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ON VALUATIONS OF NEARFIELDS

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Consider two projective planes P and P' coordinatized by planar ternary rings (S, t) and (S', t'), respectively. Either of these coordinatizations is essentially determined by ordering a four-point coordinate frame V, U, O, E and V′, U′, O′, E′, respectively. Every epimorphismus (if any) of the projective plane P onto P' induces a mapping $\Phi : S$ into $S' \cup \{\infty\}$ which becomes a place of fields in the commonly used sence, if P and P' are Pappian planes and (S, t), (S', t) are fields. This problem was most generally discussed in [2] and [5]. The place of alternative fields was investigated in [6].

This article deals with the place theory of nearfields. It appears, namely, that from the point of view of the place and its connections with valuations, the nearfields are close to skewfields. In more great details: there exists a one-to-one correspondence between the classes of equivalent places, valuation nearrings and valuations of nearfields, respectively. The same concluding has been reached by J. L. Zemmer in [5]. Our article considers the algebraic problems. For completeness, let us point out that a planar ternary ring (S, t) coordinatizing the plane P is a planar nearfield exactly if the plane P is simultaneously translative; if (V, x)-transitive for every line x passing through the point U and if (U, y)-transitive for every line y passing through the point V.

0. Introduction

For codification reasons, let us first introduce the axioms for a nearring, a nearfield and planar nearfields; unlike to [3] we will require from the beginning the commutativity of addition.

Let $\boldsymbol{NR}$ be a nonempty set

$$(a, b) \to a + b, \qquad (a, b) \to a \,.\, b$$

two binary operations on $\boldsymbol{NR}$ called addition and multiplication, respectively. ($a + b$ and $a \,.\, b$ are, respectively, sum and product of elements $a, b \in \boldsymbol{NR}$). The set $\boldsymbol{NR}$ together with both binary operations are called a *nearring* if the following axioms hold:

$$\forall\, a, b \in \boldsymbol{NR} \qquad a + b = b + a, \tag{1}$$

$$\forall\, a, b, c \in \boldsymbol{NR} \qquad a + (b + c) = (a + b) + c, \tag{2}$$

$$\exists\, 0 \in \boldsymbol{NR}, \forall\, a \in \boldsymbol{NR} \qquad a + 0 = a, \tag{3}$$

$$\forall\, a \in \boldsymbol{NR}, \exists\, -a \in \boldsymbol{NR} \qquad a + (-a) = 0, \tag{4}$$

$$\forall\, a \in \boldsymbol{NR} \qquad a \,.\, 0 = 0, \tag{5}$$

$$\forall\, a, b, c \in \boldsymbol{NR} \qquad a \,.\, (b \,.\, c) = (a \,.\, b) \,.\, c, \tag{6}$$

$$\forall\, a, b, c \in \boldsymbol{NR} \qquad (a + b) \,.\, c = a \,.\, c + b \,.\, c, \tag{7}$$

$$\exists\, 1 \in \boldsymbol{NR}, \forall\, a \in \boldsymbol{NR} \qquad a \,.\, 1 = 1 \,.\, a = a, \tag{8}$$

where 0 in (3) denotes a zero element and $-a$ in (4) is written for an opposite element to a and $1 \neq 0$ is valid. Immediate consequences are

(a) $\forall\, x \in \boldsymbol{NR} \qquad 0 \,.\, x = 0,$

(b) $\forall\, x, y \in \boldsymbol{NR} \qquad (-x) \,.\, y = -(x \,.\, y).$

The nearring $\boldsymbol{NF}$ is called a *nearfield* if the set of its nonzero elements together with multiplication is a group, i.e.

$$\forall\, a \in \boldsymbol{NF}, a \neq 0 \qquad \exists\, a^{-1} \in \boldsymbol{NF} \qquad a \,.\, a^{-1} = a^{-1} \,.\, a = 1, \tag{9}$$

where a^{-1} denotes an inverse element to a.

If $\boldsymbol{NF}$ is a nearfield of charakteristic $\neq 2$, i.e. $\forall\, x \in \boldsymbol{NF}$, $x \neq 0$ is $x + x \neq 0$, then

(α) $\forall\, a \in \boldsymbol{NF} \qquad a \,.\, (-1) = -a.$

For the proof see [7], p. 348, whence

(β) $\forall\, a, b \in \boldsymbol{NF} \qquad a \,.\, (-b) = -(a \,.\, b).$

Besides it holds for arbitrary nearfield

(c) $\forall\, a, b \in \boldsymbol{NF};\ a \neq b \qquad \exists\, !x \in \boldsymbol{NF} \qquad a \,.\, x = b \,.\, x + c.$

Proof: $a \,.\, x = b \,.\, x + c \Leftrightarrow a \,.\, x - b \,.\, x = c \Leftrightarrow (a - b) \,.\, x = c$, however, such an x exists exactly one.

(d) Let $\boldsymbol{NF}$ be a nearfield. Then $\forall\, x, y, x', y' \in \boldsymbol{NF}$; $x \neq x'$

$$\exists!\, (a, b) \in \boldsymbol{NF} \times \boldsymbol{NF} : \begin{aligned} x \,.\, a + b &= y, \\ x' \,.\, a + b &= y'. \end{aligned} \tag{1}$$

Proof: If (1) is valid, then

$$(x - x') \,.\, a = y - y'. \tag{2}$$

Conversely, if (2) is true for an $a \in \boldsymbol{NF}$, then putting $b = y - x \,.\, a$, we get $x' \,.\, a + b = (x' \,.\, a - x \,.\, a) + y = -(x \,.\, a - x' \,.\, a) + y = -(x - x') \,.\, a + y = y' - y + y = y'$. Since $x \neq x'$, a is uniquely determined by condition (2) as well as b is so by $x \,.\, a + b = y$.

(e) $\forall\, a, b, c, x' \in \boldsymbol{NF}, \quad a \neq b : x \,.\, a = x \,.\, b + c \quad x' \,.\, a = x' \,.\, b + c \Rightarrow \Rightarrow x = x'$.

Proof: $(x - x') \,.\, a = (x - x') \,.\, b$; if for instance $b = 0$, then $a \neq 0 \Rightarrow x - x' = 0$; if $a \neq 0$, $b \neq 0$, then there must be again $x - x' = 0$.

The nearfield is called *planar* if

$$\forall\, a, b, c \in \boldsymbol{NF}, \quad a \neq b \qquad \exists\, x \in \boldsymbol{NF} \qquad x \,.\, a = x \,.\, b + c. \tag{10}$$

We understand an ideal of the nearring $\boldsymbol{NF}$ any of its nonempty subset $\mathscr{J}$ having the following properties:

$$a, b \in \mathscr{J} \Rightarrow a + b \in \mathscr{J}, \tag{1}$$

$$a \in \mathscr{J}, \quad c \in \boldsymbol{NR} \Rightarrow a \,.\, c \in \mathscr{J}, \tag{2}$$

$$a, b \in \boldsymbol{NR}, \quad u \in \mathscr{J} \Rightarrow a \,.\, (b + u) - a \,.\, b \in \mathscr{J}. \tag{3}$$

The definition of a maximal ideal is analogous to that for rings. Zorn's lemma can equally well be used to show that every ideal $\mathscr{J}$ of $\boldsymbol{NR}$ and different from $\boldsymbol{NR}$, is contained in a maximal ideal.

1. Places of Nearfields

Let $\boldsymbol{NF}$ and $\boldsymbol{NF}'$ be nearfields, and ∞ be an element not belonging to $\boldsymbol{NF}'$. Likewise, as we did in case of fields, we extend the addition and multiplication in $\boldsymbol{NF}'$ via formulas

$$a' + \infty = \infty + a' = \infty \qquad a' \in \boldsymbol{NF}',$$

$$a' \,.\, \infty = \infty \,.\, a' = \infty \qquad a' \in \boldsymbol{NF}',\ a' \neq 0,$$

$$\infty \,.\, \infty = \infty.$$

Thus $\infty + \infty$, $0 \,.\, \infty$, $\infty \,.\, 0$ are undefined.

By a *place* (more precisely $\boldsymbol{NF}'$ *place*) of the nearfield $\boldsymbol{NF}$ we call every mapping

$$U : \boldsymbol{NF} \to \boldsymbol{NF}' \cup \{\infty\},$$

for which

$$U(a + b) = U(a) + U(b),$$

if $U(a) + U(b)$ is defined (i.e. if there is not $U(a) = U(b) = \infty$);

$$U(a \,.\, b) = U(a) \,.\, U(b),$$

if $U(a)$. $U(b)$ is defined (i.e. there is not $U(a) = \infty \wedge U(b) = 0$ or $U(a) = 0 \wedge \wedge U(b) = \infty$);

$$U(1) = 1'.$$

Proposition 1.1. Let U: $\mathbf{NF} \to \mathbf{NF}' \cup \{\infty\}$ be a place. Putting $\mathbf{NF}^* = \{x' \in \mathbf{NF}' \mid \exists\, x \in \mathbf{NF}, x' = U(x)\}$, then $\mathbf{NF}$ is a nearfield.

Proof: The validity of axioms (1), (2), (5)–(8) is clear. However, there is also $1' = U(1)$, which leads to $U(1 + 0) = U(1) + U(0) = 1' + U(0) \Rightarrow U(0) = 0' \Rightarrow \Rightarrow 0' \in \mathbf{NF}^*$.

Letting $a' \in \mathbf{NF}^* \Rightarrow a' = U(a)$; $a \in \mathbf{NF}$. Then $0' = U(0) = U[a + (-a)] = U(a) + U(-a) \Rightarrow U(-a) = -U(a) = -a' \Rightarrow -a' \in \mathbf{NF}^*$.

Corollary. The set $\mathbf{NF}^* \cup \{\infty\}$ may be taken to be a codomain of the place U, whereby U: $\mathbf{NF} \to \mathbf{NF}^* \cup \{\infty\}$ becomes a surjective mapping.

Besides, we have found in the proof of Proposition 1.1, that $U(0) = 0'$, $U(-a) = -U(a)\ \forall\, a \in \mathbf{NF}$.

Proposition 1.2. Let $U: \mathbf{NF} \to \mathbf{NF}' \cup \{\infty\}$ be a place of the nearfield $\mathbf{NF}$. Then the following implication $U(a) = U(b) = \infty \wedge U(x\,.\,a + b) \in \mathbf{NF}' \Rightarrow U(x) = U[-(b\,.\,a^{-1})]$ holds for every $x, a, b \in \mathbf{NF}$.

Proof: Letting $y = x\,.\,a + b$, $s = -(b\,.\,a^{-1}) \Rightarrow b = (-s)\,.\,a$ leads to $y = x\,.\,a + (-s)\,.\,a = (x - s)\,.\,a$; because $U(y) \in \mathbf{NF}'$ and $U(a) = \infty$ must be $U(x - s) = 0 \Rightarrow U(x) = U(s)$.

Theorem 1.3. Let $U: \mathbf{NF} \to \mathbf{NF}' \cup \{\infty\}$ be a place of the nearfield $\mathbf{NF}$. Then the following two conditions are equivalent:

(**A**) $$\forall\, a, m, x \in \mathbf{NF}: U(x\,.\,a\,.\,x^{-1}) = U(x\,.\,m\,.\,x^{-1}) \wedge U(x) \neq 0 \wedge \wedge U(x\,.\,a - x\,.\,m) \in \mathbf{NF}' \Rightarrow U(a) = U(m).$$

(**B**) $$\forall\, a, m, x \in \mathbf{NF}: U(x) = U(x\,.\,m) = \infty \wedge U(x\,.\,a - x\,.\,m) \in \mathbf{NF}' \Rightarrow \Rightarrow U(a) = U(m).$$

Remark: Changing the assumption $U(x\,.\,m) = \infty$ by the condition $U(x\,.\,m) \in \mathbf{NF}'$ in (**B**) gives $U(m) = 0$, so that $U(x\,.\,a) = U[(x\,.\,a - x\,.\,m) + x\,.\,m] = U(x\,.\,a - x\,.\,m) + U(x\,.\,m) \in \mathbf{NF}'$. However, because of $U(x) = \infty$, there must be $U(a) = 0$ and $U(a) = U(m)$ always when $U(x) = \infty$, $U(x\,.\,a - x\,.\,m) \in \mathbf{NF}'$ and $U(x\,.\,m) \in \mathbf{NF}'$.

Proof: (**A**) $\Rightarrow$ (**B**).

Let us put $b = x\,.\,m - x\,.\,a \Rightarrow U(b) \in \mathbf{NF}' \Rightarrow U(b\,.\,x^{-1}) = U(b)$. $U(x^{-1}) = 0$; since $b\,.\,x^{-1} = x\,.\,m\,.\,x^{-1} - x\,.\,ax^{-1}$ it holds $x\,.\,a\,.\,x^{-1} + b\,.\,x^{-1} = x\,.\,m\,.\,x^{-1} \Rightarrow \Rightarrow U(x\,.\,a\,.\,x^{-1}) = U(x\,.\,m\,.\,x^{-1})$. By relation (**A**) $U(a) = U(m)$.

(**B**) $\Rightarrow$ (**A**).

Let first $U(x) \in \mathbf{NF}'$. Then $U(x^{-1}) \neq 0, \infty$, hence $U(x^{-1}) \in \mathbf{NF}'$. Now $U(x)\,.\,U(a)\,.\,[U(x)]^{-1} = U(x\,.\,a\,.\,x^{-1}) = U(x\,.\,m\,.\,x^{-1}) = U(x)\,.\,U(m)\,.\,[U(x)]^{-1} \Rightarrow$

$\Rightarrow U(a) = U(m)$. Let $U(x) = \infty$. If also $U(x . m) = \infty$, then by (**B**) $U(a) = U(m)$. Let $U(x . m) \in NF'$. Then $U(m) = 0$, but $U(x . a) = U[(x . a - x . m) + x . m] = U(x . a - x, m) + U(x . m) \in NF' \Rightarrow U(a) = 0$, i.e. $U(a) = U(m)$ again.

Theorem 1.4. Let the place U: $NF \rightarrow NF' \cup \{\infty\}$ fulfil either of the conditions (**A**), (**B**) given in Theorem 1.3. Then $\forall\, a, b, x \in NF : U(a) = U(b) = U(x) = U(x . a + b) = \infty \Rightarrow U[x^{-1} . (x . a + b)] = \infty \vee U(b . a^{-1}) = \infty$.

Proof: Be assumed that the assumption of our Theorem are fulfilled and $U(b . a^{-1}) \in NF'$. Since $x \neq 0$, there exists an $m \in NF$ so that $x . m = x . a + b \Rightarrow$ $\Rightarrow m = x^{-1} . (x . a + b) \Rightarrow x . m . a^{-1} = (x . a + b) . a^{-1} = x + b . a^{-1} \Rightarrow$ $\Rightarrow x . m . a^{-1} . x^{-1} = 1 + b . a^{-1} . x^{-1}$, but $1 = x . 1 . x^{-1}$, $U(b . a^{-1} . x^{-1}) = U(b . a^{-1}) . U(x^{-1}) = 0$, thus $U[x . (m . a^{-1}) . x^{-1}] = U(x . 1 . x^{-1})$ and by condition (**A**) $U(m . a^{-1}) = 1' \Rightarrow U(m) = \infty$, for $U(a^{-1}) = 0$.

Theorem 1.5. Let $U : NF \rightarrow NF' \cup \{\infty\}$ be a surjective place of the planar nearfield NF, with U fulfilling either of the equivalent conditions from Theorem 1.3. Then NF' is a planar nearfield.

Proof: Let $a', b', c' \in NF'$, $a' \neq b'$. Because of the surjectivity of the mapping U there exist $a, b, c \in NF$ so that $a' = U(a)$, $b' = U(b)$, $c' = U(c)$. As $a \neq b$ and with respect to the planarity of the nearfield NF, $\exists\, x \in NF$ so that

$$x . a = x . b + c.$$

Let first $U(x) \in NF'$. It then follows from (1) that $U(x) . a' = U(x) . b' + c'$. Let next $U(x) = \infty$ and besides also $U(x . a) = \infty$. We have then $U(x . a - x . b) = U(c) = c' \in NF'$ and following the condition (**B**) from Theorem 1.3. $a' = U(a) = U(b) = b'$, which is a contradiction.

Let as assume $U(x) = \infty$, $U(xa) \in NF'$. Then $U(a) = 0$ and $U(xa - xb) = U(c) \in NF'$. But $xbx^{-1} + cx^{-1} = xax^{-1}$. As $U(cx^{-1}) = 0$, $U(xbx^{-1}) = U(xax^{-1})$ holds. According to the condition (A) from Theorem 1.3. $a' = U(a) = U(b) = b'$ which is a contradiction.

2. Valuation Nearrings

Let U: $NF \rightarrow NF' \cup \{\infty\}$ be a place of the nearfield NF. Write

$$\begin{aligned} NR &= \{x \in NF \mid U(x) \in NF'\}, \\ V &= \{x \in NF \mid U(x) \in NF' \wedge U(x) \neq 0\}, \\ M &= \{x \in NF \mid U(x) = 0\}. \end{aligned} \tag{I}$$

Clearly, NR is $V \cup M$. As in the case of field, we can easily find that: NF is a nearring, V is a set of its units, M is a set of its noninvertible elements being the single maximal ideal of the nearfield NR.

It evidently holds:

(a) $x \in \mathbf{NF}, x \notin \mathbf{NR} \Rightarrow x^{-1} \in \mathbf{NR}$,

(b) $a, b, x \in \mathbf{NR} \wedge x^{-1} \notin \mathbf{NR} \Rightarrow [a . (b + x) - a . b]^{-1} \notin \mathbf{NR}$.

The nearring $\mathbf{NR}$ is called the *valuation nearring* of the nearfield $\mathbf{NF}$ relative to the place.

If we define the equivalence of two places equally as in the case of fields, we find that two places of the same nearfield $\mathbf{NF}$ are equivalent if and only if they have same valuation nearrings. Generally, let us define for an arbitrary nearfield $\mathbf{NF}$:

Subring $\mathbf{NR}$ of the nearfield $\mathbf{NF}$ is called its valuation nearring if it has the properties (a), (b).

The definition of sets $\mathbf{V}$ and $\mathbf{M}$ from (I) may be rewritten for the valuation nearring of an arbitrary nearfield in the form:

$$\begin{aligned} \mathbf{V} &= \{x \in \mathbf{NR} \mid x^{-1} \in \mathbf{NR}\}, \\ \mathbf{M} &= \{x \in \mathbf{NR} \mid x^{-1} \in \mathbf{NF} \setminus \mathbf{NR} \vee x = 0\}. \end{aligned} \tag{II}$$

Proposition 2.1. The set $\mathbf{M}$ defined by (II) is an ideal in a nearring $\mathbf{NR}$.

Proof: Clearly, $\mathbf{M}$ is a set of all noninvertible elements from $\mathbf{NR}$, so that it follows from the condition (b) in the definition of the valuation nearring that

$$a, b, x \in \mathbf{NR}, \qquad x \in \mathbf{M} \Rightarrow a . (b + x) - a . b \in \mathbf{M}.$$

Let $a, b \in \mathbf{M}$. If any of these elements is zero, then certainly $a + b \in \mathbf{M}$. Let $a \neq 0$, $b \neq 0$. Then either $a . b^{-1} \in \mathbf{NR}$ or $b . a^{-1} \in \mathbf{NR}$. Assuming $a + b \notin \mathbf{M}$, then $a + b$ is a unit ($a + b \in \mathbf{V}$), whence it follows that $(a + b)^{-1} \in \mathbf{NR} \Rightarrow 1 + (a . b)^{-1} = b . b^{-1} + a . b^{-1} = (a + b) . b^{-1} \Rightarrow (a + b) . b^{-1} \in \mathbf{NR}$, which next yields $(a + b)^{-1} . (a + b) . b^{-1} \in \mathbf{NR} \Rightarrow b^{-1} \in \mathbf{NR} \Rightarrow b \notin \mathbf{M}$, i.e. a contradiction. Let $a \in \mathbf{M}$, $c \in \mathbf{NR}$. If $a . c \notin \mathbf{M}$, then $(a . c)^{-1} \in \mathbf{NR} \Rightarrow c^{-1} . a^{-1} \in \mathbf{NR} \Rightarrow c . c^{-1} . a^{-1} \in \mathbf{NR} \Rightarrow a^{-1} \in \mathbf{NR}$, i.e. a contradiction again.

Clearly, $\mathbf{M}$ is the only one maximal ideal in $\mathbf{NR}$. Besides this it holds for every $x \in \mathbf{NR}$, $x \notin \mathbf{M}$ that $x \in \mathbf{V}$, so that $x^{-1} + \mathbf{M}$ is a class being inverse to $x + \mathbf{M}$. Thus, the following theorem is valid:

Theorem 2.2. If $\mathbf{NR}$ is a valuation nearring of the nearfield $\mathbf{NF}$ with $\mathbf{M}$ being its maximal ideal, then $\mathbf{NR}/\mathbf{M}$ is a nearfield.

Evidently, the mapping U: $\mathbf{NF} \to \mathbf{NR}/\mathbf{M} \cup \{\infty\}$ given by the conditions $U(x) = x + \mathbf{M}$, if $x \in \mathbf{NR}$; $U(x) = \infty$, if $x \in \mathbf{NF} \setminus \mathbf{NR}$, is a place of the nearfield $\mathbf{NF}$ and $\mathbf{NR}$ is a valuation nearring belonging to the place.

Theorem 2.3. Let $\mathbf{NR}$ be a valuation nearring of the nearfield $\mathbf{NF}, \mathbf{M}_1, \mathbf{M}_2$ be its arbitrary ideals. Then

$$\mathbf{M}_1 \subset \mathbf{M}_2 \vee \mathbf{M}_2 \subset \mathbf{M}_1.$$

Proof is the same as for fields.

3. On Valuation of Nearfields

Let $\boldsymbol{NF}$ be a nearfield, $\boldsymbol{G}$ be a linearly ordered, at least two-element set with the smallest element o. The mapping

$$\nu : \boldsymbol{NF} \to \boldsymbol{G}$$

will be called the *valuation* (more precisely $\boldsymbol{G}$-*valuation* of the nearfield $\boldsymbol{NF}$) if it holds:

$$\nu(x) = o \Leftrightarrow 0, \tag{1}$$

$$\forall\, x, y, z \in \boldsymbol{NF}, \quad \nu(x) \leqq \nu(y) \Rightarrow \nu(x \,.\, z) \leqq \nu(y \,.\, z), \tag{2}$$

$$\forall\, x \in \boldsymbol{NF}, \quad \nu(x + y) \leqq \max\,[\nu(x), \nu(y)], \tag{3}$$

$$\forall\, a, b, x \in \boldsymbol{NF}, \quad \nu(a) \leqq \nu(1), \nu(b) \leqq \nu(1), \nu(x) < \nu(1) \Rightarrow \tag{4}$$

$$\Rightarrow \nu[a \,.\, (b + x) - a \,.\, b] < \nu(1).$$

In what follows we put $e = \nu(1)$. Obviously $e \neq o$. Let

$$\begin{aligned} \boldsymbol{NR} &= \{x \in \boldsymbol{NF} \mid \nu(x) \leqq e\}, \\ \boldsymbol{V} &= \{x \in \boldsymbol{NF} \mid \nu(x) = e\}, \\ \boldsymbol{M} &= \{x \in \boldsymbol{NF} \mid \nu(x) < e\}, \end{aligned} \tag{III}$$

0,1 are certainly in $\boldsymbol{NR}$. Assume that $a, b \in \boldsymbol{NR} \Rightarrow \nu(a + b) \leqq \max\,[\nu(a), \nu(b)] \leqq e \Rightarrow a + b \in \boldsymbol{NR}$. Further $\nu(a) \leqq \nu(1) \Rightarrow \nu(a \,.\, b) \leqq \nu(1 \,.\, b) = \nu(b) \leqq e \Rightarrow a \,.\, b \in \boldsymbol{NR}$.

We investigate the element $\nu(-1)$ of the set $\boldsymbol{G}$. If $\nu(-1) < \nu(1)$, then $\nu[(-1) \,.\, (-1)] < \nu[1 \,.\, (-1)] \Rightarrow \nu(1) < \nu(-1)$, yielding a contradiction.

Completely analogous we disprove that $\nu(1) < \nu(-1)$. Thus $\nu(-1) = e$, so that $-1 \in \boldsymbol{NR}$, whence with every $a \in \boldsymbol{NR}$ it is $-a \in \boldsymbol{NR}$.

This proves:

Proposition 3.1. The set $\boldsymbol{NR}$ from (III) is a subnearring of the nearfield $\boldsymbol{NF}\ \forall\ \boldsymbol{NF}$

Proposition 3.2. Let ν: $\boldsymbol{NF} \to \boldsymbol{G}$ be a valuation of the nearfield $\boldsymbol{NF}$. Then
(a) $\forall\, a, b, c \in \boldsymbol{NF}$ it holds $\nu(a) = \nu(b) \Rightarrow \nu(a \,.\, c) = \nu(b \,.\, c)$,
(b) $\forall\, a, b, c \in \boldsymbol{NF}$, $c \neq 0$ $\nu(a) < \nu(b) \Rightarrow \nu(a \,.\, c) < \nu(b \,.\, c)$.

Proof: (a) $\nu(a) = \nu(b) \Rightarrow \nu(a) \leqq \nu(b) \wedge \nu(b) \leqq \nu(a) \Rightarrow \nu(a \,.\, c) \leqq \nu(b \,.\, c) \wedge \nu(b \,.\, c) \leqq \nu(a \,.\, c)$.
(b) $\nu(a \,.\, c) \leqq \nu(b \,.\, c)$, if however $\nu(a \,.\, c) = \nu(b \,.\, c)$ then by (a) $\nu(a \,.\, c \,.\, c^{-1}) = \nu(b \,.\, c \,.\, c^{-1}) \Rightarrow \nu(a) = \nu(b)$. Our consideration leading Proposition 3.1 shows that

$$\nu(-1) = e,$$

whence

$$\nu(a) = \nu(-a)\ \forall\ a \in \boldsymbol{NF}.$$

Theorem 3.3. Let ν: $\boldsymbol{NF} \to \boldsymbol{G}$ be a valuation of the nearfield $\boldsymbol{NF}$. Then the set

$\boldsymbol{NR}$, $\boldsymbol{V}$ and $\boldsymbol{M}$ from (III) are, respectively, the valuation nearring of the nearfield $\boldsymbol{NF}$, the set of the units of the nearring $\boldsymbol{NR}$, and the maximal ideal of the nearring $\boldsymbol{NR}$.

Proof: Because of Proposition 3.1, it suffices to prove that $\boldsymbol{NR}$ meets the conditions from the definition of the valuation nearring.

Let $x \in \boldsymbol{NF} \setminus \boldsymbol{NR}$, then $\nu(x) > e = \nu(1) \Rightarrow \nu(x \,.\, x^{-1}) > \nu(x^{-1}) \Rightarrow e > \nu(x^{-1}) \Rightarrow$ $\Rightarrow x^{-1} \in \boldsymbol{NR}$ (even $x^{-1} \in \boldsymbol{M}$).

Let $a, b, x \in \boldsymbol{NR}$ and let $x^{-1} \notin \boldsymbol{NR}$. Then $\nu(a) \leqq e$, $\nu(b) \leqq e$, $\nu(x) < e \Rightarrow$ $\Rightarrow \nu[a \,.\, (b + x) - a \,.\, b] < e \Rightarrow [a \,.\, (b + x - a \,.\, b]^{-1} \in \boldsymbol{NR}$.

Other statements of our theorem are obvious.

Let us now have a nearfield $\boldsymbol{NF}$ and its valuation nearring $\boldsymbol{NR}$. Let $\boldsymbol{V}$ be a set of units $\boldsymbol{NR}$. Putting $\boldsymbol{NF}^* = \boldsymbol{NF} \setminus \{0\}$, then $\boldsymbol{NF}^*$ together with the multiplication is a group, $\boldsymbol{V}$ is its subgroup (not necessarily normal). Let $\boldsymbol{G}^*$ be a set of all right classes of the group $\boldsymbol{NF}$ with respect to the subgroup $\boldsymbol{V}$. Let $0 \notin \boldsymbol{G}^*$, $\boldsymbol{G} = \boldsymbol{G}^* \cup \{0\}$. We introduce the relation $\leqq$ on $\boldsymbol{G}$ as follows:

(1) $\forall\, a \in \boldsymbol{NF}^* \quad 0 \leqq \boldsymbol{V} \,.\, a (\Rightarrow 0 < \boldsymbol{V} \,.\, a); 0 \leqq 0,$

(2) $\forall\, a, b \in \boldsymbol{NF}^* \quad \boldsymbol{V} \,.\, a \leqq \boldsymbol{V} \,.\, b \Leftrightarrow a \,.\, b^{-1} \in \boldsymbol{NR}.$

We prove that $\leqq$ is a linear ordering on $\boldsymbol{G}$. $\forall\, a \in \boldsymbol{NF}^*$ $\boldsymbol{V} \,.\, a \leqq \boldsymbol{V} \,.\, a$ for $a \,.\, a^{-1} =$ $= 1 \in \boldsymbol{NR}$.

Let $a, b \in \boldsymbol{NF}^*$ and $\boldsymbol{V} \,.\, a \leqq \boldsymbol{V} \,.\, b \wedge \boldsymbol{V} \,.\, b \leqq \boldsymbol{V} \,.\, a \Rightarrow a \,.\, b^{-1} \in \boldsymbol{NR} \wedge b \,.\, a^{-1} \in \boldsymbol{NR} \Rightarrow$ $\Rightarrow a \,.\, b^{-1} \in \boldsymbol{V}$. Further $a = (a \,.\, b^{-1}) \,.\, b \Rightarrow \boldsymbol{V} \,.\, a = \boldsymbol{V} \,.\, b$.

Let $a, b, c \in \boldsymbol{NF}$ and let $\boldsymbol{V} \,.\, a \leqq \boldsymbol{V} \,.\, b$ and $\boldsymbol{V} \,.\, b \leqq \boldsymbol{V} \,.\, c$, then $a \,.\, b^{-1} \in \boldsymbol{NR} \wedge$ $\wedge\, b \,.\, c^{-1} \in \boldsymbol{NR} \Rightarrow a \,.\, c^{-1} \in \boldsymbol{NR} \Rightarrow \boldsymbol{V} \,.\, a \leqq \boldsymbol{V} \,.\, c$. Let $a, b \in \boldsymbol{NF}^*$, then either $a \,.\, b^{-1} \in$ $\in \boldsymbol{NR}$ or $b \,.\, a^{-1} \in \boldsymbol{NR} \Rightarrow \boldsymbol{V} \,.\, a \leqq \boldsymbol{V} \,.\, b \vee \boldsymbol{V} \,.\, b \leqq \boldsymbol{V} \,.\, a$.

Let $x, y, z \in \boldsymbol{NF}^*$ and $\boldsymbol{V} \,.\, x \leqq \boldsymbol{V} \,.\, y \Rightarrow x \,.\, y^{-1} \in \boldsymbol{NR} \Rightarrow x \,.\, z \,.\, z^{-1} \,.\, y^{-1} \in \boldsymbol{NR} \Rightarrow$ $\Rightarrow (x \,.\, z) \,.\, (y \,.\, z)^{-1} \in \boldsymbol{NR} \Rightarrow \boldsymbol{V} \,.\, (x \,.\, z) \leqq \boldsymbol{V} \,.\, (y \,.\, z)$, thus $\boldsymbol{V} \,.\, x \leqq \boldsymbol{V} \,.\, y \Rightarrow \boldsymbol{V} \,.\, (x \,.\, z) \leqq$ $\leqq \boldsymbol{V} \,.\, (y \,.\, z)$. Let $x, y \in \boldsymbol{NF}^*$ and let, say $\boldsymbol{V} \,.\, x \leqq \boldsymbol{V} \,.\, y \Rightarrow x \,.\, y^{-1} \in \boldsymbol{NR} \Rightarrow (x + y) \,.$ $.\, y^{-1} = x \,.\, y^{-1} + 1 \in \boldsymbol{NR} \Rightarrow \boldsymbol{V} \,.\, (x + y) \leqq \boldsymbol{V} \,.\, y$. Therefore $\forall\, x, y \in \boldsymbol{NF}^*$ is $\boldsymbol{V} \,.\, (x + y) \leqq \max (\boldsymbol{V} \,.\, x, \boldsymbol{V} \,.\, y)$. Let finally $a, b, x \in \boldsymbol{NF}^*$ and $\boldsymbol{V} \,.\, a \leqq \boldsymbol{V}$, $\boldsymbol{V} \,.\, b \leqq \boldsymbol{V}$, $\boldsymbol{V} \,.\, x < \boldsymbol{V} \Rightarrow a \in \boldsymbol{NR}, b \in \boldsymbol{NR}, x \in \boldsymbol{NR}$. If x is a unit in $\boldsymbol{NR}$, then $x \in \boldsymbol{V} \Rightarrow \boldsymbol{V} \,.\, x = \boldsymbol{V}$, which is a contradiction. Hence it is that $x^{-1} \notin \boldsymbol{NR} \Rightarrow [a \,.\, (b + x) - a \,.\, b]^{-1} \notin$ $\in \boldsymbol{NR} \Rightarrow \boldsymbol{V} \,.\, [a \,.\, (b + x) - ab] < \boldsymbol{V}$ and therefore $\boldsymbol{V} \,.\, a \leqq \boldsymbol{V}$, $\boldsymbol{V} \,.\, b \leqq \boldsymbol{V}$, $\boldsymbol{V} \,.\, x < \boldsymbol{V} \Rightarrow$ $\Rightarrow \boldsymbol{V} \,.\, [a \,.\, (b + x) - a \,.\, b] < \boldsymbol{V}$. This however implies that the mapping $\nu : \boldsymbol{NF} \rightarrow$ $\rightarrow \boldsymbol{G}$ for which $\nu(a) = \boldsymbol{V} \,.\, a$, if $a \in \boldsymbol{NF}^*$ and $\nu(0) = 0$ is a valuation of the nearfield $\boldsymbol{NF}$ for which

$$\boldsymbol{NR} = \{x \in \boldsymbol{NF} \mid v(x) \in \boldsymbol{V}\}.$$

Let ν: $\boldsymbol{NF} \rightarrow \boldsymbol{G}$ be a valuation of the nearfield $\boldsymbol{NF}$. Then we may take the set $\boldsymbol{G}' =$ $= \{y \in \boldsymbol{G} \mid \exists\, x \in \boldsymbol{NF};\ y = \nu(x)\}$ as a codomain of this valuation. $\boldsymbol{G}'$ is then in a natural way linearly ordered set possessing the smallest element 0. Thus, every valuation may be considered as a surjective mapping.

If we define the equivalence of two valuations of the nearfield $\boldsymbol{NF}$ analogous to the case of the field, we find that both valuations ν and ν' are equivalent if and only if the same valuation ring belongs to them, i.e.

$$\{x \in \boldsymbol{NF} \mid \nu(x) \leqq \nu(1)\} = \{x \in \boldsymbol{NF} \mid \nu'(x) \leqq \nu'(1)\}.$$

Thus there exists a one-to-one correspondence between the class of equivalent places of the given nearfield and its valuation nearrings on one side, and a one-to-one correspondence between the classes of equivalent valuations and the valuation nearrings of the nearfield $\boldsymbol{NF}$ on the other. If $\boldsymbol{NF}$ is a planar nearfield and $\boldsymbol{U}$ is its $\boldsymbol{NF}'$-place being surjective, and if $\boldsymbol{NF}$ possesses any of equivalent properties (**A**), (**B**) from Theorem 1.3, then $\boldsymbol{NF}'$ is planar as well, and $\boldsymbol{U}$ in a natural way induces an epimorphism of the projective planes coordinatized by the nearfields $\boldsymbol{NF}$ and $\boldsymbol{NF}'$.

НОРМИРОВАНИЯ ПОЧТИ-ТЕЛ

Резюме

В статье доказано существование взаимно однозначного отношения между точками, почти-кольцами нормирования и нормированиями правых почти-тел аналогично, как тому в случае полей.

По геометрической причине особенно рассматриваются планарные почти-тела.

VALUACE SKOROTĚLES

Souhrn

V článku je dokázána existence 1 — 1 korespondence mezi umístěními valuačními skorookruhy a valuacemi pravého skorotělesa analogicky, jako je tomu u komutativních těles. Z geometrických důvodů je zvláštní zřetel vzat na planární skorotělesa.

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