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ON ALGEBRA-LATTICES

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O. STEINFELD in [1] studies absorbents of elements of groupoid-lattices. This note generalizes some results of [1] for algebra-lattices.

1. We say that $\mathfrak{A} = (A, F, \leqq)$ is an *ordered algebra* if

(1) (A, F) is an algebra with a set F of finitary operations;
(The set of all n-ary operations of F ($n \in N$) is denoted by F_n.)

(2) $(A, \leqq)$ is an ordered set;

(3) $a_i \leqq a_i'$ implies $f(a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_n) \leqq f(a_1, \dots, a_{i-1}, a_i', a_{i+1}, \dots, a_n)$ for all $n \in N, f \in F_n, a_1, \dots, a_{i-1}, a_i, a_i', a_{i+1}, \dots, a_n \in A$.

2. An ordered algebra $\mathfrak{A} = (A, F, \leqq)$ is called an *algebra-lattice* if

(4) $f(a, \dots, a) \leqq a$ for all $n \geqq 1, f \in F_n, a \in A$;

(5) $(A, \leqq)$ is a complete lattice;
(We shall denote the smallest element of this lattice by o, the greatest element by e.)

(6) $f(\underbrace{e, \dots, e}_{i-1 \text{ times}}, o, e, \dots, e) = o$ for all $n \geqq 1, f \in F_n, i \in \{1, \dots, n\}$.

Throughout the paper, $\mathfrak{A} = (A, F, \leqq)$ always mean an algebra-lattice.

3. Note. a) It is clear that $f(a_1, \dots, a_{i-1}, o, a_{i+1}, \dots, a_n) = o$ for all $n \geqq 1$, $f \in F_n$, $i \in \{1, \dots, n\}$, $a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n \in A$.

b) If $n \geqq 1$, $f \in F_n$, $a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n, b_\gamma \in A$, $\gamma \in \Gamma$, then for any $i \in \{1, \dots, n\}$

$$f(a_1, \dots, a_{i-1}, \bigwedge_{\gamma \in \Gamma} b_\gamma, a_{i+1}, \dots, a_n) \leqq \bigwedge_{\gamma \in \Gamma} f(a_1, \dots, a_{i-1}, b_\gamma, a_{i+1}, \dots, a_n),$$

$$f(a_1, \dots, a_{i-1}, \bigvee_{\gamma \in \Gamma} b_\gamma, a_{i+1}, \dots, a_n) \geqq \bigvee_{\gamma \in \Gamma} f(a_1, \dots, a_{i-1}, b_\gamma, a_{i+1}, \dots, a_n).$$

4. a) Let $f \in F_n$, $i \in \{1, \dots, n\}$. Then an element $b \in A$ is called an *$f^{(i)}$-absorbent of an element* $a \in A$ if $b \leqq a$ and $f(\underbrace{a, \dots, a}_{i-1 \text{ times}}, b, a, \dots, a) \leqq b$.

b) b is called an *f-absorbent of a* if b is an $f^{(i)}$-absorbent of a for each $i \in \{1, \dots, n\}$.

c) b is called an *f-quasiabsorbent of a* if $b \leqq a$ and $\bigwedge_{i=1}^{n} f(\underbrace{a, \ldots, a}_{i-1 \text{ times}}, b, a, \ldots, a) \leqq b$.

5. a) An element $b \in A$ is called an *absorbent of an element* $a \in A$ if b is an f-absorbent of a for each $f \in F$.

b) b is called a *quasiabsorbent of* a if b is an f-quasiabsorbent of a for each $f \in F$.

6. Note. By the definition of an algebra-lattice and by 3, it is clear that o and a are absorbents of a for each $a \in A$.

7. *If $f \in F_n$ and if b_i $(i = 1, \ldots, n)$ is an $f^{(i)}$-absorbent of an element $a \in A$, then $\bigwedge_{i=1}^{n} b_i$ is an f-quasiabsorbent of a.*

Proof. It is

$$\bigwedge_{j=1}^{n} f(\underbrace{a, \ldots, a}_{j-1 \text{ times}}, \bigwedge_{i=1}^{n} b_i, a, \ldots, a) \leqq \bigwedge_{j=1}^{n} f(\underbrace{a, \ldots, a}_{j-1 \text{ times}}, b_j, a, \ldots, a) \leqq \bigwedge_{j=1}^{n} b_j .$$

8. a) *If $f \in F_n$, $i \in \{1, \ldots, n\}$ and if b_γ $(\gamma \in \Gamma)$ are $f^{(i)}$-absorbents (f-absorbents, f-quasiabsorbents) of an element $a \in A$, then $\bigwedge_{\gamma \in \Gamma} b_\gamma$ is an $f^{(i)}$-absorbent (an f-absorbent, an f-quasiabsorbent) of a.*

b) *If b_γ $(\gamma \in \Gamma)$ are absorbents (quasiabsorbents) of an element $a \in A$, then $\bigwedge_{\gamma \in \Gamma} b_\gamma$ is an absorbent (a quasiabsorbent) of a.*

Proof. It holds

$$f(\underbrace{a, \ldots, a}_{i-1 \text{ times}}, \bigwedge_{\gamma \in \Gamma} b_\gamma, a, \ldots, a) \leqq f(\underbrace{a, \ldots, a}_{i-1 \text{ times}}, b_\gamma, a, \ldots, a) \leqq b_\gamma$$

for each $\gamma \in \Gamma$, hence

$$f(\underbrace{a, \ldots, a}_{i-1 \text{ times}}, \bigwedge_{\gamma \in \Gamma} b_\gamma, a, \ldots, a) \leqq \bigwedge_{\gamma \in \Gamma} b_\gamma .$$

The remainder parts can be proved analogously.

9. *If $f \in F_n$ and if b_i $(i = 1, \ldots, n)$ is an $f^{(i)}$-absorbent of an element $a \in A$, then*

$$f(b_1, \ldots, b_n) \leqq \bigwedge_{i=1}^{n} b_i .$$

Proof. It holds

$$f(b_1, \ldots, b_{i-1}, b_i, b_{i+1}, \ldots, b_n) \leqq f(\underbrace{a, \ldots, a}_{i-1 \text{ times}}, b_i, a, \ldots, a) \leqq b_i$$

for each $i \in \{1, \ldots, n\}$, thus $f(b_1, \ldots, b_n) \leqq \bigwedge_{i=1}^{n} b_i$.

10. a) Let $f \in F_n$, $i \in \{1, \dots, n\}$. We say that an element $a \in A$ *satisfies the condition* $(P_{f^{(i)}})$ if for each $f^{(i)}$-absorbent b of a and for each element $x \leqq a$ of A, $f(\underbrace{b, \dots, b}_{i-1 \text{ times}}, x, b, \dots, b)$ is an $f^{(i)}$-absorbent of a.

b) Let $f \in F_n$. We say that $a \in A$ *satisfies the condition* (P_f) if a satisfies the condition $(P_{f^{(i)}})$ for each $i \in \{1, \dots, n\}$.

c) We say that $a \in A$ *satisfies the condition* (P) if a satisfies the condition (P_f) for each $f \in F$.

11. *If $f \in F_n$ and if an element $a \in A$ satisfies the condition (P_f), then the following conditions are equivalent:*

(1) *a has exactly the trivial f-quasiabsorbents (i.e. o and a) and $f(a, \dots, a) \neq o$.*

(2) *a has exactly the trivial $f^{(i)}$-absorbents for each $i \in \{1, \dots, n\}$ and $f(a, \dots, a) \neq o$.*

(3) *$f(\underbrace{a, \dots, a}_{i-1 \text{ times}}, x, a, \dots, a) = a$ for each $o \neq x \leqq a$ and for each $i \in \{1, \dots, n\}$.*

(4) *For each $i \in \{1, \dots, n\}$, any $f^{(i)}$-absorbent x of a is f-idempotent (i.e. $f(x, \dots, x) = x$) and for each $f^{(i)}$-absorbents $b^{(i)} \neq o$, $b_1^{(i)} \neq o$, $b_2^{(i)} \neq o$ of a,*

$$f(\underbrace{b_1^{(i)}, \dots, b_1^{(i)}}_{i-1 \text{ times}}, b^{(i)}, \dots, b_1^{(i)}) = f(\underbrace{b_2^{(i)}, \dots, b_2^{(i)}}_{i-1 \text{ times}}, b^{(i)}, b_2^{(i)}, \dots, b_2^{(i)})$$

implies $b_1^{(i)} = b_2^{(i)}$.

Proof. $1 \Rightarrow 2$: Trivial.

$2 \Rightarrow 3$: Since a satisfies the condition (P_f), $f(\underbrace{a, \dots, a}_{i-1 \text{ times}}, x, a, \dots, a)$ is an $f^{(i)}$-absorbent of a, hence it is equal to o or a. Let $f(\underbrace{a, \dots, a}_{i-1 \text{ times}}, x, a, \dots, a) = o$.

Then x is an $f^{(i)}$-absorbent of a, and so $f(a, \dots, a) = o$, a contradiction. Therefore $f(\underbrace{a, \dots, a}_{i-1 \text{ times}}, x, a, \dots, a) = a$.

$3 \Rightarrow 1$: Let $o \neq b$ be an f-quasiabsorbent of a. Then

$$a = \bigwedge_{i=1}^{n} f(\underbrace{a, \dots, a}_{i-1 \text{ times}}, b, a, \dots, a) \leqq b,$$

hence $b = a$.

$2 \Rightarrow 4$: Trivial.

$4 \Rightarrow 2$: Let $o \neq b$ be an $f^{(i)}$-absorbent of a. Then $f(b, \dots, b) = b$ and

$$f(b, \dots, b) \leqq f(\underbrace{a, \dots, a}_{i-1 \text{ times}}, b, a, \dots, a) \leqq b.$$

Consequently

$$f(\underbrace{b, \ldots, b}_{i-1 \text{ times}}, b, b, \ldots, b) = f(\underbrace{a, \ldots, a}_{i-1 \text{ times}}, b, a, \ldots, a),$$

hence $b = a$.

12. Let $f \in F_n$, $i \in \{1, \ldots, n\}$. Then an $f^{(i)}$-absorbent (an f-quasiabsorbent) b of an element $a \in A$ is called *minimal* if b is a minimal element in the ordered set of all non-zero $f^{(i)}$-absorbents (f-quasiabsorbents) of a.

13. *Let $f \in F_n$, let an element $a \in A$ satisfy the condition (P_f) and let b_i $(i = 1, \ldots, n)$ be a minimal $f^{(i)}$-absorbent of a. Then $b = \bigwedge_{i=1}^{n} b_i$ is either equal to o or it is a minimal f-quasiabsorbent of a.*

Proof. Let $b \neq o$. Then by 7, b is an f-quasiabsorbent of a. Let $o < b' < b$ be an f-quasiabsorbent of a. Since a satisfies (P_f), $f(\underbrace{a, \ldots, a}_{i-1 \text{ times}}, b', \ldots, a)$ is an $f^{(i)}$-absorbent of a and $f(\underbrace{a, \ldots, a}_{i-1 \text{ times}}, b', a, \ldots, a) \leqq f(\underbrace{a, \ldots, a}_{i-1 \text{ times}}, b_i, a, \ldots, a) \leqq b_i$. Hence $f(\underbrace{a, \ldots, a}_{i-1 \text{ times}}, b', a, \ldots, a)$ is equal to o or b_i. In the first case, b' is an $f^{(i)}$-absorbent of a and $o < b' < b \leqq b_i$, a contradiction. Thus $f(\underbrace{a, \ldots, a}_{i-1 \text{ times}}, b', a, \ldots, a) = b_i$. This implies

$$b = \bigwedge_{i=1}^{n} b_i = \bigwedge_{i=1}^{n} f(\underbrace{a, \ldots, a}_{i-1 \text{ times}}, b', a, \ldots, a) \leqq b',$$

a contradiction.

14. Let $f \in F_n$, $i \in \{1, \ldots, n\}$, $a, b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n \in A$. We shall denote by $(a : b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n)_{f^{(i)}}$ such element of A that $x \leqq (a : b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n)_{f^{(i)}}$ if and only if $f(b_1, \ldots, b_{i-1}, x, b_{i+1}, \ldots, b_n) \leqq a$ for each element $x \in A$.

$(a : b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n)_{f^{(i)}}$ is called an *$f^{(i)}$-division of a* and $b_1, \ldots, b_{i-1}, b_{i+1}, b_n$.

15. a) Let $f \in F_n$, $i \in \{1, \ldots, n\}$. Then $\mathfrak{A}$ is called an *$f^{(i)}$-division algebra-lattice* if there exists an $f^{(i)}$-division of any a and $b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n$ of A.

b) $\mathfrak{A}$ is called an *f-division algebra-lattice* if $\mathfrak{A}$ is an $f^{(i)}$-division algebra-lattice for each $i = 1, \ldots, n$.

c) $\mathfrak{A}$ is called a *division algebra-lattice* if $\mathfrak{A}$ is an f-division algebra-lattice for each $f \in F$.

16. a) Let $f \in F_n$, $i \in \{1, \dots, n\}$. Then we say that $\mathfrak{A}$ is an *$f^{(i)}$-complete distributive algebra-lattice* if

$$f(a_1, \dots, a_{i-1}, \bigvee_{\gamma \in \Gamma} b_\gamma, a_{i+1}, \dots, a_n) = \bigvee_{\gamma \in \Gamma} f(a_1, \dots, a_{i-1}, b_\gamma, a_{i+1}, \dots, a_n)$$

for each $a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n, b_\gamma$ $(\gamma \in \Gamma)$ of A.

b) If $f \in F_n$, then we say that $\mathfrak{A}$ is an *f-complete distributive algebra-lattice* if $\mathfrak{A}$ is an $f^{(i)}$-complete distributive algebra-lattice for each $i = 1, \dots, n$.

c) We say that $\mathfrak{A}$ is a *complete distributive algebra-lattice* if $\mathfrak{A}$ is an f-complete distributive algebra-lattice for each $f \in F$.

17. *$\mathfrak{A}$ is a division algebra-lattice if and only if $\mathfrak{A}$ is a complete distributive algebra-lattice.*

Proof. "$\Rightarrow$": Let $\mathfrak{A}$ be a division algebra-lattice, $f \in F_n$, $i \in \{1, \dots, n\}$, $a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n, b_\gamma$ $(\gamma \in \Gamma)$ of A. Let us suppose that c is an element of A such that $f(a_1, \dots, a_{i-1}, b_\gamma, a_{i+1}, \dots, a_n) \leqq c$ for each $\gamma \in \Gamma$. Then $b_\gamma \leqq (c : a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n)_{f^{(i)}}$ for each $\gamma \in \Gamma$. This means that

$$\bigvee_{\gamma \in \Gamma} b_\gamma \leqq (c : a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n)_{f^{(i)}}$$

hence we obtain

$$f(a_1, \dots, a_{i-1}, \bigvee_{\gamma \in \Gamma} b_\gamma, a_{i+1}, \dots, a_n) \leqq c,$$

therefore $\mathfrak{A}$ is $f^{(i)}$-complete distributive. Since f is an arbitrary operation of F, $\mathfrak{A}$ is a complete distributive algebra-lattice.

"$\Leftarrow$": Let $\mathfrak{A}$ be a complete distributive algebra-lattice, $f \in F_n$, $i \in \{1, \dots, n\}$, $a, b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_n \in A$. Let c_γ $(\gamma \in \Gamma)$ be all elements of A such that $f(b_1, \dots, b_{i-1}, c_\gamma, b_{i+1}, \dots, b_n) \leqq a$.
Then

$$f(b_1, \dots, b_{i-1}, \bigvee_{\gamma \in \Gamma} c_\gamma, b_{i+1}, \dots, b_n) = \bigvee_{\gamma \in \Gamma} f(b_1, \dots, b_{i-1}, c_\gamma, b_{i+1}, \dots, b_n) \leqq a,$$

hence

$$f(b_1, \dots, b_{i-1}, \bigvee_{\gamma \in \Gamma} c_\gamma, b_{i+1}, \dots, b_n) = (a : b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_n)_{f^{(i)}}.$$

This means that $\mathfrak{A}$ is a division algebra-lattice.

REFERENCE

[1] O. Steinfeld: *Über Gruppoid-Verbände I,* Acta Sci. Math., 31 (1970), 203–218.

Souhrn

O SVAZOVÝCH ALGEBRÁCH

JIŘÍ RACHŮNEK

V článku jsou studovány absorbenty prvků ve svazových algebrách. Je tím dosaženo zobecnění výsledků získaných O. Steinfeldem pro svazové grupoidy.

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Резюме

К РЕШЕТОЧНЫМ АЛГЕБРАМ

И. РАХУНЕК

В статье рассматриваются абсорбенты элементов в решеточных алгебрах. Этим достигается обобщения результатов полученных О. Штейнфельдом для решеточных группоидов.