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The solution of integral equations on an analog computer

Terms of use:

THE SOLUTION OF INTEGRAL EQUATIONS
ON AN ANALOG COMPUTER

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Physical problems are often described by integral equations. In general, the integral equation may be written in the form

$$u(x) + \int_a^b K(x, t)\, u(t)\, \mathrm{d}t = f(x).$$

For $f(x) = 0$ is the given equation homogeneous. If the unknown function $u(x)$ is linear, then a linear equation is concerned. In the contrary case the equation is nonlinear. We may distinquish four cases of integral equations:

1. Fredholm's equation of the first kind

$$\int_a^b K(x, t)\, u(t)\, \mathrm{d}t = f(x), \qquad a \text{ and } b = \text{constants}.$$

2. Fredholm's equation of the second kind

$$u(x) + \int_a^b K(x, t)\, u(t)\, \mathrm{d}t = f(x).$$

3. Volterra's equation of the first kind

$$\int_0^x K(x, t)\, u(t)\, \mathrm{d}t = f(x).$$

4. Volterra's equation of the second kind

$$u(x) + \int_0^x K(x, t)\, u(t)\, \mathrm{d}t = f(x).$$

Programming of integral equations

Integral equations with the kernel $k(t - x)$, i.e. equations of the form

$$Au(t) + B \int_0^t k(t - x) u(x) \, dx = Cf(t), \tag{1}$$

where $u(t)$ is the unknown funcion, $f(t)$ and $k(t)$ are the given functions, A, B, $C =$ = constants, $B, C \neq 0$, may be solved on an analog computer without any additional device (generators of functions of two variables, analog memories).

Such equations may be programmed to advantage in utilizing the Laplace transform (with $A = 0$ or $A \neq 0$ meaning respectively the integral equation of the 1st or of the 2nd kind). Suppose

$$L(u(t)) = U(s), \qquad L(k(t)) = K(s), \qquad L(f(t)) = F(s).$$

Because of the convolution theorem

$$L[\int_0^t k(t - x) u(x) \, dx] = K(s) U(s),$$

the Laplace transform of (1) is of the form

$$AU(s) + BK(s) U(s) = CF(s).$$

From (2) the transform of the unknown function may be determined as

$$U(s) = \frac{CF(s)}{A + BK(s)}. \tag{3}$$

The Laplace transform of the unknown function for the first kind integral equation with $A = 0$ is of the form

$$U(s) = \frac{CF(s)}{BK(s)}. \tag{4}$$

So, in both cases the Laplace transform of the function being the solution of (1) is now know. This solution may be assigned to generating the function given by its transform Example 1: Write a programming diagram for solving equation

$$u(t) + \int_0^t (t - x) u(x) \, dx = t. \tag{5}$$

In our case $k(t) = t$, $f(t) = t$, the transforms of these functions are

$$F(s) = K(s) = \frac{1}{s^2}.$$

Thus the transform of the function as the solution of (5) will be obtained from the relation

$$U(s) + \frac{1}{s^2}\, U(s) = \frac{1}{s^2},$$

where

$$U(s) = \frac{\dfrac{1}{s^2}}{\dfrac{1}{s^2} + 1} = \frac{1}{1 + s^2}. \tag{6}$$

As known, the function $u(t)$ whose Laplace transform is given by (6) represents the output value of thecircuit with a transfer

$$H(p) = \frac{u}{z} = \frac{p}{1 + p^2}, \qquad \text{where } z \text{ is a unit step.}$$

The programming diagram of the circuit with the transfer cited above is given in figure 1. The equation $u'' + u = z'$ is programmed by the method of successive integration:

$$u'' = z' - u,$$
$$u' = z + u_1, \qquad u_1 = \int -u \, dt,$$
$$u = u_2, \qquad u_2 = \int (z + u_1) \, dt.$$
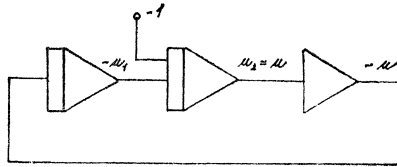
Fig. 1

From figure 1 we see that

$$u = -\int (-1 - u_1)\, dt,$$
$$u_1 = \int u \, dt,$$

i.e.

$$u' = 1 + u_1,$$
$$u_1' = -u,$$
$$u'' = u_1' = -u,$$
$$u'' + u = 0, \qquad u(0) = u_1(0) = 0, \tag{7}$$
$$u'(0) = 1.$$

Equation (7) is equivalent to equation $u'' + u = z'$. Thus the function $u = \sin t$ is the solution of (7) and (5) as well, as it follows from (6).

The correctness may be verified by inserting the relation $u = \sin t$ into equation (5):

$$\sin t + \int_0^t (t - x) \sin x \, \mathrm{d}x = t,$$

$$\int_0^t (t - x) \sin x \, \mathrm{d}x = [(t - x)(-\cos x) - \int (-1)(-\cos x) \, \mathrm{d}x]_0^t =$$
$$= [-t \cos x + x \cos x - \sin x]_0^t =$$
$$= -t \cos t + t \cos t - \sin t + t = -\sin t + t,$$
$$\sin t - \sin t + t = t,$$
$$t = t.$$

Example 2. Write a programming diagram for solving equation

$$u(t) + \int_0^t \sin(t - x) u(x) \, \mathrm{d}x = t^2. \tag{8}$$

In this case $k(t) = \sin t$, $f(t) = t^2$, so that $K(s) = \dfrac{1}{1 + s^2}$,

$$F(s) = \frac{2}{s^3}.$$

Then the transform of equation (8) is

$$U(s) + K(s) U(s) = F(s),$$

i.e.

$$U(s) = \frac{F(s)}{1 + K(s)} = \frac{\dfrac{2}{s^3}}{1 + \dfrac{1}{1 + s^2}} = \frac{\dfrac{2}{s^3}}{\dfrac{2 + s^2}{1 + s^2}} = \frac{2(s^2 + 1)}{s^3(s^2 + 2)}.$$

The function $u(t)$ becomes the output value of the circuit with the transfer

$$\frac{u}{z} = \frac{2p^3 + 2p}{p^5 + 2p^3}, \tag{9}$$

where z is a unit step function.

Rewriting equation (9) into the form

$$u^{(5)} + 2u^{(3)} = 2z''' + 2z', \tag{10}$$

let us program it by the method of lowering the order of the derivative in introducing a new variable. We decompose the equation into a system of two equations

$$v^{(5)} + 2v^{(3)} = z,$$
$$2v^{(3)} + 2v' = u. \tag{11}$$

The programming diagram for solving system (11) is given in figure 2.

From figure 2 we see that

$$x'' = -1 - 2x, \tag{12}$$

i.e.

$$x'' + 2x = -1, \qquad \text{whereby} \qquad x(0) = 0, x'(0) = 0.$$

The roots of the characteristic equation

$$\lambda^2 + 2 = 0$$

are $\lambda_{1,2} = \pm i\sqrt{2}$, so that the solution of equation (12) may be written as

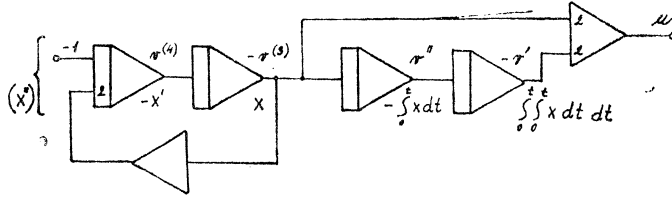$$x = c_1 \sin\sqrt{2}t + c_2 \cos\sqrt{2}t + c_3,$$

Fig. 2

where c_3 is a constant. This constant may be determined by inserting it into equation (12), i.e.

$$2c_3 = -1,$$

$$c_3 = \frac{1}{2},$$

so that $x = c_1 \sin\sqrt{2}t + c_2 \cos\sqrt{2}t - \dfrac{1}{2}$.

The coefficients c_1 and c_2 will be determined from the initial conditions $x(0) = 0$, $x'(0) = 0$:

$$x(0) = c_2 - \frac{1}{2} = 0 \qquad \text{i.e.} \qquad c_2 = \frac{1}{2},$$

$$x'(0) = \sqrt{2}c_1 \cos\sqrt{2}t - \sqrt{2}c_2 \sin\sqrt{2}t,$$

$$x'(0) = \sqrt{2}c_1 = 0 \qquad \text{i.e.} \qquad c_1 = 0.$$

Thus the function

$$x = \frac{1}{2}(\cos\sqrt{2}t - 1) \qquad (13)$$

becomes a solution of equation (12).

From figure 2 we can also see then

$$v'' = -\int_0^t x \, dt + v''(0) = -\frac{1}{2}\left(\frac{1}{\sqrt{2}}\sin\sqrt{2}t - t\right),$$

$$-v' = \int_0^t \int_0^t x \, dt \, dt - v'(0) = \int_0^t \frac{1}{2}\left(\frac{1}{\sqrt{2}}\sin\sqrt{2}t - t\right)dt =$$

$$= +\frac{1}{2}\left(-\frac{1}{2}\cos\sqrt{2}t - \frac{1}{2}t^2 + \frac{1}{2}\right).$$

The output value u of the summator is given by

$$u = -2x + 2v' = -\cos\sqrt{2}t + 1 + \frac{1}{2}\cos\sqrt{2}t + \frac{1}{2}t^2 - \frac{1}{2} =$$

$$= -\frac{1}{2}\cos\sqrt{2}t + \frac{1}{2}t^2 + \frac{1}{2}, \qquad (14)$$

$$L(u) = -\frac{1}{2}\frac{s}{s^2+2} + \frac{1}{2}\frac{2}{s^3} + \frac{1}{2s} = \frac{-s^4 + 2s^2 + 4 + s^4 + 2s^2}{2(s^2+2)s^3} =$$

$$= \frac{2(s^2+1)}{s^3(s^2+2)},$$

being equal to the Laplace transform of the solution of equation (8). To be definite about it, let us insert relation (14) into (8):

$$-\frac{1}{2}\cos\sqrt{2}t + \frac{1}{2}t^2 + \frac{1}{2} + \int_0^t \sin(t-x)\left(-\frac{1}{2}\cos\sqrt{2}x + \frac{1}{2}x^2 + \frac{1}{2}\right)dx = t^2, \qquad (15)$$

$$\int_0^t \sin(t-x)\left(-\frac{1}{2}\cos\sqrt{2}x\right)dx =$$

$$= -\left[\sin(t-x)\frac{1}{\sqrt{2}}\sin\sqrt{2}x - \frac{1}{2}\cos(t-x)\cos 2x\right]_0^t =$$

$$= \frac{1}{2}\cos\sqrt{2}t - \frac{1}{2}\cos t, \qquad (16)$$

$$\int_0^t \sin(t-x)\frac{1}{2}x^2 dx = \frac{1}{2}[2x\sin(t-x) + (x^2-2)\cos(t-x)]_0^t =$$

$$= \frac{1}{2}(t^2 - 2 + 2\cos t), \qquad (17)$$

$$\int_0^t \sin(t-x)\frac{1}{2}dx = \frac{1}{2}[\cos(t-x)]_0^t = \frac{1}{2} - \frac{1}{2}\cos t. \qquad (18)$$

After substituting expressions (16), (17) and (18) into (15), we find this equation to be satisfied:

$$-\frac{1}{2}\cos\sqrt{2}t + \frac{1}{2}t^2 + \frac{1}{2} + \frac{1}{2}\cos\sqrt{2}t - \frac{1}{2}\cos t + \frac{1}{2}t^2 -$$

$$- 1 + \cos t + \frac{1}{2} - \frac{1}{2}\cos t = t^2,$$

i.e. $t^2 = t^2$.

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Solving nonhomogeneous integral equations by the method of successive approximations

Applying the method of successive approximation in solving the equation

$$u(x) + \lambda \int_a^b k(x, t)\, u(t)\, \mathrm{d}t = f(x),$$

let us pass to the iterative formula

$$u_k(x) = f(x) - \lambda \int_a^b K(x, t)\, u_{k-1}(t)\, \mathrm{d}t. \tag{19}$$

But this method requires an additional device able to store memory and generate functions again.

Approximation the kernel of equations

1. The first form:
The kernel is approximated by a function of two variables as the end sum of functions, each of which depends on one variable, only. This means

$$K(x, t) = \sum_{i=1}^{n} a_i(x)\, b_i(t).$$

Integral equations whose kernels are expressible in the above form are called IE with a degenerate kernel. Equation (19) may then be written as

$$u(x) + \lambda \sum_{i=1}^{n} a_i(x) \int_a^b b_i(t)\, u(t)\, \mathrm{d}t = f(x). \tag{20}$$

This relation will be of practical importance if the error of approximation with a not large n will keep within the acceptable limits.

2. The second form:
Another (more universal) form is approximating the kernel by a set of functions $K(x_i, t)$, where each of these functions is dependent on a unique variable t, the second variable having a constant value. In such a case the solution of the equation will be carried over to determining the approximate solution at discrete points. Equation (19) will be solved in the form

$$u(x_i) + \lambda \int_a^b K(x_i, t)\, u(t)\, \mathrm{d}t = f(x_i) \qquad i = 1, 2, \ldots, m. \tag{21}$$

This way of solution requires m sufficiently large ($m = 10$ to 20 in general).

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Solving nonhomogeneous Fredholm's equation with the first form of approximating the kernel by the method of successive approximations

Using the method of successive approximations we pass to the iterative formula

$$u_k(x) = f(x) - \lambda \sum_{i=1}^{n} a_i(x) \int_a^b b_i(t) \, u_{k-1}(t) \, \mathrm{d}t. \tag{22}$$

To realize this process it is necessary to construit a model for (22) and to introduce a program of successive approximations for the unknown solution by means of a device capable of storing memory and generating functions again (enumerated approximations in our case).

For simplicity we write (22) as

$$u_k(x) = f(x) - \lambda \sum_{i=1}^{n} c_{k-1,i} a_i(x), \tag{23}$$

where

$$c_{k,i} = \int_a^b b_i(t) \, u_k(t) \, \mathrm{d}t \qquad (= \text{constant}). \tag{24}$$

If we preassign the initial approximations of the functionals c_i (choosing the zeroth approximation of the solution), we get the first approximation as the function of time and on integration within the limits from a to b we get new values of the functionals etc. Inserting (23) into (24), we find that no distinct notation is necessary for independent variables:

$$u_k(t) = f(t) - \lambda \sum_{i=1}^{n} c_{k-1,i} a_i(t),$$

$$c_{k,i} = \int_a^b b_i(t) \underbrace{\left[f(t) - \lambda \sum_{i=1}^{n} c_{k-1,i} a_i(t) \right]}_{u_k(t)} \mathrm{d}t, \tag{25}$$

For example, the memory device may consists even of potentiometers wtereat the voltage is fixed corresponding to the output values $c_{0,1}$ and $c_{0,2}$.
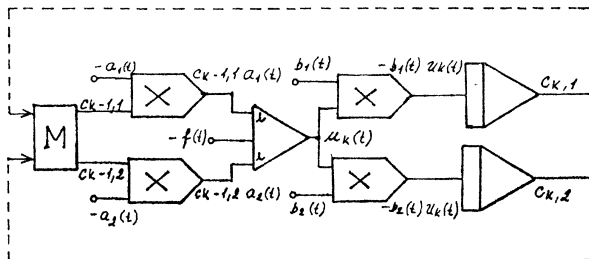

Fig. 3

The model is set into working order and the solution is interrupted at $t = b$ (assume $a = 0$).

Values $c_{0,1}$ and $c_{0,2}$ are interchanged for $c_{1,1}$ and $c_{1,2}$ measured oft at the outputs of integrators.

Further cycles proceed in sequence until sufficiently accurate solution is obtained, i.e. till the values $c_{k-1,i}$, $c_{k,i}$ cease to change with a sufficient accuracy. Picture 3a shows a programming diagram for a solution of equation

$$u(x) - \int_0^1 xe^t u(t)\, dt = e^x - x, \tag{24a}$$

Whose solution is the function

$$u(x) = e^{-x}, \tag{24b}$$

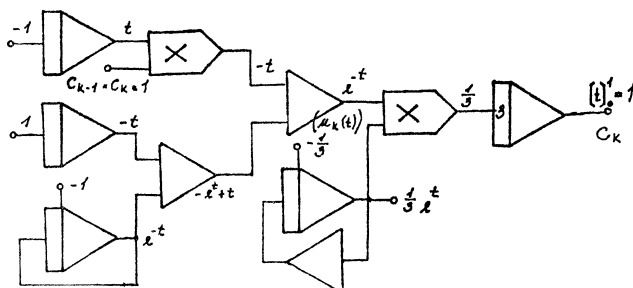as may be verified by substitution relation (24b) into (24a), i.e.

Fig. 3a

$$e^{-x} - x \int_0^1 e^t e^{-t}\, dt = e^{-x} - x,$$

$$e^{-x} - x[t]_0^1 = e^{-x} - x, \quad \text{i.e.} \quad e^{-x} - x = e^{-x} - x.$$

The corriect solution is obtained on achieving

$$c_k = c_{k-1} = 1.$$

Example: We look for a solution of equation

$$u(x) - \lambda \int_0^2 xe^{-0,2t} u(t)\, dt = e^{-2x}. \tag{26}$$

The kernel $K(x, t)$ may be written in the form

$$K(x, t) = xe^{-0,2t} = a_1(x)\, b_1(t),$$

i.e.

$$a_1 = x; \quad b_1 = e^{-0,2t}.$$

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Similarly to (23) and (24) let us rewrite equation (26) as

$$u_k(x) = f(x) + \lambda\, x c_{k-1},\qquad (27)$$

where

$$c_k = \int_0^2 e^{-0,2t} u(t)\, \mathrm{d}t = \text{constant}.$$

Following (27) then

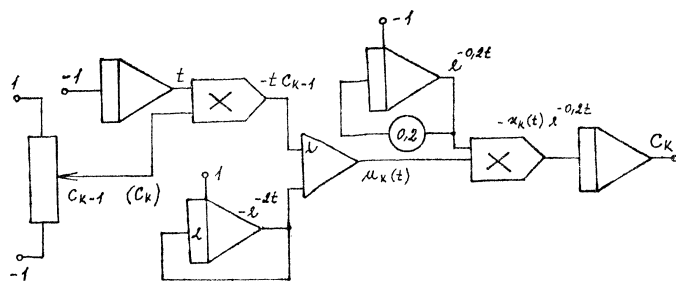$$u_k(t) = f(t) + \lambda t c_{k-1} = e^{-2t} + \lambda t c_{k-1}.$$

Fig. 4

Example: We look for a solution of equation

$$u(x) - \lambda \int_0^1 e^{-(\alpha x + t)} u(t)\, \mathrm{d}t = \sin x.\qquad (28)$$

The kernel

$$K(x, t) = e^{-\alpha x} e^{-t},$$

i.e.

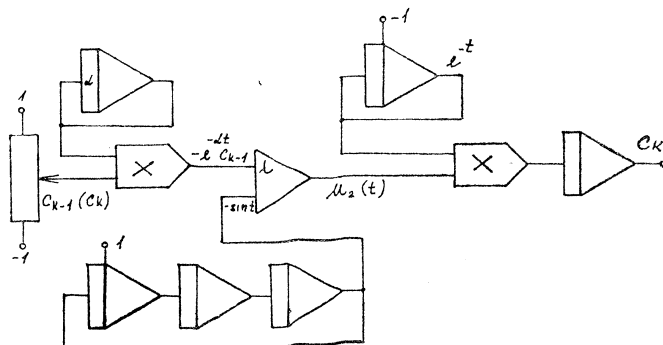$$a_1(x) = e^{-\alpha x}; \qquad b_1(t) = e^{-t}.$$

Fig. 5

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Equation (28) will be solved in the form

$$u_k(x) = f(x) + \lambda e^{-\alpha x} c_{k-1} = \sin x + \lambda e^{-\alpha x} c_{k-1},$$

$$c_k = \int_0^1 e^{-t} u_k(t)\, dt,$$

$$u(t) = f(t) + \lambda e^{-\alpha t} c_{k-1} = \sin t + \lambda e^{-\alpha t} c_{k-1}.$$

Example: Write a programming diagram for solving equation

$$u(x) = x + \int_0^{1/2} u(t)\, dt. \tag{29}$$

It holds $a_1(x) = b_1(t) = 1$ for the kernel $K(x, t) = 1$. Equation (29) will be written in the form

$$u_k(x) = f(x) + c_{k-1} = x + c_{k-1},$$

$$c_k = \int_0^{1/2} u_k(t)\, dt,$$

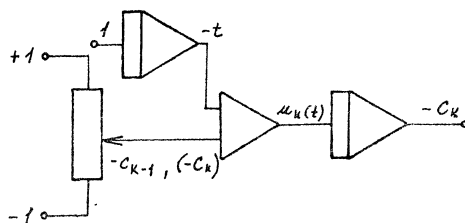$$u_k(t) = f(t) + c_{k-1} = t + c_{k-1}.$$

Fig. 6

Example: Write a programming diagram for solving equation

$$u(x) - \frac{1}{3} \int_0^1 (x + t)\, u(t)\, dt = \frac{5}{6} x - \frac{1}{9}. \tag{30}$$

The kernel $K(x, t)$ may be written in the form

$$K(x, t) = x + t = x \cdot 1 + 1 \cdot t$$

i.e.

$$a_1(x) = x; \qquad a_2(x) = 1; \qquad b_1(t) = 1; \qquad b_2(t) = t.$$

Equation (30) will be solved in the form

$$u_k(x) = f(x) + \frac{1}{3} \sum_{i=1}^{2} c_{k-1,i} a_i(x),$$

where

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$$c_{k,i} = \int_0^1 b_i(t)\, u_k(t)\, dt,$$

$$u(t) = f(t) + \frac{1}{3} \sum_{i=1}^2 c_{k-1,i} a_i(t).$$
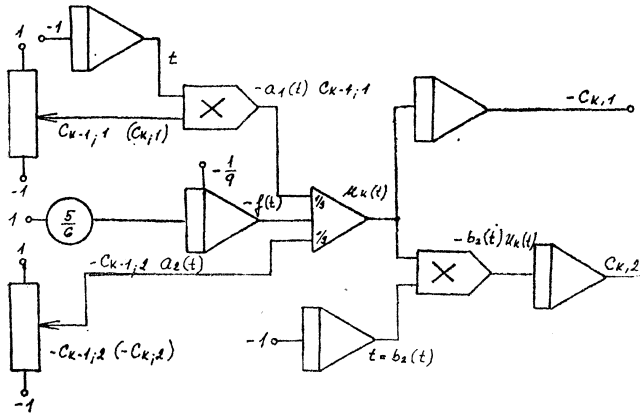

Fig. 7

Solving Fredholm's equation of the second kind with the second form of approximating the kernel

Solving integral equations by the method of successive approximations with the second form of approximating the kernel, the iterative process will be realized according the iterative formula

$$u_k(x_i) = f(x_i) - \lambda \int_0^a K(x_i, t)\, u_{k-1}(t)\, dt. \tag{31}$$

Figure (8) illustrates a block-diagram realizing the method of successive approxima-tion in the sense of (31). Here are determined the coordinales of the approximate solution at discrete points x_i. In the process of the repeated reading of the foregoing
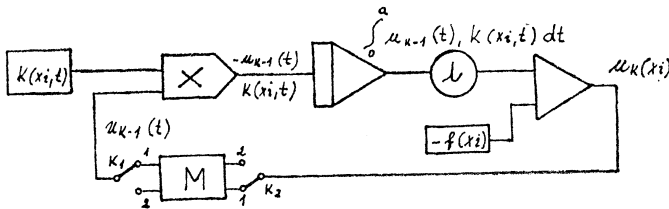

Fig. 8

$k - 1$ approximation from the memory device **M** there are in succession enumerated the coordinates of the k-th approximation to secure the unknown function and are recorded in the memory. For instance, the points of the k-th approximation are registred on the paper, a curve is drawm trough these points and we get a graph of the k-th approximation $u_k(x)$. This k-th approximation is then generated (for instance, by means of a follower of curves) as $u_k(t)$. After computation and recording all coordinates of the new approximation the positions of keys K_1 and K_2 are interchanged and the process will be reprated. Every change of the position of keys corresponds to one iteration step. The iterative process is at the end if two consecutive approximations are sufficiently near. The Gauss iterative process converges often more rapidly and may be written for solving Fredholm's integral equation of the second kind in the form of an algorithm

$$u_k(x_i) = f(x_i) - \lambda \int\limits_0^a K(x_i, t) u(t) \, dt, \tag{32}$$

where

$$u(t) = \begin{cases} u_k(t) & \text{for } t < x_i, \\ u_{k-1}(t) & \text{for } t > x_i. \end{cases}$$

The above process is however based on a special memory unit and further units (such as comparatories) being available, enabling its realization. This process could also be realized by a follower of curves and by tracing the particular approximations, but it would be too lengthy. The iterative process with the second form of approximating the kernel consists of two types of iterative steps:

a) from a slow iterative step, whereby from the k-1st approximation a k-th approximation is generated;

b) from rapid iterative step, whereby the coordinates of the new approximation at the points x_i are computed.

Dividing the independent variable x into $n + 1$ points from x_0 through x_n, then the slow iterative step consist of computing the functional values of the new approximation at the $n + 1$ points from x_0 through x_n.

Example:

$$u(x) = \frac{5}{6} x - \frac{1}{9} + \frac{1}{3} \int\limits_0^1 (t + x) u(t) \, dt, \tag{33}$$

$$u_k(x_i) = \frac{5}{6} x_i - \frac{1}{9} + \frac{1}{3} \int\limits_0^1 (t + x_i) u_{k-1}(t) \, dt.$$

Let us devide the interval $\langle 0,1 \rangle$ into 5 sections, i.e. $x_i = 0{,}2i$, $i = 0, 1, ..., 5$. Choosing $u_0(x) = 0$ yields

$$u_1(0) = \frac{5}{6} \cdot 0 - \frac{1}{9} + \frac{1}{3} \int\limits_0^1 (t + 0) u_0(t) \, dt =$$

$$= \frac{5}{6} \cdot 0 - \frac{1}{9} + \frac{1}{3} \int (t + 0) 0 \, dt = \frac{-1}{9},$$

$$u_1(0,2) = \frac{5}{6} \cdot 0,2 - \frac{1}{9} + \frac{1}{3} \int_0^1 (t + 0,2) \cdot 0 \, dt = \frac{1}{6} - \frac{1}{9} = \frac{1}{18},$$

$$u_1(0,4) = \frac{5}{6} \, 0,4 - \frac{1}{9} + \frac{1}{3} \int_0^1 (t + 0,4) \cdot 0 \, dt = \frac{2}{6} - \frac{1}{9} = \frac{4}{18},$$

$$u_1(0,6) = \frac{5}{6} \, 0,6 - \frac{1}{9} + \int_0^1 (t + 0,6) \cdot 0 \, dt = \frac{3}{6} - \frac{1}{9} = \frac{7}{18},$$

$$u_1(0,8) = \frac{5}{6} \, 0,8 - \frac{1}{9} + \frac{1}{3} \int_0^1 (t + 0,8) \cdot 0 \, dt = \frac{4}{6} - \frac{1}{9} = \frac{10}{18},$$

$$u_1(1) = \frac{5}{6} - \frac{1}{9} + \frac{1}{3} \int_0^1 (t + 1) \cdot 0 \, dt = \frac{5}{6} - \frac{1}{9} = \frac{13}{18}.$$

Functional values at the respective points are traced and a curve is drawn through the points (figure 9). Evidently, the graph of a function is the straight line with equation

$$u_1(x) = \frac{5}{6} x - \frac{1}{9} \tag{34}$$

and therefore
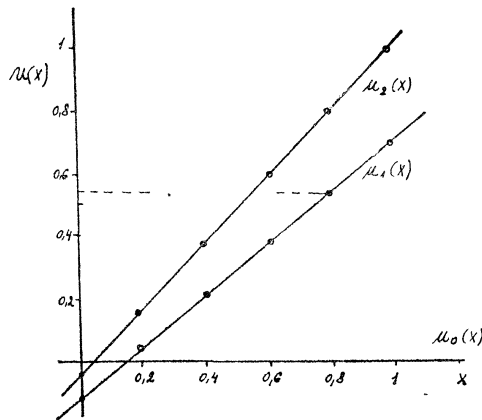
$$u_1(t) = \frac{5}{6} t - \frac{1}{9}.$$

Fig. 9

This function may also be generated by solving an appropriate differential equation without using any follower of curves. We now will compute the points of the second approximation on the basis of rolation

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$$u_2(x_i) = \frac{5}{6} x_i - \frac{1}{9} + \frac{1}{3} \int_0^1 (t + x_i) u_1(t) \, dt, \tag{35}$$

$$u_2(0) = \frac{5}{6} \cdot 0 - \frac{1}{9} + \frac{1}{3} \int_0^1 (t + 0)\left(\frac{5}{6} t - \frac{1}{9}\right) dt =$$

$$= -\frac{1}{9} + \frac{1}{3}\left[\frac{5}{6}\frac{t^3}{3} - \frac{1}{9}\frac{t^2}{2}\right]_0^1 = -\frac{1}{9} + \frac{5}{54} - \frac{1}{54} = \frac{-6 + 4}{54} = -\frac{1}{27},$$

$$u_2(0,2) = \frac{5}{6} 0,2 - \frac{1}{9} + \frac{1}{3} \int_0^1 (t + 0,2)\left(\frac{5}{6} t - \frac{1}{9}\right) dt =$$

$$= \frac{5}{6} 0,2 - \frac{1}{9} + \frac{1}{3}\left\{\left[\frac{5}{6}\frac{t^3}{3} - \frac{1}{9}\frac{t^2}{2}\right]_0^1 + 0,2\left[\frac{5}{6}\frac{t^2}{2} - \frac{1}{9}t\right]_0^1\right\} =$$

$$= \frac{5}{6} 0,2 - \frac{1}{9} + \frac{4}{54} + 0,2\left(\frac{5}{36} - \frac{1}{27}\right) =$$

$$= 0,2\left(\frac{5}{6} + \frac{5}{36} - \frac{1}{27}\right) - \frac{1}{9} + \frac{4}{54} =$$

$$= 0,2\frac{90 + 15 - 4}{108} + \frac{-6 + 4}{54} = 0,2\frac{101}{108} - \frac{1}{27},$$

$$u_2(0,4) = 0,4\frac{101}{108} - \frac{1}{27},$$

$$u_2(0,6) = 0,6\frac{101}{108} - \frac{1}{27},$$

$$u_2(0,8) = 0,8\frac{101}{108} - \frac{1}{27},$$

$$u_2(1) = \frac{101}{108} - \frac{1}{27}$$

$$u_2(x) = \frac{101}{108} x - \frac{1}{27}, \qquad u_2(t) = \frac{101}{108} t - \frac{1}{27}. \tag{36}$$

We continue in repeating this procedure till approaching the desired accuracy. The correct solution is

$$u(x) = x. \tag{37}$$

A part of the network for solving equation (33) is shown in figure 9a. The kernel $t + x_i$ is generated in solving equation

$$v' = 1; \qquad v_{(0)} = x_i.$$

Let us assume that from the funetional values of the function $u_k(x)$ at the points x_i we determine an analytic expression (or approximation) of this function, this will then be generated also by solving an appropriate differential equation. Respecting the fact that $K(x, t) \in \langle 0,2 \rangle$ holds for $x, t \in \langle 0; 1 \rangle$, we will generate the expres-

sion $\dfrac{K(x,t)}{2}$. The equation for generating the kernel will then be of the form

$$V' = 0,5: \qquad V_{(0)} = \frac{x_i}{2}.$$

We find step by step that

$$u_1(t) = \frac{5}{6}t - \frac{1}{9},$$

$$u_2(t) = \frac{101}{108}t - \frac{1}{27},$$
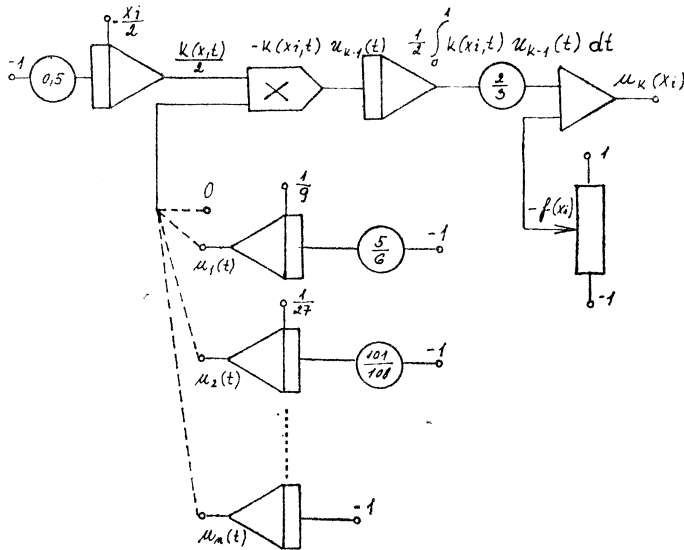
$$\vdots$$

$$u_n(t) = t$$

Fig. 9a

Same results will be obtained in using the method of successive approximations without applying the computer. Choosing $u_0(x) = 0$, yields $u_{(0)}(t) = 0$ and

$$u_1(x) = \frac{5}{6}x - \frac{1}{9} + \frac{1}{3}\int_0^1 (t + x) \cdot 0 \cdot dt = \frac{5}{6}x - \frac{1}{9},$$

i.e.

$$u_1(t) = \frac{5}{6}t - \frac{1}{9},$$

$$u_2(x) = \frac{5}{6}x - \frac{1}{9} + \frac{1}{3}\int_0^1 (t + x)\left(\frac{5}{6}t - \frac{1}{9}\right) dt =$$

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$$= \frac{5}{6}x - \frac{1}{9} + \frac{1}{3} \int_0^1 \left(\frac{5}{6}t^2 + \frac{5}{6}tx - \frac{1}{9}t - \frac{1}{9}x \right) dt =$$

$$= \frac{5}{6}x - \frac{1}{9} + \frac{1}{3} \left[\frac{5}{6}\frac{t^3}{3} + \frac{5}{6}\frac{t^2}{2}x - \frac{1}{9}\frac{t^2}{2} - \frac{1}{9}tx \right]_0^1 =$$

$$= \frac{5}{6}x\frac{1}{9} + \frac{1}{3} \left(\frac{5}{6} \cdot \frac{1}{3} + \frac{5}{12}X - \frac{1}{18} - \frac{1}{9}x \right) =$$

$$= \frac{5}{6}x - \frac{1}{9} + \frac{5}{54} + \frac{5}{36}x - \frac{1}{54} - \frac{1}{27}x =$$

$$= \frac{5}{6} + \frac{5}{36}x - \frac{1}{27}x - \frac{1}{9} + \frac{4}{54} =$$

$$= x \frac{90 + 15 - 4}{108} + \frac{-6 + 4}{54} = \frac{101}{108}x - \frac{1}{27}.$$

The correct solution is $u_{(x)} = x$, $(u(t) = t)$ as may be verified upon substituting in equation (33)

$$x = \frac{5}{6}x - \frac{1}{9} + \frac{1}{3} \int_0^1 (t + x)\, t\, dt,$$

$$x = \frac{5}{6}x - \frac{1}{9} + \frac{1}{3} \int_0^1 (t^2 + tx)\, dt = \frac{5}{6}x - \frac{1}{9} + \frac{1}{3} \left[\frac{t^3}{3} + \frac{t^2}{2}x \right]_0^1,$$

$$x = \frac{5}{6}x - \frac{1}{9} + \frac{1}{3} \left(\frac{1}{3} + \frac{x}{2} \right),$$

$$x = x.$$

Solving nonhomogeneous Solterra's equation with the first form of approximating the kernel

The equation is of the form

$$u(x) + \lambda \int_0^x K(x, t)\, u(t)\, dt = f(x). \tag{38}$$

The fact that the equation contains an integral with a variable upper limit enables in some cases to use integrators in the most natural regime for them, i.e. in a regime of a continuous integration. One of such cases is the solution with the first form of approximating the kernel whereby the kernel may be written in the form (or we approximate it by the expression):

$$K(x, t) = \sum_{i=1}^n a_i(x)\, b_i(t).$$

In applying this form of approximation, equation (38) becomes the form

$$u_{(}x) + \lambda \sum_{i=1}^n a_i(x) \int_0^x b_i(t)\, u(t)\, dt = f(x) \tag{39}$$

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in case of a nonlinear equation

$$u(x) + \lambda \int_0^x K(x, t) F(u(t)) \, dt = f(x)$$

generally becomes the form

$$u(x) + \lambda \sum_{i=1}^n a_i(x) \int_0^x b_i(t) F(u(t)) \, dt = f(x). \tag{40}$$

The integral transform

$$\int_0^x b_i(t) F(u(t)) \, dt$$

will be realized over the function of one argument and integrating may be continuous with respect to the variable x. The result of this transform is merely a function
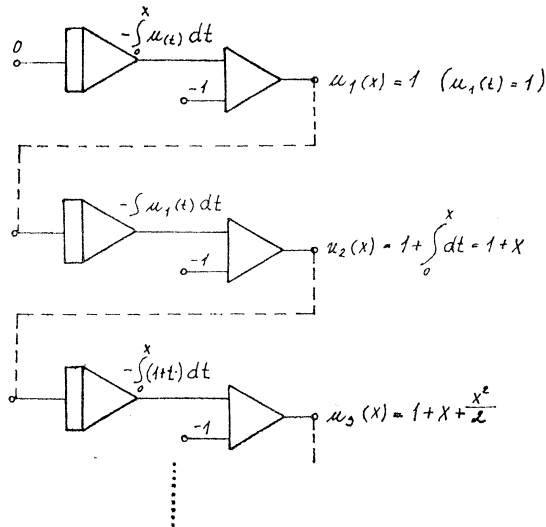

Fig. 10

of x. In this way all functions of the equation become dependent on one argument, which means that we may write a diagram for (40) enabling a direct solution as it is the case with ordinary differential equations with initial conditions. For instance, in solving equation

$$u(x) = 1 + \int_0^x u(t) \, dt \tag{41}$$

on the basis of the method of successive approximations we could proceed as shown below, if we start from $u_0(t) = 0$. The programming diagram may be simplified to the form given in figure 11.

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The programming diagram is described by equations:

$$u(t) = 1 + \int u(t)\,dt,$$

$$u'(t) - u(t) = 0, \qquad u_{t(0)} = 1, \qquad u(t) = e^t.$$

Example:

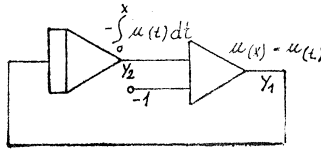$$u(x) - \int_0^x xu(t)\sin u(t)\,dt = e^x.$$

Fig. 11

The kernel $K(x, t)$ may be written in the form

$$K(x, t) = \sum_1^1 a_i(x)\,b_i(t),$$

where $a_1(x) = x$, $b_1(t) = u(t)$.

Then the equation may be rewritten as follows

$$u(x) - x \int_0^x u(t)\sin u(t)\,dt = e^x.$$

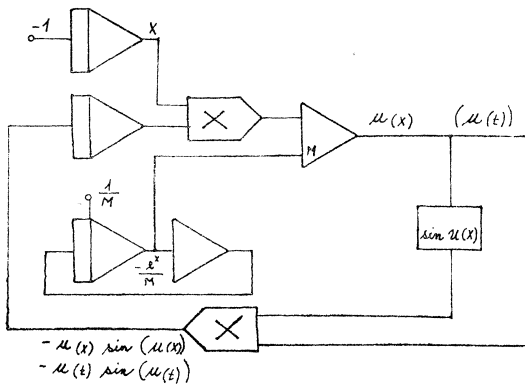The programming diagram is shown in figure 12.

Fig. 12

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Example:

$$u(x) = 1 + \int_0^x (t - x)\, u(t)\, dt.$$

The kernel $t - x$ may be written in the form

$$t - x = a_1(x)\, b_1(t) + a_2(x)\, b_2(t),$$

where

$$a_1(x) = 1, \qquad b_1(t) = t,$$
$$a_2(x) = -x, \qquad b_2(t) = 1,$$

$$u(x) = 1 + \int_0^x t u(t)\, dt - x \int_0^x u(t)\, dt,$$

$$u(t) = 1 + \int_0^t t u(t)\, dt - t \int_0^t u(t)\, dt,$$

$$u(t) = 1 + \int_0^t t u(t) + t \int_0^t u(t).$$

The solution is

$$u(t) = \cos t$$

$$\cos t = 1 + \int_0^t t \cos t\, dt + t \int_0^t \cos t\, dt.$$

The programming diagram is given in figure 13. The solution is the function $u(t) = \cos t$.

$$\int_0^t t \cos t\, dt = \left[t \sin t - \int \sin t\, dt \right]_0^t = \left[t \sin t + \cos t \right]_0^t,$$
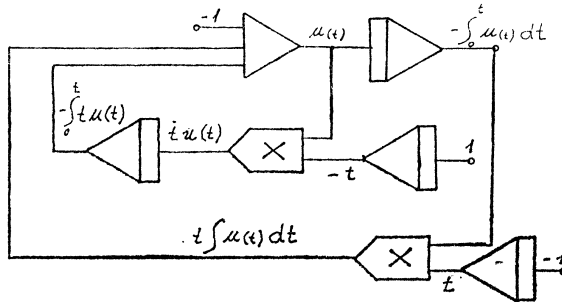

Fig. 13

$$\cos t = 1 + \left[t \sin t + \cos t \right]_0^t - t \left[\sin t \right]_0^t =$$
$$= 1 + t \sin t + \cos t - 1 - t \sin t,$$
$$\cos t = \cos t.$$

LITERATURE

Verlant, A. F.: *Metody rešenija integralnych uravnenij na analogovych vyčislitelnych mašinach.* *Naučnaja dumka, Kijev 1972*

ANALOGOVÉ ŘEŠENÍ INTEGRÁLNÍCH ROVNIC

Souhrn

V práci je podán přehled analogového řešení integrálních rovnic s aplikacemi a příklady.

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РЕШЕНИЕ ИНТЕГРАЛЬНЫХ УРАВНЕНИЙ НА АНАЛОГОВЫХ МАШИНАХ

Резюме

В работе описаны методы решения интегральных уравнений с аппликациями и примерами.