p_{|R}$ is a closed geodesic in Y. Therefore, there exists an isometry $\sigma \in G$ which leaves $\widetilde{h}$ invariant. Obviously, σ is a hyperbolic isometry of $\widetilde{X}$ which has the geodesic $\widetilde{h}$ as an axis. Therefore σ translates $\widetilde{e}$ till its end-points in ∂R are arbitrary close to $\widetilde{h}(+\infty)$. So, there exists $n \in \mathbb{N}$ such that the end-points of $\sigma^n(\widetilde{e})$ lie in the interior of $\mathrm{arc}[zw]$ (see for example Proposition 7.2 in [3]).

By repeating the same procedure with an arc $\mathrm{arc}[z'w'] \subset \partial R \setminus \mathrm{arc}[zw]$ instead of $\mathrm{arc}[zw]$, we may easily accomplish the proof of lemma. □

Now, we are able to prove the following:

Theorem 5. *Let X be an ideal polyhedron of dimension 2 and we assume that X contains at least one singular edge. Then we have:*
For every $\eta, \xi \in \partial\widetilde{X}$, there exists an uncountable number of distinct subspaces C_i of $\partial\widetilde{X}$, which are homeomorphic to S^1 and which contain η and ξ. Furthermore, every C_i is the visual boundary of a subpolyhedron P_i of $\widetilde{X}$ which is isometric to H^2.

Proof: Let $\eta, \xi \in \partial\widetilde{X}$ and $\widetilde{g} : (-\infty, +\infty) \to \widetilde{X}$ be a geodesic, with $\widetilde{g}(-\infty) = \eta$ and $\widetilde{g}(+\infty) = \xi$. Firstly, we assume that $\operatorname{Im}\widetilde{g}$ does not coincide with an edge of $\widetilde{X}$ and let $\{T_i\}_{i\in I}$ be the ordered sequence of ideal triangles, intersected by $\operatorname{Im}\widetilde{g}$. We will construct a subpolyhedron P of $\widetilde{X}$ such that:

(1) P contains the geodesic $\widetilde{g}$,
(2) P is isometric to H^2.

The whole construction will be based upon the geodesic and its way through the triangles $\{T_i\}_{i\in I}$. We will use a procedure of *cutting* along all singular edges in $\widetilde{X}$ and *re-gluing*, in order to eliminate all singular edges of $\widetilde{X}$. In this way we find a surface P with the above mentioned properties.

We consider all singular edges e_j of $\widetilde{X}$ of index k with $k > 3$, and we cut $\widetilde{X}$ along such e_j. The resulting space, say Z, has ideal triangles with free sides i.e. sides that belong only to one ideal triangle. We glue back these ideal triangles of Z by isometries, along their free sides, as follows:
Let the index of e_j be k_j. The cutting operation along e_j creates the free sides $\{e_{j,i}\}_i$, $i \in \{1, \dots, k_j\}$. Let $T_{j,i}$ be the ideal triangle which contains $e_{j,i}$. From Lemma 3, all these triangles $T_{j,i}$ are distinct for all $i \in \{1, \dots, k_j\}$. If $\operatorname{Im} \widetilde{g} \cap e_j \neq \emptyset$, then from Lemma 2, there are exactly two triangles of $\{T_{j,i}\}$, $i \in \{1, \dots, k_j\}$, which are transversed by $\widetilde{g}$ and, without loss of generality, we may assume that $T_{j,1}$, $T_{j,2}$ are these triangles. We distinguish two cases:

(1) *if the order of singularity is an even number*, then we glue $T_{j,1}$, $T_{j,2}$ via an isometry which identifies $e_{j,1}$ with $e_{j,2}$ and then, we pair-wise glue the other triangles $T_{j,i}$, by isometries along their free sides.

(2) *if the order of singularity is an odd number*, then we glue $T_{j,1}$, $T_{j,2}$, $T_{j,3}$, via an isometry which identifies $e_{j,1}$, $e_{j,2}$, $e_{j,3}$ and then, we pair-wise glue the other triangles $T_{j,i}$, by isometries along their free sides.

Notice that the isometries above (as well as the isometries in the sequel) which glue two ideal triangles, are exactly the same isometries which glue them in $\widetilde{X}$.

In this way, we construct a connected ideal polyhedron, say R, which contains $\widetilde{g}$ and all its singular edges are of index 3. Obviously R is equipped with an ideal triangulation $\widetilde{\mathcal{T}}_R$, induced by $\widetilde{\mathcal{T}}$. Denote the singular edges of R by d_j.

Label the three ideal triangles incident with the edge d_j, by $T_{j,1}$, $T_{j,2}$, $T_{j,3}$. We may assume, after relabeling if necessary, that $\widetilde{g}$ passes, as time increases, from $T_{j,1}$ to $T_{j,2}$. Consider four copies of R, say $R^{(a)}$, $R^{(b)}$, $R^{(c)}$ and $R^{(d)}$, with triangles $T^{(a)}_{j,i}$, $T^{(b)}_{j,i}$, $T^{(c)}_{j,i}$, $T^{(d)}_{j,i}$, $i \in \{1, 2, 3\}$, respectively. Cut along all singular edges in all copies of R. Then, for each singular edge d_j in R we have a total of twelve triangles, each having one free side. We glue these triangles by isometries along their free sides, as follows:

$$\begin{array}{ccc}
T^{(a)}_{j,1} & \text{with} & T^{(b)}_{j,2} \\
T^{(b)}_{j,1} & \text{with} & T^{(a)}_{j,2} \\
T^{(c)}_{j,1} & \text{with} & T^{(a)}_{j,3} \\
T^{(b)}_{j,3} & \text{with} & T^{(c)}_{j,2} \\
T^{(d)}_{j,1} & \text{with} & T^{(c)}_{j,3} \\
T^{(d)}_{j,3} & \text{with} & T^{(d)}_{j,2}
\end{array}$$

The resulting space is a surface S (not necessarily connected) since the index of every edge of S is 2. In the above construction, we take care so that there exists a component P of S which contains the geodesic $\widetilde{g}$. Indeed, since we have assumed that $\widetilde{g}$ passes from $T_{j,1}$ to $T_{j,2}$ in R, by gluing $T_{j,1}^{(a)}$ with $T_{j,2}^{(b)}$ and $T_{j,1}^{(b)}$ with $T_{j,2}^{(a)}$, such a component P exists.

In the case when $\operatorname{Im}\widetilde{g}$ coincides with an edge e of $\widetilde{X}$, we may proceed with the same method for the construction of P. In this special case, we do no need to pay any attention when we pair-wise glue the triangles along their free sides.

The surface P is of curvature less than or equal to -1, which means in this case that P has Riemannian curvature equal to -1. From the construction of P, we have a local isometry $f : P \to \widetilde{X}$. By Lemma 1, we have that P is simply connected and hence, f an isometry. By identifying P with its image, via f, we may consider P as a subpolyhedron of $\widetilde{X}$. By means of the Uniformization Theorem, we conclude also that P is isometric to the hyperbolic plane H^2. Also, as we have remarked after Lemma 1, the visual boundary ∂P, which is homeomorphic to the circle S^1, is embedded in $\partial\widetilde{X}$. So, we have found one circle $C = \partial P$ which satisfies the requirements of the theorem.

Let now $Y = p(P)$, where $p : \widetilde{X} \to X$ is the covering projection. Y is a finite ideal polyhedron contained in X, which contains necessarily a singular edge e_1 of X. This follows because P contains, by construction, a singular edge $\widetilde{e_1}$ of $\widetilde{X}$. From Lemma 4, we have that there is countable number of disjoint arcs $\operatorname{arc}[z_k w_k]$ in ∂P, $k \in \mathbb{N}$, such that:

(i) (z_k, w_k) are singular edges of $\widetilde{X}$,
(ii) the points $z_1, w_1, z_2, w_2, \ldots, z_k, w_k, \ldots$ are consecutive in $\partial P \approx S^1$.

Now, let $\widetilde{e}_i = (z_i, w_i)$ be such a singular edge of $\widetilde{X}$. Obviously, there is an ideal triangle $T_i \in \widetilde{\mathcal{T}}$ such that $T_i \cap P = \widetilde{e}_i$ and there exists also a geodesic $\widetilde{h} : (-\infty, \infty) \to \widetilde{X}$ which intersects $\widetilde{e}$ once and passes through T_i. Using the gluing construction above, we may construct a subpolyhedron P_i of $\widetilde{X}$ such that: $\operatorname{Im}\widetilde{h} \subset P_i$, $T_i \subset P_i$ and P_i is isometric to H^2. Denote by P_i^+ the subpolyhedron of P_i (and hence of $\widetilde{X}$) which contains T_i and such that $\operatorname{bd}(P_i^+) = \widetilde{e}_i$, where $\operatorname{bd}(P_i^+)$ denotes the topological boundary of P_i^+. Obviously P_i^+ is isometric to H^+ and since P_i^+ is a convex subset of $\widetilde{X}$, from Lemma 1, the visual boundary ∂P_i^+ of P_i^+ is a subspace of $\partial\widetilde{X}$ homeomorphic to $[0, 1]$. The boundary points of ∂P_i^+ in $\partial\widetilde{X}$ are z_i and w_i, so we denote ∂P_i^+ by $\operatorname{arc}[z_i w_i]$.

By applying Lemma 4 once again, we may find disjoint arcs $\operatorname{arc}[z_{i,k} w_{i,k}]$ in $\operatorname{arc}[z_i w_i]$, $k \in \mathbb{N}$, which satisfy properties (i), (ii) above. For each $i, k \in \mathbb{N}$, we continue the same procedure, with $\operatorname{arc}[z_{i,k} w_{i,k}]$ in the place of $\operatorname{arc}[z_i w_i]$ and we construct disjoint arcs $\operatorname{arc}[z_{k,i,j} w_{k,i,j}]$ in $\operatorname{arc}[z_{k,i} w_{k,i}]$, $j \in \mathbb{N}$, and so on.

By combining the arcs $\operatorname{arc}[z_{i_1,i_2,\ldots,i_k} w_{i_1,i_2,\ldots,i_k}]$, $i_1, i_2, \ldots, i_k \in \mathbb{N}$ and $k \to \infty$, we may find an uncountable number of simple paths in $\partial\widetilde{X}$ which join η and ξ. Therefore, the proof of theorem may easily be accomplished. □

Finally, we remark that if we remove from $\partial\widetilde{X}$ two points z, w with the property that the geodesic line (z,w) in $\widetilde{X}$ is an edge of $\widetilde{X}$, then $\partial\widetilde{X} - \{z,w\}$ is not connected. However, as an immediate application of Theorem 5 we have the following

Corollary 6. *There exists an infinite subset A of $\partial\widetilde{X}$ with* $\mathrm{Int}(A) = \emptyset$ *such that $\partial\widetilde{X} - A$ is arc connected.*

Proof: With the notation at the end of Theorem 5, we may construct the set A by picking up a point from the interior of each arc$[z_i w_i]$ in ∂P, $i \in \mathbb{N}$. Obviously $\partial\widetilde{X} - A$ is arc connected. □

References

[1] Benakli N., *Polyédres hyperboliques, passage du local au global*, Thése, Université de Paris-Sud, 1992.

[2] Bourdon M., *Structure conforme au bord et flot géodésique d'un* CAT(−1)*-espace*, Enseign. Math. **41** (1995), 63–102.

[3] Coornaert M., *Sur les groupes proprement discontinus d'isometries des espaces hyperboliques au sens de Gromov*, Thèse, U.L.P., 1990.

[4] Charitos C., Papadopoulos A., *The geometry of ideal* 2*-dimensional simplicial complexes*, Glasgow Math. J. **43** (2001), 39–66.

[5] Charitos C., Papadopoulos A., *Hyperbolic structures and measured foliations on* 2*-dimensional complexes*, Monatsh. Math. **139** (2003), 1–17.

[6] Charitos C., Papadopoulos A., *On the isometries of ideal polyhedra*, Rend. Circ. Mat. Palermo (2) **54** (2005), no. 1, 71–80.

[7] Charitos C., Tsapogas G., *Geodesic flow on ideal polyhedra*, Canad. J. Math. **49** (1997), no. 4, 696–707.

[8] Charitos C., Tsapogas G., *Complexity of geodesics on* 2*-dimensional ideal polyhedra and isotopies*, Math. Proc. Camb. Phil. Soc. **121** (1997), 343–358.

[9] Ghys E., de la Harpe P., *Sur les groupes hyperboliques d'après Mikhael Gromov*, Progress in Mathematics, vol. 83, Birkhäuser, Boston, 1990, pp. 1–25.

[10] Gromov M., *Structures Métriques pour les Variétés Riemanniennes*, J. Lafontaine and P. Pansu, Eds., Fernand Nathan, Paris, 1981.

[11] Gromov M., *Hyperbolic Groups*, in Essays in Group Theory, Math. Sci. Res. Inst. Publ., vol. 8, Springer, New York, 1987, pp. 75–263.

[12] Haglund F., *Les polyédres de Gromov*, Thése, Université de Lyon I, 1992.

[13] Kapovich I., Benakli N., *Boundaries of hyperbolic groups*, Combinatorial and Geometric Group Theory (New York, 2000/Hoboken, NJ, 2001), Contemp. Math. 296, Amer. Math. Soc., Providence, RI, 2002, pp. 39-93.

[14] Paulin F., *Constructions of hyperbolic groups via hyperbolizations of polyhedra*, in: Group Theory from a Geometrical Viewpoint, ICTP, Trieste, Italy, 1990, E. Ghys and A. Haefliger, Eds., World Sci. Publishing, River Edge, NJ, 1991, pp. 313–372.

[15] Thurston W.P., *Three-dimensional Geometry and Topology*, Princeton University Press, Princeton, NJ, 1997.

University of Patras, Department of Mathematics, Patras 26500, Greece
and
The American College of Greece, Pierce College, 6 Gravias Street, Athens 15342, Greece

E-mail: mvrontakis@acgmail.gr

(*Received* November 4, 2004, *revised* October 6, 2005)