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# ON A GENERALIZATION OF NETS

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This paper is concerned with the study of nets, where groupoids participate in an algebraical expression. These problems vere studied by M.A.Taylor, V.D.Belousov, N.I.Prodan, particularly for 3-nets. This paper studies the nets of an arbitrary degree. It appears that in general the apparatus of groupoids are not sufficient to this purpose and certain admissible relations are to be used. Under cartain conditions it becomes possible to pass from them over to the groupoids or even to quasi-groups with the help of homotopies. These problems were investigated by V.Havel, G.Čupona, J.Ušan, Z. Stojaković in general, and we are investigating them for the nets of dimension 2.

A (general) net is defined as a quadruplet $(\mathcal{P}, \mathcal{L}, (\mathcal{L}_\iota)_{\iota \in \mathcal{J}}, I)$, where $\mathcal{P}$ is a non-empty set of elements called points, $\mathcal{L}$ is a non-empty set of elements called lines and $(\mathcal{L}_\iota)_{\iota \in \mathcal{J}}$ a system of mutually disjoint subsets of $\mathcal{L}$ (called pencils), the union of which is $\mathcal{L}$ . $\mathcal{J}$ is a set of indices $\# \mathcal{J} \geqq 3$, $I \subset \mathcal{P} \times \mathcal{L}$ is an incidence relation and the following conditions are satisfied:

(i) $\forall \ell \in \mathcal{L} \quad \exists P \in \mathcal{L}$ , $PI\ell$ ,

(ii) $\forall P \in \mathcal{P} \quad \forall \iota \in \mathcal{J} \; \exists \ell \in \mathcal{L}_\iota$ , $PI\ell$,

(iii) $\forall_{\iota \in \mathcal{J}} \quad \forall k,h \in \mathcal{L}_\iota ;\ k \neq h \ \{X \in \mathcal{P} \mid XIk,\ XIh\} = \emptyset$,

(iiii) $\forall \alpha, \beta, \gamma \in \mathcal{J};\ \alpha \neq \beta \neq \gamma \quad \forall \ell_1, \ell_2, \ell_3 \quad \ell_1 \in \mathcal{L}_\alpha$, $\ell_2 \in \mathcal{L}_\beta$, $\ell_3 \in \mathcal{L}_\gamma \ \# \{X \in \mathcal{P} \mid XI\ell_1, \ell_2, \ell_3\} \leqq 1$ .

Lines of the same pencil (distinct pencils) are called parallel (non-parallel) and we write $a \parallel b$ ($a \nparallel b$). The points $A_1, A_2, \ldots$ are called collinear if there is e line a such that $A_1I$ , $A_2I$ , .... We denote it by $\overline{A_1A_2\ldots}$ . The lines $a_1, a_2$, ... are called concurrent, if there is a point P such that $a_1IP$ , $a_2IP$ , .... We denote it by $\overline{a_1, a_2, \ldots}$ .

We denote by $[\ell] := \{X \in \mathcal{P} \mid XI\ell\}$ for every $\ell \in \mathcal{L}$ .
By a degree of a net the cardinality of the set $\mathcal{J}$ is meant.

Under a homomorphism of net $\mathcal{N} = (\mathcal{P}, \mathcal{L}, (\mathcal{L}_\iota)_{\iota \in \mathcal{J}}, I)$ onto a net $\mathcal{N}' = (\mathcal{P}', \mathcal{L}', (\mathcal{L}'_\iota)_{\iota \in \mathcal{J}}, I')$ we understand a couple $(\pi, \lambda)$ of surjective mappings $\pi : \mathcal{P} \to \mathcal{P}'$, $\lambda : \mathcal{L} \to \mathcal{L}'$ such that

(i) $PI\ell \Rightarrow \pi(P)I'\lambda(\ell)$ ,

(ii) $(\ell \in \mathcal{L}_\iota \Longrightarrow \lambda(\ell) \in \mathcal{L}'_\iota) \quad \forall_\iota \in \mathcal{J}$ .

If $\pi$, $\lambda$ are injections, then $(\pi, \lambda)$ is called an isomorphism.

We say $\mathcal{N} = (\mathcal{P}, \mathcal{L}, (\mathcal{L}_\iota)_{\iota \in \mathcal{J}}, I)$ is solvable if for every two lines a,b from distinct pencils $[a] \cap [b] \neq \emptyset$. $\mathcal{N}$ is uniquely solvable if there exists one uniquely determined point $P = [a] \cap [b]$. The uniquely solvable nets are called as classical nets.

We say $\mathcal{N}$ is a Cartesian net if there exist two significant pencils denote $\mathcal{L}_1$ , $\mathcal{L}_2$ $1,2 \in \mathcal{J}$ such that

$a \in \mathcal{L}_1$, $\quad b \in \mathcal{L}_2 \Rightarrow (\exists! P \in \mathcal{P}$ $PIa$ , $PIb)$ .

Let $(S_\iota)_{\iota\in\mathcal{J}}$ be a family of non-empty sets with the index set $\mathcal{J}$, $\#\mathcal{J} \geqq 3$. A subset $\sigma \subset \underset{\iota\in\mathcal{J}}{\times} S_\iota$ is called an admissible relation if

(1) $\mathrm{proj}_\iota \sigma = S_\iota$ for every $\iota \in \mathcal{J}$,

(2) for every $(s_\iota)_{\iota\in\mathcal{J}}$, $(z_\iota)_{\iota\in\mathcal{J}} \in \sigma$ if there exist $\alpha$, $\beta$, $\gamma$; $\alpha \neq \beta \neq \gamma \neq \alpha$ such that $s_\alpha = z_\alpha$, $s_\beta = z_\beta$, $s_\gamma = z_\gamma$, then $(s_\iota)_{\iota\in\mathcal{J}} = (z_\iota)_{\iota\in\mathcal{J}}$.

<u>Theorem 1</u>. For every admissible relation $\sigma \subset \underset{\iota\in\mathcal{J}}{\times} S$ (where $S_\iota$ are mutually disjoint) there exists a net $\mathcal{N} = (\mathcal{P}, \mathcal{L}, (\mathcal{L}_\iota)_{\iota\in\mathcal{J}}, I)$, such that $\mathcal{P} = \sigma$,

$$\mathcal{L} = \bigcup_{\iota\in\mathcal{J}} S_\iota \qquad \mathcal{L}_\iota = S_\iota \qquad \forall_\iota \in \mathcal{J},$$
$$(s_\iota)_{\iota\in\mathcal{J}} \; I y \Longleftrightarrow y \in \{s_\iota\}_{\iota\in\mathcal{J}}.$$

(Notation: $\mathcal{N}$ is a net over $\sigma$ ).

T o  t h e  p r o o f. The condition (i) from the definition of net is satisfied since $\mathrm{proj}_\iota \sigma = S_\iota$ for every $\iota \in \mathcal{J}$. The conditions (ii) and (iii) folow from the fact that $\sigma \subset \underset{\iota\in\mathcal{J}}{\times} S_\iota$ and if(1) holds. The condition (iiii) follows from (2) of the definition of $\sigma$.

<u>Theorem 2</u>. For every net $\mathcal{N} = (\mathcal{P}, \mathcal{L}, (\mathcal{L}_\iota)_{\iota\in\mathcal{J}}, I)$ there exists an admissible relation $\sigma \subset \underset{\iota\in\mathcal{J}}{\times} \mathcal{L}$ such that $(a_\iota)_{\iota\in\mathcal{J}} \in \sigma$ iff all lines of the set $\{a_\iota\}$ are concurrent.
(Notation : $\sigma$ is associated to $\mathcal{N}$ ).

T o  t h e  p r o o f. Property (1) of the admissible relation follows from condition (i) of the definition of a net. Property (2) follows from condition (iiii).

Under a homotopy (an isotopy) of an admissible relation $\sigma \subset \underset{\iota\in\mathcal{J}}{\times} S_\iota$ onto an admissible relation $\sigma' \subset \underset{\iota\in\mathcal{J}}{\times} S'_\iota$ we mean a system of surjective (bijective) mappings $\varphi_\iota: S_\iota \to S'_\iota$ such that

$$(s_\iota)_{\iota\in\mathcal{J}} \in \sigma \Longrightarrow (\varphi_\iota(s_\iota))_{\iota\in\mathcal{J}} \in \sigma'.$$

The isotopy of admissible relations with the same index set is an equivalence relation; two admissible relations from the same equivalence class are said to be isotopic.

Theorem 3. Let $\mathcal{N} = (\mathcal{P}, \mathcal{L}, (\mathcal{L}_\iota)_{\iota \in \mathcal{J}}, I)$ be a net. Then the net $\mathcal{N}'$ over the admissible relation associated to $\mathcal{N}$ is isomorphic to $\mathcal{N}$. Let $\sigma \subset \underset{\iota \in \mathcal{J}}{\times} S_\iota$ be an admissible relation. Then the admissible relation $\sigma'$ associated to the net over $\sigma$ is isotopic to $\sigma$.

T o  t h e  p r o o f. The isomorphism of nets is $(\mathcal{T}, id_{\mathcal{L}})$, where $\mathcal{T}$ maps every point $A \in \mathcal{P}$ onto the point $(a_\iota)_{\iota \in \mathcal{J}}$, where $AIa_\iota$ for every $\iota \in \mathcal{J}$. The isotopy of admissible relations is $(id_{S_\iota})_{\iota \in \mathcal{J}}$.

Let $\sigma \subset \underset{\iota \in \mathcal{J}}{\times} S_\iota$ be an admissible relation. We say $\sigma$ is solvable (uniquely solvable) if for every two elements $s_\alpha$, $s_\beta$ ; $s_\alpha \in S_\alpha$, $s_\beta \in S_\beta$ there exist (uniquely determined) elements $s_\iota$ $\iota \in \mathcal{J} \setminus \{\alpha, \beta\}$ such that $(s_\iota)_{\iota \in \mathcal{J}} \in \sigma$.

Clearly the solvable (uniquely solvable) admissible relation is associated to a solvable (uniquely solvable) net. And coversely: every solvable (uniquely solvable) net is a net over a solvable (uniquely solvable) admissible relation.

Let $\sigma \subset \underset{\iota \in \mathcal{J}}{\times} S_\iota$ be an admissible relation, $\sim_\iota$ an equivalence relation on $S_\iota$ $\iota \in \mathcal{J}$. We denote $S_\iota / \sim_\iota = \bar{S}_\iota$ and define the relation $\bar{\sigma} \subset \underset{\iota \in \mathcal{J}}{\times} \bar{S}_\iota$ as follows:

$$\forall X_\iota \in \bar{S}_\iota \quad (X_\iota)_{\iota \in \mathcal{J}} \in \bar{\sigma} \iff \forall \iota \in \mathcal{J} \quad \exists s_\iota \in X_\iota \in \bar{S}_\iota \quad (s_\iota)_{\iota \in \mathcal{J}} \in \sigma.$$

Then $\bar{\sigma}$ is an admissible relation called a factor relation, written as $\bar{\sigma} = \sigma / (\sim_\iota)_{\iota \in \mathcal{J}}$.

One example of the equivalence relation: Let $\sigma \subset \underset{\iota \in \mathcal{J}}{\times} S_\iota$ be an admissible relation. For every $\alpha \in \mathcal{J}$ we define the relation $\sim_\alpha$ on $S_\alpha$ as : $a \sim_\alpha b$ means that for every $\beta, \gamma \in \mathcal{J}$ $\alpha \neq \beta \neq \gamma \neq \alpha$ $(x_\iota)_{\iota \in \mathcal{J}} \in \sigma$, where $x_\alpha = a$, $x_\beta = x$, $x_\gamma = y$ if and only if $(y_\iota)_{\iota \in \mathcal{Y}} \in \sigma$, where $y_\alpha = b$, $y_\beta = x$, $y_\gamma = y$ .

Let $\sigma \subset \underset{\iota \in \mathcal{Y}}{\times} S_\iota$ be an admissible relation. We put $\sigma_{\alpha\beta\gamma}(x_\alpha, x_\beta) = \{x_\gamma \in S_\gamma \mid (x_\iota)_{\iota \in \mathcal{Y}} \in \sigma\}$ for every triple $(\alpha, \beta, \gamma) \in \mathcal{Y}^3$ and for every $x_\alpha \in S_\alpha$, $x_\beta \in S_\beta$. If $\sigma$ is a solvable relation then $\sigma_{\alpha\beta\gamma}(x_\alpha, x_\beta) \neq \emptyset$ for every $(\alpha, \beta, \gamma) \in \mathcal{Y}^3$ and for every $x_\alpha \in S_\alpha$, $x_\beta \in S_\beta$ .

The solvable relation $\sigma$ is called regular if for every triple $(\alpha, \beta, \gamma) \in \mathcal{Y}^3$ the set $\{\sigma_{\alpha\beta\gamma}(x_\alpha, x_\beta) \mid x_\alpha \in S_\alpha,\ x_\beta \in S_\beta\}$ is a decomposition of the set $S_\gamma$ (corresponding to some equivalence relation $\sim_\gamma$ on $S_\gamma$).

Theorem 4. Let $\sigma \subset \underset{\iota \in \mathcal{Y}}{\times} S_\iota$ be a regular admissible relation. Then $\sigma/(\sim_\iota)_{\iota \in \mathcal{Y}}$ is a uniquely solvable admissible relation.

T o  t h e  p r o o f. Since $\sigma$ is a solvable admissible relation, then with respect to the definition of regularity, $\bar{\sigma} = \sigma/(\sim_\iota)_{\iota \in \mathcal{Y}}$ is solvable, too.

Analogously to the set $\sigma_{\alpha\beta\gamma}(x_\alpha, x_\beta)$ for $\sigma$ we now define the sets $\bar{\sigma}_{\alpha\beta\gamma}(X_\alpha, X_\beta)$ for the relation $\bar{\sigma} = \sigma/(\sim_\iota)_{\iota \in \mathcal{Y}}$ . With regard to the definition of a regular relation it becomes evident that $\# \bar{\sigma}_{\alpha\beta\gamma}(X_\alpha, X_\beta) = 1$ for every $(\alpha, \beta, \gamma) \in \mathcal{Y}^3$ and for every $X_\alpha \in \bar{S}_\alpha$, $X_\beta \in \bar{S}_\beta$ .

Hence it follows that the relation $\bar{\sigma}$ is uniquely solvable.

Corollary : Any regular admissible relation can be mapped onto a uniquely solvable admissible relation by a homotopy.

T o  t h e  p r o o f. We define a system of surjections $\varphi_\iota : S_\iota \to \bar{S}_\iota$ such that

$$\varphi(x_{\iota_0}) = X_{\iota_0}, \qquad x_{\iota_0} \in X_{\iota_0}$$
$$x_{\iota_0} \in S_\iota \qquad X_{\iota_0} \in \bar{S}_\iota .$$

There may be found to all properties of the admissible relation corresponding properties and corresponding theorems for nets. The most important of them are considered below.

**Theorem 5.** Let $\mathcal{N} = (\mathcal{P}, \mathcal{L}, (\mathcal{L}_\iota)_{\iota \in \mathcal{J}}, I)$ be a regular net. Then there exists a homomorphism which maps the net to a classical net.

Under a three-basic groupoid we understand a quadruplet $(A_1, A_2, A_3, \cdot)$, where $A_1, A_2, A_3$ are non-empty sets and $\cdot$ is a mapping $\cdot : A_1 \times A_2 \longrightarrow A_3 \quad (a,b) \longmapsto a.b$ .

**Theorem 6.** Let $\mathcal{N}$ be a Cartesian net, $\sigma$ an admissible relation associated to $\mathcal{N}$. Then there exists a system of three-basic groupoids $((S_1, S_2, S_\alpha, \cdot_\alpha)_{\alpha \in \mathcal{J} \setminus \{1,2\}})$ canonically determined by $\sigma$ .

T o t h e p r o o f. With regard to the definition of a Cartesian net $a \in \mathcal{L}_1$ , $b \in \mathcal{L}_2 \Rightarrow \exists! P \in \mathcal{P}$ PIa, PIb and to the axiom (ii) of the definition of net (exactly one line from every pencil is incident with P) for relation $\sigma$ there must hold $s_1 \in S_1$, $s_2 \in S_2 \Rightarrow \exists! s_\alpha \in S_\alpha$ $\alpha \in \mathcal{J} \setminus \{1,2\} (s_\iota)_{\iota \in \mathcal{J}} \in \sigma$

and the mappings $\cdot_\alpha : S_1 \times S_2 \longrightarrow S_\alpha$ $\alpha \in \mathcal{J}$ are given.

Let $\sigma \subset \underset{\iota \in \mathcal{J}}{\times} S_\iota$ be an admissible relation associated to a Cartesian net such that $\# S_\iota$ = constant for every $\iota \in \mathcal{J}$ Then $\sigma$ determines a system of three-basic groupoids and let S be any set the cardinality of which is $\# S_\iota$ . The system of bijections $\varphi_\iota : S_\iota \longrightarrow S$ for every $\iota \in \mathcal{J}$ maps our system of three-basic groupoids by isotopism onto a system groupoids $((S, \cdot_\alpha)_{\alpha \in \mathcal{J} \setminus \{1,2\}})$. Hence it holds :

If $\mathcal{N}$ is a Cartesian net for which $\# \mathcal{L}_\iota$ = constant for every $\iota \in \mathcal{J}$, then the corresponding algebraical system is a family of groupoids. If the net $\mathcal{N}$ is a solvable Cartesian net, then the equations $s_1 \cdot_\alpha x = s_\alpha$ , $y \cdot_\alpha s_2 = s_\alpha$ are solvable for every $s_1 \in S_1$ , $s_2 \in S_2$ , $s_\alpha \in S_\alpha$ for every $\alpha \in \mathcal{J} \setminus \{1,2\}$ and we obtain the family of D-groupoids (PRODAN) called also solvable groupoids, or groupoids with division. If the net $\mathcal{N}$ is a uniquely solvable (obviously it is a Cartesian net and $\# \mathcal{L}_\iota$ = constant), i.e. a classical net, then all groupoids are uniquely solvable and we obtain the family of quasigroups.

<u>Conclusion</u> : Let $\mathcal{N}$ be a regular net and $\sigma$ an admissible relation associated to $\mathcal{N}$. Then the system of three-basic groupoids determined by $\sigma$ is homotopic to a system of quasigroups.

Thus we have found an important class of regular solvable nets. The fact that they can pass from the epimorphism to the classical nets guarantees that this generalization is reasonable and is worth of investigating their properties.

<u>Example</u> of a regular solvable admissible relation :
Let $S_k$ be the sets of complex numbers having absolute value equal to $k$ for $k \in \{2,3,6,12,18,36,72,108,216,432,648,1296,\dots\} = \mathcal{J}$ where $k = 2^m.3^n$ , $|m-n| \leqq 1$ $m,n \in N \cup \{0\}$ .
(N is a set of natural numbers.)

We put $\sigma = \{(s_\iota)_{\iota \in N} \mid s_\iota = s_1^m . s_2^n \ , \ \iota = m+2n \ \ |m-n| \leqq 1\} \subset$

$$\subset S_2 \times S_3 \times S_6 \times \dots \times S_k \times \dots$$

Then $\sigma$ has all sequested properties.
The corresponding equivalence relation $\sim_k$ $k \in \mathcal{J}$ are the relations of diametrally opposite points on $S_k$. The factor relation $\sigma/(\sim_k)_{k \in \mathcal{J}}$ is isotopic to

$$\{(x_\iota)_{\iota \in N} \mid |x_1| = 2 \ , \ |x_2| = 3, \dots, |x_\iota| = 2^m.3^n \ , \dots$$

$$x_\iota = x_1^m . x_2^n \quad \text{if} \quad \iota = m+2n \quad |m-n| \leqq 1 \quad m,n \in N \cup \{0\} \ .$$

Now we make a system of three-basic groupoids.

$((S_2, S_3, S_k, \cdot_k)_{k \in \mathcal{J}})$, $\cdot k$ is defined the following way:

$$s_\iota = s_1 \cdot_k s_2 \iff s_\iota = s_1^m \bullet s_2^n \ , \quad s_1 \in S_2 \ , \ s_2 \in S_3 \ , \quad s_\iota \in S_k \ ,$$

$\iota = m+2n$ $|m-n| \leqq 1$ , $\bullet$ is product of complex numbers.

## REFERENCES


[1] B e l o u s o v, V.D., P r o d a n, N.I.: Cartesian solution nets and groupoids with division (in Russian), Mat.Issl. (Kišiněv), fasc.43, 1976, 30-38.

[2] H a v e l, V.: General nets and their associated groupoids, Proseedings of the symp. n-ary structures (Skopje) 1982, 229-241.

[3] H a v e l, V.: Nets and groupoids, CMUC (Praha) 1967, vol 8, fasc.3, 435-451.

[4] P r o d a n, N.I.: Regular groupoids with division (in Russian), Issledovanija po teorii kvazigrup i lup, Kišiněv 1973, 139-145.

[5] T a y l o r, M.A.: Classical, Cartesian and solutions nets, Mathematicy (Cluj) 13(36), 1, 1971, 151-166.

[6] T a y l o r, M.A.: Cartesian Nets and groupoids, Canad. Math.Bull. 16(3), 1973, 347-362.


## O JEDNOM ZOBECNĚNÍ TKÁNÍ


Souhrn

V článku se zavádí pojem obecné tkáně a k ní příslušného algebraického protějšku - přípustné relace. Specielně kartézským tkáním odpovídají tzv. tříbázové grupoidy. Ukazuje se, že třída regulárních řešitelných tkání je významná tím, že každou její tkáň je možno epimorfně zobrazit na klasickou tkáň.

## ОБ ОДНОМ ОБОБЩЕНИИ СЕТЕЙ

**Резюме**

Статья описывает понятие общей сети и соответствующего алгебраического выражения. Показывается класс сетей, которые отображаются помощью эпиморфизма на классические сети.

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