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A NOTE ON VARIETIES WITH DISTRIBUTIVE SUBALGEBRA LATTICES

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Abstract: A necessary and sufficient term condition characterizing varieties whose members have distributive lattices of subalgebras is given.

Key words: variety of algebras, term condition, hamiltonian variety, distributive lattice, subalgebra lattice.

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T. Evans and B. Ganter [1] gave a ∀∃-term characterization of varieties of algebras having modular subalgebra lattices. The aim of this short note is to use a similar method to characterize varieties having distributive subalgebra lattices.

Let A be an algebra. Denote by $SubA$ the lattice of all subalgebras of A (orderer by set inclusion). The operation join (or meet) in $SubA$ will be denoted by $\vee$ (or $\wedge$, respectively). A is said to have *distributive subalgebra lattice* if $SubA$ is distributive. A variety V is *subalgebra distributive* if each member $A \in V$ has distributive subalgebra lattice.

If $a_1, \dots, a_n$ are elements of an algebra A, denote by $Gen(a_1, \dots, a_n)$ the subalgebra of A generated by the set

$\{a_1, \ldots, a_n\}$. If V is a variety of algebras, denote by $F_n(V)$ the free algebra of V generated by n free generators.

An algebra A is *hamiltonian* (see [2]) if every subalgebra of A is a class of some congruence on A. A variety V is *hamiltonian* if each A V has this property.

Proposition 1 (**Theorem 1.4.** in [3]). *If B, C are subalgebras of an algebra A in a hamiltonian variety and $x \in B \vee C$, then there exist $b_1, b_2 \in B$ and $c_1, c_2 \in C$ such that $x \in Gen(b_1, b_2, c_1, c_2)$.*

Proposition 2 (**Theorem** 1 in [1]). *If the lattice of subalgebras of every free algebra in a variety V is modular, then V is hamiltonian.*

Now, we are ready to prove the following.

Theorem. *For a variety V, the following conditions are equivalent*:

(1) *V is subalgebra distributive;*

(2) *V is hamiltonian and for every 4-ary term p there exist a 4-ary term q and unary terms r_i, s_i $(i=1,2,3,4)$ such that*

$$p(x_1, x_2, x_3, x_4) = q(s_1(x_1), s_2(x_2), s_3(x_3), s_4(x_4))$$
$$r_i(p(x_1, x_2, x_3, x_4)) = s_i(x_i) \text{ for } i=1,2,3,4.$$

Proof. (1)⇒(2): Let V be a subalgebra distributive variety and $A=F_4(V)$, where x_1, x_2, x_3, x_4 are free generators of A. Let p be a 4-ary **term** (over V). Let C be a **subalgebra** of A such that $C=Gen(p(x_1, x_2, x_3, x_4))$ and B_i be a subalgebra of A such that $B_i=Gen(x_i)$ for $i=1,2,3,4$. Then, evidently,

$$p(x_1, x_2, x_3, x_4) \in C \wedge (B_1 \vee B_2 \vee B_3 \vee B_4).$$

By (1), **we** have

$$p(x_1, x_2, x_3, x_4) \in (C \wedge B_1) \vee (C \wedge B_2) \vee (C \wedge B_3) \vee (C \wedge B_4).$$

By Proposition 2, V **is hamiltonian and , by Proposition** 1, **there exist elements** $b_i \in (C \wedge B_i)$, $i=1,2,3,4$ **such that**

$$p(x_1, x_2, x_3, x_4) \in Gen(b_1, b_2, b_3, b_4),$$

since every B_i **as well as** C **is generated by the unique generator. Hence, there exists a 4-ary term** q **such that**

$$p(x_1, x_2, x_3, x_4) = q(b_1, b_2, b_3, b_4)$$

and $b_i \in C\wedge B_i$ gives

$$b_i = r_i(p(x_1,x_2,x_3,x_4)) = s_i(x_i), \quad i=1,2,3,4$$

for some unary terms r_i, s_i.

(2)⇒(1): Let $A\in V$ and R,S,Q be subalgebras of A. Let $a\in R\wedge(S\vee Q)$. Then $a\in R$ and, by **Proposition** 1, there **exists** a 4-ary term p and elements $b_1,b_2\in S$, $c_1,c_2\in Q$ such that

$$a=p(b_1,b_2,c_1,c_2).$$

By (2), there exist a 4-ary term q and unary terms r_i, s_i ($i=1,2,3,4$) such that

$$a=q(s_1(b_1),s_2(b_2),s_3(c_1),s_4(c_2)),$$

where

$$s_1(b_1)=r_1(a),\quad s_2(b_2)=r_2(a)$$

$$s_3(c_1)=r_3(a),\quad s_4(c_2)=r_4(a).$$

Hence,

$$s_1(b_1)\in S\wedge R,\quad s_2(b_2)\in S\wedge R,$$

$$s_3(c_1)\in Q\wedge R,\quad s_4(c_2)\in Q\wedge R,$$

thus $a\in(S\wedge R)\vee(Q\wedge R)$, proving (1). ∎

Example. Every at most unary variety is subalgebra distributive.

It is evident that every at **most** unary variety is hamiltonian (see the ∀∃-**term** characterization of hamiltonian varietis in [2]) and also the ∀∃-**term** condition of (2) is clearly satisfied if p is at **most** unary.

References

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