Commentationes Mathematicae Universitatis Carolinae

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Commentationes Mathematicae Universitatis Carolinae, Vol. 48 (2007), No. 4, 607--622

Persistent URL: `http://dml.cz/dmlcz/119684`

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Abstract. Let L, M be Archimedean Riesz spaces and $\mathcal{L}_b(L,M)$ be the ordered vector space of all order bounded operators from L into M. We define a Lamperti Riesz subspace of $\mathcal{L}_b(L,M)$ to be an ordered vector subspace $\mathcal{L}$ of $\mathcal{L}_b(L,M)$ such that the elements of $\mathcal{L}$ preserve disjointness and any pair of operators in $\mathcal{L}$ has a supremum in $\mathcal{L}_b(L,M)$ that belongs to $\mathcal{L}$. It turns out that the lattice operations in any Lamperti Riesz subspace $\mathcal{L}$ of $\mathcal{L}_b(L,M)$ are given pointwise, which leads to a generalization of the classic Radon-Nikodým theorem for Riesz homomorphisms. We then introduce the notion of maximal Lamperti Riesz subspace of $\mathcal{L}_b(L,M)$ as a generalization of orthomorphisms. In this regard, we show that any maximal Lamperti Riesz subspace of $\mathcal{L}_b(L,M)$ is a band of $\mathcal{L}_b(L,M)$, provided M is Dedekind complete. Also, we extend standard transferability theorems for orthomorphisms to maximal Lamperti Riesz subspace of $\mathcal{L}_b(L,M)$. Moreover, we give a complete description of maximal Lamperti Riesz subspaces on some continuous function spaces.

Keywords: continuous functions spaces, disjointness preserving operator, Lamperti Riesz subspace, order bounded operator, orthomorphism, Radon-Nikodým, Riesz space

Classification: 06F20, 47B65

1. Introduction and preliminaries

We take the standard monographs [17], [25] as a starting point to which we refer the reader for unexplained terminology and notation. Throughout this paper, L and M are Archimedean Riesz spaces (also called vector lattices). The ordered vector space of all order bounded operators from L into M is denoted by $\mathcal{L}_b(L,M)$ and briefly by $\mathcal{L}_b(M)$ whenever $L = M$. In general, $\mathcal{L}_b(L,M)$ is not a Riesz space, unless M is Dedekind complete, for instance. We henceforth need to extend to $\mathcal{L}_b(L,M)$ some terminologies usually used in the context of Riesz spaces. We call a vector subspace $\mathcal{L}$ of $\mathcal{L}_b(L,M)$ a *Riesz subspace* of $\mathcal{L}_b(L,M)$ if for all $S,T \in \mathcal{L}$, the pair $\{S,T\}$ has a supremum in $\mathcal{L}_b(L,M)$ that belongs to $\mathcal{L}$. We define a Riesz subspace $\mathcal{I}$ of $\mathcal{L}_b(L,M)$ to be an *ideal* of $\mathcal{L}_b(L,M)$ whenever $0 \leq S \leq T$ in $\mathcal{L}_b(L,M)$ and $T \in \mathcal{I}$ imply $S \in \mathcal{I}$. Notice that we retrieve the usual definitions of Riesz subspaces and ideals if $\mathcal{L}_b(L,M)$ is a Riesz space.

An operator T from L into M is said to be *disjointness preserving* if $|Tf| \wedge |Tg| = 0$ in M whenever $|f| \wedge |g| = 0$ in L. A positive disjointness preserving operator is called *a Riesz* (or *lattice*) *homomorphism*. Observe that $T \in \mathcal{L}_b(L,M)$

is a lattice homomorphism if and only if $|Tf| = T|f|$ for all $f \in L$. This paper deals with Riesz subspaces of $\mathcal{L}_b(L, M)$ the elements of which are disjointness preserving operators. For the sake of simpleness, we call such Riesz spaces *Lamperti Riesz subspaces* of $\mathcal{L}_b(L, M)$. This terminology comes from [5] by Arendt in which order bounded disjointness preserving operators are called *Lamperti operators*. Ideals of $\mathcal{L}_b(L, M)$ the elements of which preserve disjointness are called *Lamperti ideals* of $\mathcal{L}_b(L, M)$. Lamperti Riesz subspaces of $\mathcal{L}_b(L, M)$ was used in [6] by the author and Boulabiar to give an alternative proof of the existence of the modulus of a complex order bounded disjointness preserving operator. Lamperti ideals of $\mathcal{L}_b(L, M)$ were investigated in [7] by the same authors.

Our approach in this paper relies heavily on the following fundamental result due to Meyer [18] (see also [4, Theorem 8.6]). If $T \in \mathcal{L}_b(L, M)$ preserves disjointness then the absolute value $|T|$ of T in $\mathcal{L}_b(L, M)$ exists and satisfies

$$|Tf| = |T\,|f|| = |T|\,|f| \quad \text{for all} \quad f \in L.$$

Elementary proofs of the Meyer's theorem can be found in [8] by Bernau and [21] by de Pagter. For more background on disjointness preserving operators we refer to [3] by Abramovich and Kitover, and [19] by Meyer-Nieberg.

Orthomorphisms on M form a fundamental class of disjointness preserving operators. Indeed, $T \in \mathcal{L}_b(M)$ is called an *orthomorphism* if $|Tf| \wedge |g| = 0$ whenever $|f| \wedge |g| = 0$. The set of all orthomorphisms on M is denoted by $\mathrm{Orth}(M)$. It is well-known that $\mathrm{Orth}(M)$ is a Riesz space the lattice operations of which are given pointwise, that is,

$$(S \vee T)f = Sf \vee Tf \quad \text{and} \quad (S \wedge T)f = Sf \wedge Tf$$

for all $S, T \in \mathrm{Orth}(M)$ and $f \in M^+$. Surveys on orthomorphisms can be found in [20] by de Pagter, and [25] by Zaanen.

It follows quickly from the formulas above that a pair of orthomorphisms on M has a supremum in $\mathcal{L}_b(M)$ that coincides with its supremum in $\mathrm{Orth}(M)$. Hence, $\mathrm{Orth}(M)$ is a Lamperti Riesz subspace of $\mathcal{L}_b(M)$. Now, let $\mathcal{L}$ be an arbitrary Lamperti Riesz subspace of $\mathcal{L}_b(M)$ that contains $\mathrm{Orth}(M)$. In particular, the identity operator I of M belongs to $\mathcal{L}$. Hence $I + T$ is a Riesz homomorphism from L into M for every positive operator $T \in \mathcal{L}$. From Problem 3.3.1 in [2], it follows that T is an orthomorphism on M. In other words, $\mathrm{Orth}(M)$ is a maximal element in the set of all Lamperti Riesz subspaces of $\mathcal{L}_b(M)$. Surprisingly, it turns out that most of the classical properties of $\mathrm{Orth}(M)$ are based on its maximality as a Lamperti Riesz subspace of $\mathcal{L}_b(M)$.

We proceed now to a brief synopsis of the main results of this paper. In the second section we prove that the lattice operations in any Lamperti Riesz subspace of $\mathcal{L}_b(L, M)$ are given pointwise. As a nice consequence we obtain a generalization of the classic Radon-Nikodým theorem for Riesz homomorphisms (see [16] by

Luxemburg and Schep). Maximal Lamperti Riesz subspaces of $\mathcal{L}_b(L,M)$ are introduced in the third section as a generalization of orthomorphisms. The first result we get in this direction is that any maximal Lamperti Riesz subspace of $\mathcal{L}_b(L,M)$ is a band of $\mathcal{L}_b(L,M)$, provided M is Dedekind complete. The last part of this section deals with the transferability of various order properties from M into any maximal Lamperti Riesz subspace of $\mathcal{L}_b(L,M)$, generalizing the vast literature on transferability theorems in the context of orthomorphisms [4], [9], [11], [24], [25]. A complete description of maximal Lamperti Riesz subspaces on some continuous function spaces is furnished in the last section of this paper.

We end this section with a basic lemma, the proof of which is analogous to the demonstration of the sufficient condition in Theorem 1.14 in [4] by Aliprantis and Burkinshaw.

Lemma 1. *Let $\mathcal{D}$ be a nonempty directed upward subset of $\mathcal{L}_b(L,M)$. If the set $\{Tf : T \in \mathcal{D}\}$ has a supremum in M for all $f \in L^+$ then $\mathcal{D}$ has a supremum in $\mathcal{L}_b(L,M)$ and*

$$(\sup \mathcal{D})f = \sup\{Tf : T \in \mathcal{D}\} \quad \textit{for all} \quad f \in L^+.$$

Notice that in Lemma 1 we do not assume M to be Dedekind complete.

2. Lamperti Riesz subspaces

As observed in the previous section, $\mathrm{Orth}(M)$ is a Lamperti Riesz subspace of $\mathcal{L}_b(M)$ the lattice operations in which are given pointwise. Our first theorem states that the latter remains valid for an arbitrary Lamperti Riesz subspace of $\mathcal{L}_b(L,M)$. It is a direct consequence of the Meyer's result and for this reason its proof is omitted.

Theorem 1. *Let $\mathcal{L}$ be a Lamperti Riesz subspace of $\mathcal{L}_b(L,M)$. Then the lattice operations in $\mathcal{L}$ are given pointwise, that is,*

$$(S \vee T)f = (Sf) \vee (Tf) \quad \textit{and} \quad (S \wedge T)\, f = (Sf) \wedge (Tf)$$

for all $S, T \in \mathcal{L}$ and $f \in L^+$.

Next we furnish necessary and sufficient conditions on two operators in a Lamperti Riesz subspace of $\mathcal{L}_b(L,M)$ to be disjoint.

Proposition 1. *Let $\mathcal{L}$ be a Lamperti Riesz subspace of $\mathcal{L}_b(L,M)$. For $S, T \in \mathcal{L}$ the following conditions are equivalent.*

(i) *$S \perp T$.*
(ii) *$Sf \perp Tf$ for all $f \in L$.*
(iii) *$Sf \perp Tg$ for all $f, g \in L$.*

Proof: (i) ⇒ (ii) Let $f \in L$ and observe that

$$|Sf| \wedge |Tf| = |S|\,|f| \wedge |T|\,|f| = (|S| \wedge |T|)\,(|f|) = 0$$

so $Sf \perp Tf$.

(ii) ⇒ (iii) Let $f, g \in L$. Then

$$0 \leq |Sf| \wedge |Tg| = |S|\,|f| \wedge |T|\,|g| \leq |S|\,(|f| + |g|) \wedge |T|\,(|f| + |g|) = 0.$$

Therefore $|Sf| \wedge |Tg| = 0$.

(iii) ⇒ (i) If $|Sf| \wedge |Tg| = 0$ for all $f, g \in L$ then

$$(|S| \wedge |T|)\, f = |S| f \wedge |T| f = |Tf| \wedge |Sf| = 0$$

for all $f \in L^+$. Hence $|T| \wedge |S| = 0$. □

Now we turn our attention to Radon-Nikodým type theorems on Riesz homomorphisms. To prove the main theorem in this direction we need the following proposition, which is of an independent interest on its own.

Proposition 2. *Let $\mathcal{L}$ be a Lamperti Riesz subspace of $\mathcal{L}_b(L, M)$. We consider the following statements for $S, T \in \mathcal{L}$.*

(i) *$Sf \in \{Tf\}^{\mathrm{dd}}$ for all $f \in L$.*
(ii) *$S(B) \subset \{T(B)\}^{\mathrm{dd}}$ for all bands B in L.*
(iii) *$S\big(\{f\}^{\mathrm{dd}}\big) \subset \big\{T\big(\{f\}^{\mathrm{dd}}\big)\big\}^{\mathrm{dd}}$ for all $f \in L$.*
(iv) *$S(L) \subset \{T(L)\}^{\mathrm{dd}}$.*
(v) *$S \in \{T\}^{\mathrm{dd}}$, where $\{T\}^{\mathrm{dd}}$ is the principal band generated by T in $\mathcal{L}$.*
(vi) *The set $\{|S| \wedge n|T| : n \in \mathbb{N}\}$ has $|S|$ as a supremum in $\mathcal{L}_b(L, M)$.*

Then (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) ⇒ (v)*. Moreover, if M has the principal projection property and $\mathcal{L}$ is an ideal of $\mathcal{L}_b(L, M)$ then* (v) ⇒ (vi) ⇒ (i)*, so that all the properties above are equivalent.*

Proof: The implications (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) are obvious.

(iv) ⇒ (v) Let $\{T\}^{\mathrm{d}}$ denote the disjoint complement of T in $\mathcal{L}$ and let $R \in \{T\}^{\mathrm{d}}$. By Proposition 1, $R(L) \perp T(L)$ and then $R(L) \subset \{T(L)\}^{\mathrm{d}} \subset \{S(L)\}^{\mathrm{d}}$. This implies $\{T\}^{\mathrm{d}} \subset \{S\}^{\mathrm{d}}$, where we use Proposition 1 once more. We conclude that $S \in \{T\}^{\mathrm{dd}}$.

Assume now M to have the principal projection property and $\mathcal{L}$ to be an ideal of $\mathcal{L}_b(L, M)$.

(v) ⇒ (vi) Clearly, we may suppose S, T to be positive. Let $f \in L^+$ and observe that

$$(S \wedge nT)f = (Sf) \wedge n(Tf) \quad \text{for all} \quad n \in \mathbb{N}.$$

where $\mathbb{N} = \{1, 2, \dots\}$ is the set of all natural numbers. Hence $\{(S \wedge nT)f : n \in \mathbb{N}\}$ has a supremum in M because M has the principal projection property. By Lemma 1, $\{S \wedge nT : n \in \mathbb{N}\}$ has a supremum R in $\mathcal{L}_b(L, M)$. Obviously, $0 \leq R \leq S$ in $\mathcal{L}_b(L, M)$. However, $\mathcal{L}$ is an ideal of $\mathcal{L}_b(L, M)$ so $R \in \mathcal{L}$. It follows that R is the supremum of $\{S \wedge nT : n \in \mathbb{N}\}$ in $\mathcal{L}$. On the other hand, S is the supremum of $\{S \wedge nT : n \in \mathbb{N}\}$ in $\mathcal{L}$ as $S \in \{T\}^{\mathrm{dd}}$. Thus $R = S$, which gives the desired result.

The implication (vi) $\Rightarrow$ (i) can be obtained in the same way. □

We proceed to a short historical account on Radon-Nikodým type theorems on Riesz homomorphisms. Let $S : L \to M$ be a positive operator and $T : L \to M$ be a Riesz homomorphism. Consider the following assertions.

(i) $Sf \in \{Tf\}^{\mathrm{dd}}$ for all $f \in L$.
(ii) $S(B) \subset \{T(B)\}^{\mathrm{dd}}$ for all bands B in L.
(iii) $S\big(\{f\}^{\mathrm{dd}}\big) \subset \big\{T\big(\{f\}^{\mathrm{dd}}\big)\big\}^{\mathrm{dd}}$ for all $f \in L$.
(iv) The pair $\{S, nT\}$ has a infimum in $\mathcal{L}_b(L, M)$ for all $n \in \mathbb{N}$ and the set $\{S \wedge nT : n \in \mathbb{N}\}$ has S as a supremum in $\mathcal{L}_b(L, M)$.

The equivalences (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) have been established in 1991 by Huijsmans and de Pagter [14] when both L and M are Dedekind complete. One year later, Huijsmans and Luxemburg [13] have proved the equivalence (i) $\Leftrightarrow$ (iv) even if the assumption of Dedekind completeness is imposed only on M. Under this same condition, de Pagter and Schep [22] have proved that all the equivalences (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) remain valid. The latter result is obtained next under weaker assumptions.

Theorem 2. *If M has the principal projection property then* (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) *and* (i) $\Rightarrow$ (iv). *Moreover, if M in addition is uniformly complete then* (iv) $\Rightarrow$ (i), *so that* (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv).

Proof: The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are straightforward.

For a positive operator $U \in \mathcal{L}_b(L, M)$, we set

$$\mathcal{I}_U = \{R \in \mathcal{L}_b(L, M) : -nU \leq R \leq nU, \quad \text{for some} \quad n \in \mathbb{N}\}.$$

If U is a Riesz homomorphism then $\mathcal{I}_U$ is a Lamperti ideal of $\mathcal{L}_b(L, M)$. Indeed, it is readily verified that $\mathcal{I}_U$ is a vector subspace of $\mathcal{L}_b(L, M)$. Let $R \in \mathcal{I}_U$ and choose $n \in \mathbb{N}$ so that $-nU \leq R \leq nU$. Hence,

$$|Rf| \leq nUf \quad \text{for all} \quad f \in L^+.$$

It follows that if $f, g \in L$ with $|f| \wedge |g| = 0$ then

$$0 \leq |Rf| \wedge |Rg| \leq n(U|f| \wedge U|g|) = 0.$$

Consequently, R is disjointness preserving. We derive that all operators in $\mathcal{I}_U$ have absolute values in $\mathcal{L}_b(L,M)$ which are given pointwise. Clearly, $|R| \in \mathcal{I}_U$ for all $R \in \mathcal{I}_U$. This shows that $\mathcal{I}_U$ is a Lamperti Riesz subspace of $\mathcal{L}_b(L,M)$. Now let $V, W \in \mathcal{L}_b(L,M)$ with $0 \leq V \leq W$ and $W \in \mathcal{I}_U$. If $n \in \mathbb{N}$ is such that $-nU \leq W \leq nU$ then $-nU \leq V \leq nU$ so $V \in \mathcal{I}_U$. Therefore $\mathcal{I}_U$ is a Lamperti ideal of $\mathcal{L}_b(L,M)$.

We prove the implication (iii) $\Rightarrow$ (i). Let $f, g \in L$ with $|f| \wedge |g| = 0$ and observe that $\{f\}^{\mathrm{dd}} \perp \{g\}^{\mathrm{dd}}$. Since T is a Riesz homomorphism, we get $\{T(\{f\}^{\mathrm{dd}})\}^{\mathrm{dd}} \perp \{T(\{g\}^{\mathrm{dd}})\}^{\mathrm{dd}}$. Using (iii) we obtain $S(\{f\}^{\mathrm{dd}}) \perp S(\{g\}^{\mathrm{dd}})$ and $S(\{f\}^{\mathrm{dd}}) \perp T(\{g\}^{\mathrm{dd}})$. In particular $|Sf| \wedge |Sg| = 0$ and $|Sf| \wedge |Tg| = 0$. Now it is easy to prove that $U = S + T$ is a Riesz homomorphism from L into M. The Lamperti Riesz subspace $\mathcal{I}_U$ of $\mathcal{L}_b(L,M)$ is an ideal and contains S, T. By Proposition 2, $Sf \in \{Tf\}^{\mathrm{dd}}$ for all $f \in L$.

We proceed to (i) $\Rightarrow$ (iv). The same argument as previously used to prove the implication (iii) $\Rightarrow$ (i) yields that $U = S+T$ is a Riesz homomorphism from L into M and then $\mathcal{I}_U$ is a Lamperti ideal of $\mathcal{L}_b(L,M)$. Using once more Proposition 2, the set $\{S \wedge nT : n \in \mathbb{N}\}$ has S as a supremum in $\mathcal{L}_b(L,M)$.

Now we show (iv) $\Rightarrow$ (i), when M in addition is uniformly complete (and then Dedekind σ-complete). Let $f \in L^+$ and observe that the sequence $((S \wedge nT)f)_{n=1}^{\infty}$ is increasing in M and $(S \wedge nT)f \leq Sf$ for all $n \in \mathbb{N}$. Since M is Dedekind σ-complete, the set $\{(S \wedge nT)f : n \in \mathbb{N}\}$ has a supremum in M. Using Lemma 1, we see that the set $\{(S \wedge nT) : n \in \mathbb{N}\}$ has a supremum in $\mathcal{L}_b(L,M)$ which is given pointwise. But $S = \sup\{(S \wedge nT) : n \in \mathbb{N}\}$ in $\mathcal{L}_b(L,M)$ and then

$$Sf = \sup\{(S \wedge nT)f : n \in \mathbb{N}\} \quad \text{for all} \quad f \in L^+.$$

It follows straightforwardly that $Sf \in \{Tf\}^{\mathrm{dd}}$ for all $f \in L^+$. This completes the proof. □

3. Maximal Lamperti Riesz subspaces

We have pointed out that $\mathrm{Orth}(M)$ is a maximal Lamperti Riesz subspace of $\mathcal{L}_b(M)$. It seems to be natural therefore to introduce and study the notion of maximal Lamperti Riesz subspaces of $\mathcal{L}_b(L,M)$ as a generalization of orthomorphisms. A Lamperti Riesz subspace $\mathcal{M}$ of $\mathcal{L}_b(L,M)$ is said to be *maximal* if $\mathcal{M} = \mathcal{L}$ whenever $\mathcal{M} \subset \mathcal{L}$ for some Lamperti Riesz subspace $\mathcal{L}$ of $\mathcal{L}_b(L,M)$. Our next purpose is to extend standard facts on orthomorphisms to arbitrary maximal Lamperti Riesz subspaces. In this direction, we prove that any maximal Lamperti Riesz subspace of $\mathcal{L}_b(L,M)$ is a band, provided that M is Dedekind complete. We first need the following lemma.

Lemma 2. *Let $S \in \mathcal{L}_b(L,M)$ be a disjointness preserving operator and let $\mathcal{M}$ be a Lamperti Riesz subspace of $\mathcal{L}_b(L,M)$. If $S \in \mathcal{M}$ then $|Sf| \wedge |Tg| = 0$ for* *all $T \in \mathcal{M}$ and $|f| \wedge |g| = 0$. Conversely, if $\mathcal{M}$ is maximal and an order bounded disjointness preserving operator S satisfies $|Sf| \wedge |Tg| = 0$ for all $T \in \mathcal{M}$ and $|f| \wedge |g| = 0$ then $S \in \mathcal{M}$.*

Proof: Assume that $S \in \mathcal{M}$. Let $T \in \mathcal{M}$ and $f, g \in L$ with $|f| \wedge |g| = 0$. Hence

$$\begin{aligned} 0 &\leq |Sf| \wedge |Tg| = |S|\,|f| \wedge |T|\,|g| \\ &\leq (|S| + |T|)\,|f| \wedge (|S| + |T|)\,|g| \\ &= (|S| + |T|)\,(|f| \wedge |g|) = 0. \end{aligned}$$

So $|Sf| \wedge |Tg| = 0$. Assume now that $\mathcal{M}$ is maximal and that S satisfies $|Sf| \wedge |Tg| = 0$ for all $T \in \mathcal{M}$ and $|f| \wedge |g| = 0$. It is easily seen that $T + n|S|$ is a Riesz homomorphism for all $T \in \mathcal{M}^+$ and $n \in \mathbb{N}$. Let $\mathcal{I}$ denote the vector subspace of $\mathcal{L}_b\,(L, M)$ defined by

$$\mathcal{I} = \left\{R \in \mathcal{L}_b(L, M) : -T - n|S| \leq R \leq T + n|S| \;\text{ for some }\; T \in \mathcal{M}^+, n \in \mathbb{N}\right\}.$$

Observe that $S \in \mathcal{I}$ and $\mathcal{M} \subset \mathcal{I}$. Furthermore, using a similar argument as previously used at the beginning of the proof of Theorem 2, we show that $\mathcal{I}$ is a Lamperti Riesz subspace of $\mathcal{L}_b(L, M)$. By maximality, $\mathcal{M} = \mathcal{I}$ and then $S \in \mathcal{M}$. This completes the proof. □

Theorem 3. *Let $\mathcal{M}$ be a maximal Lamperti Riesz subspace of $\mathcal{L}_b(L, M)$. Then $\mathcal{M}$ is an ideal of $\mathcal{L}_b(L, M)$. Moreover, $\mathcal{M}$ is a band of $\mathcal{L}_b(L, M)$ if M is Dedekind complete.*

Proof: Let $S, T \in \mathcal{L}_b\,(L, M)$ with $T \in \mathcal{M}$ and $0 \leq S \leq T$. Since T is a Riesz homomorphism, so is S. Let $f, g \in L$ satisfy $|f| \wedge |g| = 0$, and let $R \in \mathcal{M}$. Then

$$0 \leq S\,|f| \wedge R\,|g| \leq T\,|f| \wedge R\,|g| = 0.$$

Hence $S \in \mathcal{M}$, where we use Lemma 2. Thus $\mathcal{M}$ is an ideal of $\mathcal{L}_b(L, M)$.

Assume now M to be Dedekind complete. Let $\mathcal{D}$ be a directed upward subset of $\mathcal{M}^+$ and suppose that $\mathcal{D}$ has a supremum T in $\mathcal{L}_b(L, M)$. We claim that $T \in \mathcal{M}$. To this end, let $f, g \in L$ with $|f| \wedge |g| = 0$. By Lemma 2, $|Rf| \wedge |Sg| = 0$ for all $R, S \in \mathcal{D}$. Using Theorem 1.14 in [4], we get

$$\begin{aligned} 0 &\leq |Tf| \wedge |Tg|| \leq T|f| \wedge T|g| \\ &= \sup\{R\,|f| : R \in \mathcal{D}\} \wedge \sup\{S\,|g| : S \in \mathcal{D}\} \\ &= \sup\{R|f| \wedge S|g| : R, S \in \mathcal{D}\} = 0. \end{aligned}$$

Thus T preserves disjointness. Analogously,

$$|Tf| \wedge |Sg| = 0 \;\text{ for all }\; S \in \mathcal{M}.$$

By Lemma 2, $T \in \mathcal{M}$. The proof is complete. □

Consider now the following properties of Riesz spaces: **(1)** Relatively uniform completeness, **(2)** Principal projection property, **(3)** Dedekind σ-completeness, **(4)** Dedekind completeness, **(5)** Laterally σ-completeness, **(6)** Laterally completeness, **(7)** Universally σ-completeness, and **(8)** Universally completeness. It is well-known that $\mathrm{Orth}(M)$ has any one of the properties **(1)**–**(8)** when M has the same property (see, for instance, [4], [9], [10], [11], [20], [25]). It is therefore a natural question to ask whether the transferability of these properties holds for arbitrary maximal Lamperti Riesz subspaces of $\mathcal{L}_b(L, M)$. Examining the proofs of such transferability theorems in the context of orthomorphisms, we can see that they can be used for the more general setting of maximal Lamperti Riesz subspaces. Thus we have the following.

Theorem 4. *Let $\mathcal{M}$ be a maximal Lamperti Riesz subspace of $\mathcal{L}_b(L, M)$. If M has one of the properties* **(1)**–**(8)** *then $\mathcal{M}$ also has the same property.*

Now we turn our attention to Riesz spaces with sufficiently many projections. In [24], it is proved that $\mathrm{Orth}(M)$ has sufficiently many projections if M has sufficiently many projections. The proof uses some properties of orthomorphisms which are not true for arbitrary disjointness preserving operators. More precisely, the proof is based on the facts that $\mathrm{Orth}(M)$ has a week order unit and that the kernel of any orthomorphism is a band. In spite of that, we prove the following.

Theorem 5. *Let $\mathcal{M}$ be a maximal Lamperti Riesz subspace of $\mathcal{L}_b(L, M)$. Then $\mathcal{M}$ has sufficiently many projections whenever M has sufficiently many projections.*

Proof: Let $S \in \mathcal{M}^+$ with $S \neq 0$. Since the band $\{S(L)\}^{\mathrm{dd}}$ is nonzero and M has sufficiently many projections, there exists a nonzero projection band $B \subset \{S(L)\}^{\mathrm{dd}}$. Let Q be the band projection on B and $P = I - Q$ be the band projection on B^{d}, where I is the identity operator on M. Since $\mathcal{M}$ is an ideal of $\mathcal{L}_b(L, M)$ (see Theorem 3), it is readily checked that $PT \in \mathcal{M}$ for all $T \in \mathcal{M}$. Besides, the map

$$\widetilde{P} : \mathcal{M} \longrightarrow \mathcal{M},\ T \longmapsto PT$$

is a band projection of $\mathcal{M}$. For all $T \in \ker \widetilde{P}$, we have $PT = 0$. In other words

$$T(L) \subset \ker P = B \subset \{S(L)\}^{\mathrm{dd}} \quad \text{for all} \quad T \in \ker \widetilde{P}.$$

By Proposition 2, $\{S\}^{\mathrm{dd}}$ contains the projection band $\ker \widetilde{P}$. To finish our proof it suffices to show that $\ker \widetilde{P}$ is not trivial. Indeed, observe that $\widetilde{P}(QS) = PQS = 0$ and then $QS \in \ker \widetilde{P}$. Observe now that if $QS = 0$, then $S(L) \subset \ker Q = B^{\mathrm{d}}$. Thus $B \subset \{S(L)\}^{\mathrm{d}} \subset B^{\mathrm{d}}$, which implies that $B = \{0\}$, in contradiction with the hypotheses. This shows that $\ker \widetilde{P}$ is a nonzero projection band contained in $\{S\}^{\mathrm{dd}}$. Therefore $\mathcal{M}$ have sufficiently many projections. $\square$

The last result of this section deals with the projection property. It is shown in [25] that the projection property is heredited by $\mathrm{Orth}(M)$. The main argument of the proof is the order continuity of orthomorphisms. Since order bounded disjointness preserving operators need not be order continuous, it is plausible to think that this result cannot be extended to maximal Lamperti Riesz subspaces of $\mathcal{L}_b(L,M)$. Surprisingly, we prove in the last result of this section that any maximal Lamperti Riesz subspace of $\mathcal{L}_b(L,M)$ has the projection property if M has the projection property. The proof is based on the following lemma.

Lemma 3. *Let $\mathcal{L}$ be a Lamperti Riesz subspace of $\mathcal{L}_b(L,M)$ and let $e \in L^+$. Let $\varphi : \mathcal{L} \to M$ be defined by*

$$\varphi T = Te \quad \text{for all} \quad T \in \mathcal{L}.$$

Then the following hold.

(i) *φ is a Riesz homomorphism.*

(ii) *If M has the principal projection property and $\mathcal{L}$ is an ideal of $\mathcal{L}_b(L,M)$, then the range $\varphi(\mathcal{L})$ of φ is an order dense Riesz subspace of the ideal $M_{\varphi(\mathcal{L})}$ generated by $\varphi(\mathcal{L})$ in M.*

PROOF: (i) Clearly φ is linear. Let $T \in \mathcal{L}$ and observe that

$$|\varphi T| = |Te| = |T|e = \varphi\,|T|\,.$$

Thus φ is a Riesz homomorphism.

(ii) By (i) φ is a Riesz homomorphism and then $\varphi(\mathcal{L})$ is a Riesz subspace of M. Let $g \in M_{\varphi(\mathcal{L})}$ and $T \in \mathcal{L}^+$ with $0 < g \leq Te$. Given $\varepsilon \in (0,\infty)$, we denote p_ε the band projection on the principal band $\left\{ (g - \varepsilon Te)^+ \right\}^{\mathrm{dd}}$ in M. It follows that $p_\varepsilon\,(g - \varepsilon Te) = (g - \varepsilon Te)^+ \geq 0$ and then

$$0 \leq \varepsilon p_\varepsilon(Te) \leq p_\varepsilon g \leq g.$$

Suppose by a way of contradiction that $p_\varepsilon Te = 0$ for all $\varepsilon \in (0,\infty)$. Hence,

$$Te \wedge (g - \varepsilon Te)^+ = 0 \quad \text{for all} \quad \varepsilon \in (0,\infty)\,.$$

It yields that $Te \wedge g = 0$, which contradicts $0 < g \leq Te$. Therefore, there exists $\varepsilon \in (0,\infty)$ such that $0 < \varepsilon p_\varepsilon Te \leq g$. It is clear that $S = \varepsilon p_\varepsilon T \in \mathcal{L}$. Since $0 < Se = \varphi S \leq g$, $\varphi(\mathcal{L})$ is order dense in $M_{\varphi(\mathcal{L})}$ and we are done. $\square$

We are in position now to prove the last result of this section.

Theorem 6. *Let $\mathcal{M}$ be a maximal Lamperti Riesz subspace of $\mathcal{L}_b(L,M)$. Then $\mathcal{M}$ has the projection property if M has the projection property.*

Proof: Let $\mathcal{B}$ be a band in $\mathcal{M}$ and $S \in \mathcal{M}^+$. We define

$$\mathcal{B}_S = \{T \in \mathcal{B} : 0 \leq T \leq S\}.$$

We claim that $\mathcal{B}_S$ has a supremum in $\mathcal{M}$. By Lemma 1, it suffices to show that $\{Tf : T \in \mathcal{B}_S\}$ has a supremum in M for all $f \in L^+$. Hence, if $f \in L^+$ then

$$\{Tf : T \in \mathcal{B}_S\} \subset \{Tf : T \in \mathcal{B} \text{ and } 0 \leq Tf \leq Sf\}.$$

Conversely, choose $f \in L^+$ and let $T \in \mathcal{B}$ such that $0 \leq Tf \leq Sf$. Since $|T| \in \mathcal{B}$ and $|T|f = |Tf| = Tf$, we can assume T to be positive. Let P denote the band projection on the principal band $\{Tf\}^{\mathrm{dd}}$ and define $R = PT$. It is simple to check that $R \in \mathcal{B}$ and

$$Rf = PTf = Tf.$$

Observe now that if $g \in [0, nf]$ for some $n \in \mathbb{N}$, then

$$(Sg - Tg)^- = (S-T)^- g \leq (S-T)^- (nf) = n\,(Sf - Tf)^- = 0,$$

where we use Theorem 1. It yields that $Tg \leq Sg$ for all g in the principal order ideal generated by f in L. Consequently, if $g \in L^+$ then

$$\begin{aligned} Rg &= PTg \\ &= \sup\{Tg \wedge nTf : n \in \mathbb{N}\} \\ &= \sup\{T(g \wedge nf) : n \in \mathbb{N}\} \\ &\leq \sup\{S(g \wedge nf) : n \in \mathbb{N}\} \\ &\leq Sg. \end{aligned}$$

So $R \leq S$ and then $R \in \mathcal{B}_S$. This yields the converse inclusion

$$\{Tf : T \in \mathcal{B} \text{ and } 0 \leq Tf \leq Sf\} \subset \{Tf : T \in \mathcal{B}_S\}.$$

We shall prove now that the set $\{Tf : T \in \mathcal{B} \text{ and } 0 \leq Tf \leq Sf\}$ has a supremum in M. First of all, observe that $\mathcal{B}$ is an ideal in $\mathcal{L}_b(L,M)$. Let $f \in L^+$ and define $\varphi : \mathcal{B} \to M$ by

$$\varphi T = Tf \quad \text{for all } T \in \mathcal{B}.$$

By (ii) in Lemma 3, the range $\varphi(\mathcal{B}) = \{Tf : T \in \mathcal{B}\}$ of φ is a Riesz subspace of M, which is order dense in the projection band $\{\varphi(\mathcal{B})\}^{\mathrm{dd}}$. Denote by Q the band projection on $\{\varphi(\mathcal{B})\}^{\mathrm{dd}}$. We get quickly

$$\{g \in \varphi(\mathcal{B}) : 0 \leq g \leq Sf\} = \{g \in \varphi(\mathcal{B}) : 0 \leq g \leq QSf\}.$$

On the other hand, Theorem 3.1 in [4] yields that $\{g \in \varphi(\mathcal{B}) : 0 \leq g \leq QSf\}$ has a supremum in $\{\varphi(\mathcal{B})\}^{\mathrm{dd}}$ and

$$\sup\{g \in \varphi(\mathcal{B}) : 0 \leq g \leq QSf\} = QSf.$$

Since $\{\varphi(\mathcal{B})\}^{\mathrm{dd}}$ is a projection band in M, QSf is the supremum in M of

$$\left\{g \in \{\varphi(\mathcal{B})\}^{\mathrm{dd}} : 0 \leq g \leq QSf\right\}.$$

Therefore, QSf is the supremum in M of

$$\{g \in \varphi(\mathcal{B}) : 0 \leq g \leq Sf\}.$$

This proves that $\{Tf : T \in \mathcal{B}_S\}$ has a supremum in M for all $f \in L^+$. By Lemma 1, $\mathcal{B}_S$ has a supremum in $\mathcal{L}_b(L, M)$ and we have

$$(\sup \mathcal{B}_S) f = \sup\{Tf : T \in \mathcal{B}_S\} \quad \text{for all} \quad f \in L^+.$$

Since $\sup \mathcal{B}_S \leq S$ and according to Theorem 3, $\mathcal{M}$ is an ideal we obtain $\sup \mathcal{B}_S \in \mathcal{M}$. $\square$

4. Examples

In this last section, we describe maximal Lamperti Riesz subspaces on some continuous functions spaces. We start with some useful notations and facts. Let X be a completely regular space. As usual, we denote by $\mathbb{R}^X$ the Riesz space of all real-valued functions on X and by $C(X)$ the Riesz subspace of all continuous functions. By $\mathbf{e}_X$ we mean the function in $C(X)$ defined by $\mathbf{e}_X(x) = 1$ for all $x \in X$. Therefore, the vector lattice $C(X)$ has $\mathbf{e}_X$ as an order unit if X in addition is compact. The cozeroset of a function f in $C(X)$ is denoted by $\mathrm{coz}(f)$ and defined by

$$\mathrm{coz}(f) = \{x \in X : f(x) \neq 0\}.$$

For more background on continuous functions spaces we refer to the classical book [12] by Gillman and Jerison.

Let X, Y be completely regular spaces and let U be an algebra homomorphism from $C(X)$ into $\mathbb{R}^Y$. It is clear that

$$F_U = \{w \in C(Y) : wU \text{ maps } C(X) \text{ into } C(Y)\}$$

is a Riesz subspace of $C(Y)$. Observe also that wU is an order bounded disjointness preserving operator from $C(X)$ into $C(Y)$ for all $w \in F_U$. Our first result is a direct consequence of these two remarks.

Proposition 3. *If U is an algebra homomorphism from $C(X)$ into $\mathbb{R}^Y$ then $\mathcal{M}_U = \{wU : w \in F_U\}$ is a Lamperti Riesz subspace of $\mathcal{L}_b\left(C(X), C(Y)\right)$.*

For an algebra homomorphism U from $C(X)$ into $\mathbb{R}^Y$ we put

$$O_U = \bigcup_{T\in\mathcal{M}_U} \operatorname{coz}\left(T\mathbf{e}_X\right).$$

It is a routine matter to prove that Uf is continuous on O_U for all $f \in C(X)$. Observe also that $T = (T\mathbf{e}_X)\,U$ for all $T \in \mathcal{M}_U$. Indeed, if $T \in \mathcal{M}_U$ then there exists $w \in C(Y)$ satisfying $T = wU$. It follows that

$$Tf = wUf = wU\left(\mathbf{e}_X f\right) = w\left(U\mathbf{e}_X\right)\left(Uf\right) = \left(T\mathbf{e}_X\right)\left(Uf\right) \quad \text{for all} \quad f \in C(X),$$

and then $T = (T\mathbf{e}_X)\,U$.

We say that U is maximal whenever

$$\left[O_U \subset O_V \text{ and } (Uf)_{|O_U} = (Vf)_{|O_U} \quad \text{for all} \quad f \in C(X)\right] \Longrightarrow \left[O_U = O_V\right]$$

for all algebra homomorphisms V from $C(X)$ into $\mathbb{R}^Y$.

We claim that a subset $\mathcal{M}$ of $\mathcal{L}_b(C(X), C(Y))$ is a maximal Lamperti Riesz subspace of $\mathcal{L}_b(C(X), C(Y))$ if and only if there exists a maximal algebra homomorphism U from $C(X)$ into $\mathbb{R}^Y$ satisfying $\mathcal{M} = \mathcal{M}_U$. To prove this result we need some preparation.

Lemma 4. *Let F be an Archimedean Riesz space and let T be an order bounded disjointness preserving operator from $C(X)$ into F. Then $T = 0$ if and only if $T\mathbf{e}_X = 0$.*

Proof: It is clear that if $T = 0$ then $T\mathbf{e}_X = 0$. Conversely assume that $Te_X = 0$ and let $f \in C\left(X\right)^+$. From

$$(f - n\mathbf{e}_X)^2 = f^2 - 2nf + n^2\mathbf{e}_X \geq 0,$$

it follows that

$$f - n\mathbf{e}_X \leq 2f - n\mathbf{e}_X \leq \frac{f^2}{n}$$

and then

$$0 \leq f - f \wedge n\mathbf{e}_X = (f - n\mathbf{e}_X)^+ \leq \frac{f^2}{n}.$$

We deduce that

$$\begin{aligned} 0 \leq |Tf| &\leq \left|T\left(f - f \wedge n\mathbf{e}_X\right)\right| + \left|T\left(f \wedge n\mathbf{e}_X\right)\right| \\ &\leq \frac{1}{n}|T|\left(f^2\right) + n\left|T\mathbf{e}_X\right| = \frac{1}{n}\,|T|\left(f^2\right), \end{aligned}$$

for all natural numbers n. Since F is Archimedean, $Tf = 0$ for all $f \in C(X)^+$. This implies that $T = 0$ and the proof is finished. $\square$

Our next result furnish a complete description of Lamperti Riesz subspaces of $C(X)' = \mathcal{L}_b\left(C(X), \mathbb{R}\right)$.

Proposition 4. *If $\mathcal{M}$ is a nonzero Lamperti Riesz subspace of $C(X)'$, then there exists a nonzero algebra homomorphism U from $C(X)$ into $\mathbb{R}$ with $\mathcal{M} = \mathcal{M}_U$. In particular any nonzero Lamperti Riesz subspace of $C(X)'$ is maximal.*

PROOF: Let $\mathcal{M}$ be a nonzero Lamperti Riesz subspace of $C(X)'$. Using Lemma 4 it yields that $\{T\mathbf{e}_X : T \in \mathcal{M}\}$ is a nonzero vector subspace of $\mathbb{R}$ and consequently $\{T\mathbf{e}_X : T \in \mathcal{M}\} = \mathbb{R}$. Let $U \in \mathcal{M}$ satisfying $U\mathbf{e}_X = 1$. By Theorem 18.8 in [23] U is a nonzero algebra homomorphism from $C(X)$ into $\mathbb{R}$. Since $\mathcal{M}$ is a vector space it is not hard to see that $\mathcal{M}_U = \{wU : w \in \mathbb{R}\} \subset \mathcal{M}$. To prove the converse inclusion let $T \in \mathcal{M}$ and observe that $S = T - (T\mathbf{e}_X)U \in \mathcal{M}$ and satisfies $S\mathbf{e}_X = 0$. By Lemma 4, $S = 0$ and then $T = (T\mathbf{e}_X)U \in \mathcal{M}_U$. That is $\mathcal{M} \subset \mathcal{M}_U$ and finally $\mathcal{M} = \mathcal{M}_U$.

Assume now that $\mathcal{M} \subset \mathcal{M}'$ where $\mathcal{M}'$ is Lamperti Riesz subspace of $C(X)'$. Let U' be a nonzero algebra homomorphism from $C(X)$ into $\mathbb{R}$ with $\mathcal{M}' = \mathcal{M}_{U'}$. It follows immediately from $\mathcal{M}_U \subset \mathcal{M}_{U'}$ that $U = U'$ and then $\mathcal{M} = \mathcal{M}'$. In other words $\mathcal{M}$ is a maximal Lamperti Riesz subspace of $C(X)'$. □

We generalize the proposition above as follows.

Proposition 5. *Let $\mathcal{M}$ be a Lamperti Riesz subspace of $\mathcal{L}_b(C(X), C(Y))$. Then there exists an algebra homomorphism U from $C(X)$ into $\mathbb{R}^Y$ with $\mathcal{M} \subset \mathcal{M}_U$. If $\mathcal{M}$ in addition is maximal then $\mathcal{M} = \mathcal{M}_U$.*

PROOF: Let $y \in Y$ and observe that $\mathcal{M}(y) = \left\{\delta_y \circ T : T \in \mathcal{M}\right\}$ is a Lamperti Riesz subspace of $C(X)'$. By Proposition 4, $\mathcal{M}(y) = \{0\}$ or $\mathcal{M}(y) = \left\{wU_y : w \in \mathbb{R}\right\}$ for a nonzero algebra homomorphism U_y from $C(X)$ into $\mathbb{R}$. Let U be the mapping from $C(X)$ into $\mathbb{R}^Y$ defined by

$$(Uf)(y) = \begin{cases} U_y f & \text{if } \ \mathcal{M}(y) \neq \{0\}, \\ 0 & \text{if } \ \mathcal{M}(y) = \{0\}. \end{cases}$$

It is easily shown that U is an algebra homomorphism from $C(X)$ into $\mathbb{R}^Y$. Let $T \in \mathcal{M}$ and $y \in Y$. Observe that if $\mathcal{M}(y) \neq \{0\}$ then

$$(Tf)(y) = \left(\delta_y \circ T\right) f = (T\mathbf{e}_X)(y) \cdot U_y f = (T\mathbf{e}_X)(y) \cdot (Uf)(y) \ \text{ for all } \ f \in C(X).$$

Also $\mathcal{M}(y) = \{0\}$ implies that

$$(Tf)(y) = \left(\delta_y \circ T\right) f = 0 = (T\mathbf{e}_X)(y) \cdot (Uf)(y) \ \text{ for all } \ f \in C(X).$$

It follows that

$$(Tf)(y) = (T\mathbf{e}_X)(y) \cdot (Uf)(y) \ \text{ for all } \ f \in C(X) \ \text{ and all } \ y \in Y.$$

This shows that $T = (T\mathbf{e}_X)U \in \mathcal{M}_U$ for all $T \in \mathcal{M}$ and then $\mathcal{M} \subset \mathcal{M}_U$. This inclusion together with Proposition 3 show that if $\mathcal{M}$ is maximal then $\mathcal{M} = \mathcal{M}_U$. □

We have gathered now all of the ingredients for the main result in this section.

Proposition 6. *Let $\mathcal{M}$ be a subset of $\mathcal{L}_b(C(X),C(Y))$. Then the following are equivalent.*

(i) *$\mathcal{M}$ is a maximal Lamperti Riesz subspace of $\mathcal{L}_b\left(C(X),C(Y)\right)$.*

(ii) *There exists a maximal algebra homomorphism U from $C(X)$ into $\mathbb{R}^Y$ satisfying $\mathcal{M}=\mathcal{M}_U$.*

Proof: (i) $\Rightarrow$ (ii) Assume first that $\mathcal{M}$ is a maximal Lamperti Riesz subspace of $\mathcal{L}_b(C(X),C(Y))$. By Proposition 5, there exists an algebra homomorphism U from $C(X)$ into $\mathbb{R}^Y$ with $\mathcal{M}=\mathcal{M}_U$. Let V be an algebra homomorphism from $C(X)$ into $\mathbb{R}^Y$ satisfying $O_U\subset O_V$ and $(Uf)_{|O_U}=(Vf)_{|O_U}$ for all $f\in C(X)$. It follows that if $T\in\mathcal{M}_U$ and $y\in Y$ then

$$\begin{aligned}(Tf)(y)&=\begin{cases}\left(T\mathbf{e}_X\right)(y)(Uf)(y) & \text{if } \ y\in O_U\\ 0 & \text{if } \ y\notin O_U\end{cases}\\ &=\begin{cases}\left(T\mathbf{e}_X\right)(y)(Vf)(y) & \text{if } \ y\in O_U\\ 0 & \text{if } \ y\notin O_U\end{cases}\\ &=\begin{cases}\left(T\mathbf{e}_X\right)(y)(Vf)(y) & \text{if } \ y\in O_V\\ 0 & \text{if } \ y\notin O_V\end{cases}\\ &=\left(T\mathbf{e}_X\right)(y)(Vf)(y)\end{aligned}$$

for all $f\in C(X)$. So $T=\left(T\mathbf{e}_X\right)V\in\mathcal{M}_V$ for all $T\in\mathcal{M}_U$. This means that $\mathcal{M}_U\subset\mathcal{M}_V$. Since $\mathcal{M}_U$ is maximal we get $\mathcal{M}_U=\mathcal{M}_V$ and then $O_U=O_V$. This shows that U is maximal.

(ii) $\Rightarrow$ (i) Assume now that there exists a maximal algebra homomorphism U from $C(X)$ into $\mathbb{R}^Y$ satisfying $\mathcal{M}=\mathcal{M}_U$. From Proposition 3 it follows that $\mathcal{M}$ is a Lamperti Riesz subspace of $\mathcal{L}_b(C(X),C(Y))$. Let $\mathcal{M}'$ be a Lamperti Riesz subspace of $\mathcal{L}_b(C(X),C(Y))$ with $\mathcal{M}\subset\mathcal{M}'$. By Proposition 5, there exists an algebra homomorphism U' from $C(X)$ into $\mathbb{R}^Y$ satisfying $\mathcal{M}'\subset\mathcal{M}_{U'}$. It follows immediately that $\mathcal{M}_U\subset\mathcal{M}_{U'}$ and then $O_U\subset O_{U'}$. Let $y_0\in O_U$ and take $T\in\mathcal{M}_U$ with $\left(T\mathbf{e}_X\right)(y_0)\neq 0$. Using the inclusion $\mathcal{M}_U\subset\mathcal{M}_{U'}$, we get

$$(Tf)\left(y_0\right)=\left(T\mathbf{e}_X\right)\left(y_0\right)(Uf)\left(y_0\right)=\left(T\mathbf{e}_X\right)\left(y_0\right)\left(U'f\right)\left(y_0\right)$$

for all $f\in C(X)$. Consequently $(Uf)\left(y_0\right)=\left(U'f\right)\left(y_0\right)$ and then $(Uf)_{|O_U}=\left(U'f\right)_{|O_U}$ for all $f\in C(X)$. Since U is maximal it yields that $O_U=O_{U'}$.

A similar method to that used in the proof of the implication (i) $\Rightarrow$ (ii) shows that $\mathcal{M}_{U'}\subset\mathcal{M}_U$ and then $\mathcal{M}=\mathcal{M}'$. This proves that $\mathcal{M}$ is a maximal Lamperti Riesz subspace of $\mathcal{L}_b\left(C(X),C(Y)\right)$. □

Let τ be a function from Y into X and observe that we define an algebra homomorphism U_τ from $C(X)$ into $\mathbb{R}^Y$ by putting $Uf=f\circ\tau$ for all $f\in C(X)$.

A slight modification of the proof of the standard Theorem 7.22 in [4] shows that when X is a compact Hausdorff space any algebra homomorphism U from $C(X)$ into $\mathbb{R}^Y$ satisfies $U = U_\tau$ for a (unique) function τ from Y into X. For a sake of simpleness we write F_τ, $\mathcal{M}_\tau$, and O_τ instead of F_{U_τ}, $\mathcal{M}_{U_\tau}$, and O_{U_τ} respectively. Observe that τ is continuous on O_τ as $U_\tau f = f \circ \tau$ is continuous on O_τ for all $f \in C(X)$.

We say that τ is maximal whenever U_τ is a maximal algebra homomorphism from $C(X)$ into $\mathbb{R}^Y$. Maximal functions from Y into X can be characterized as follows.

Lemma 5. *Assume that X is a compact Hausdorff space. Then a function τ from Y into X is maximal if and only if*

$$\left[O_\tau \subset O \ \text{ and } \ \tau_{|O_\tau} \ \text{ extends to a continuous function on } \ O\right] \Longrightarrow [O = O_\tau]$$

for all open subset O of Y.

Proof: The condition is obviously sufficient. To prove necessity, assume that τ is maximal and let O be an open subset of Y satisfying

(i) $O_\tau \subset O$;
(ii) $\tau_{|O_\tau}$ extends to a continuous function $\alpha : O \to X$.

Let τ' be an arbitrary function from Y into X which extends α. Let $y_0 \in O$. Since Y is completely regular, there exists $w \in C(Y)$ satisfying $w(y_0) = 1$ and $w(y) = 0$ for all $y \in Y \setminus O$. It is not hard to prove that $T = wU_{\tau'} \in \mathcal{M}_{\tau'}$ and then $y_0 \in \operatorname{coz}(T\mathbf{e}_X) \subset O_{\tau'}$. This implies that $O \subset O_{\tau'}$ and consequently $O_\tau \subset O_{\tau'}$. On the other hand, it follows from $\tau'_{|O_\tau} = \tau_{|O_\tau}$ that $(U_\tau f)_{|O_\tau} = (U_{\tau'} f)_{|O_\tau}$ for all $f \in C(X)$. Since τ is maximal, it yields that $O_\tau = O_{\tau'}$ and then $O = O_\tau$. $\square$

Lemma 5 together with Proposition 6 leads to the following corollary which is the last result of this work.

Corollary 1. *Assume that X is a compact Hausdorff space and let $\mathcal{M}$ be a subset of $\mathcal{L}_b(C(X), C(Y))$. Then the following are equivalent.*

(i) *$\mathcal{M}$ is a maximal Lamperti Riesz subspace of $\mathcal{L}_b(C(X), C(Y))$.*
(ii) *There exists a maximal function τ from Y into X such that $\mathcal{M} = \mathcal{M}_\tau$.*

Acknowledgment. The author is very grateful to the referee for valuable comments that improved the quality of the paper.

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(*Received* June 3, 2005, *revised* August 23, 2007)