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Asymptotic and Integral Equivalence of Multivalued Differential Systems

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Abstract

We give some new general results on asymptotic and integral equivalence of multivalued differential systems.

Key words: Multivalued differential systems, asymptotic equivalence of differential systems, integral equivalence of differential systems.

MS Classification: 34A60, 34D05, 34E10

1 Introduction

The purpose of this paper is to study the asymptotic and integral equivalence of the systems

$$y'(t) = A(t)y(t) \tag{a}$$

and

$$x'(t) \in A(t)x(t) + F(t, x(t), Sx(t)) \ , \tag{b}$$

where x, y are n-dimensional vectors, $A(t)$ is an $n \times n$ matrix-function defined on $J := [0, +\infty)$ whose elements are integrable on compact subsets of J, $F(t, u, v)$ is a nonempty compact convex subset of R^n for each $(t, u, v) \in J \times R^n \times R^n$ and S is an operator which maps the set $B_\psi(J)$ of continuous and ψ-bounded functions defined on J (we shall say that a function $z(t)$ *is ψ-bounded on the interval* J iff $\sup_{t\in J} |\psi^{-1}(t)z(t)| < +\infty$) into $B_\psi(J)$ and is continuous in the following sense: if $\{x_n\} \in B_\psi(J)$ converges to x uniformly on compact subsets of J then $\{(Sx_n)(t)\}$ converges to $(Sx)(t)$ at each $t \in J$ e.g.

$$Sx(t) := \int_0^t K(t,s)\,x(s)\,ds,$$

under certain conditions on the function K(t,s). By *a solution of (b)* we mean an absolutely continuous function on some nondegenerate subinterval of J which satisfies (b) almost everywhere (a.e.).

Definition 1.1 Let $\psi(t)$ be a positive function on an interval $[t_0, +\infty) \subset J$. We shall say that *two systems* (a) *and* (b) *are ψ-asymptotically equivalent* iff for each solution $x(t)$ of (b) there exists a solution $y(t)$ of (a) such that

$$\psi^{-1}(t)|x(t) - y(t)| \to 0 \quad \text{as} \quad t \to +\infty \tag{c}$$

and conversely, for each solution $y(t)$ of (a) there exists a solution $x(t)$ of (b) such that (c) holds.

Definition 1.2 Let $\psi(t)$ be a positive function on an interval $[t_0, +\infty) \subset J$ and let $p \geq 1$. We shall say that *two systems* (a) *and* (b) *are* (ψ, p)*-integral equivalent on* $[t_0, +\infty)$ iff for each solution x(t) of (b) on $[t_0, +\infty)$ there exists a solution $y(t)$ of (a) on $[t_0, +\infty)$ such that

$$\psi^{-1}(t)|x(t) - y(t)| \in L_p([t_0, +\infty)) \tag{d}$$

and conversely, for each solution $y(t)$ of (a) on $[t_0, +\infty)$ there exists a solution $x(t)$ of (b) on $[t_0, +\infty)$ such that (d) holds.

By *restricted* (ψ, p)*-integral (asymptotic) equivalence of* (a) *and* (b) we shall mean that the relation (d) $((c))$ is satisfied for some subsets of solutions of (a) and (b), e.g. for the ψ-bounded solutions.

The problem of asymptotic equivalence has been studied by many authors. Hallam's paper [1] treats the asymptotic equivalence of ψ-bounded solutions of systems of single-valued differential equations. S. V. Seah in [6] studies ψ-asymptotical equivalence of the systems of the form (a) and (b) in the case that S is the identity operator (however the proofs of the theorems of this paper are based on Lemma 2.3 of [6], which is false). A. Haščák and M. Švec in [5] gave sufficient conditions for (ψ, p)-integral equivalence of two single-valued systems. Sufficient conditions for (ψ, p)-integral equivalence of the systems of the form (a) and (b) (in the case that S is identity operator) may be found in [2] and [3]. Our results are supplementary to that from [4].

2 Preliminary results

In this section we give some notation as well as preliminary results which will be needed later.

We shall write $|\cdot|$ for any convenient vector (matrix) norm. If A is a subset of R^n, we define

$$|A| := sup\{|a| : a \in A\}.$$

Let Y be a topological space. Let us denote by 2^Y the family of all nonempty subsets of the space Y and let $cf(Y)$ be the set of all nonempty convex and closed subsets of Y. $L_p^n(J)$ will denote the n-th Cartesian product of $L_p(J)$ and let $B(t_0)$ be the space of all continuous functions from $[t_0, +\infty)$ to R^n. The topology on $B(t_0)(= B([t_0, +\infty))$ will be then introduced by the family of semi-norms $\{p_n\}$ where for each $x \in B(t_0)$

$$p_n(x) := \sup_{t_0 \le t \le t_0+n} |x(t)|. \tag{2.1}$$

A fundamental system of neighbourhoods of the function $x(t) = 0, t \in [t_0, +\infty)$ is then given by the sets v_n, $n = 1, 2, \ldots$ where

$$v_n := \{x \in B(t_0) : p_n < \tfrac{1}{n}\}.$$

Under this topology $B(t_0)$ is a complete, locally convex and metrizable vector space. The topology is equivalent to the topology of uniform convergence on compact subsets of $[t_0, +\infty)$.

Let $\psi(t)$ be a positive continuous function on $[t_0, +\infty)$. For $z \in B(t_0)$ we denote

$$|z|_\psi := \sup_{t \ge t_0} |\psi^{-1}(t)z(t)|.$$

Let

$$B_\psi(t_0) := \{z \in B(t_0) : |z|_\psi < +\infty\}.$$

Then $B_\psi(t_0)$ with the norm $|\cdot|_\psi$ is a Banach space.

For $\rho > 0$ we denote

$$B_{\psi,\rho}(t_0) := \{z \in B_\psi(t_0) : |z|_\psi \le \rho\}.$$

Further, let $\varphi(s)$ be a positive continuous function defined on $J = [0, +\infty)$. By $L_{p,\varphi}(J)$, $1 \le p < +\infty$ we shall denote the set of all real-valued measurable functions $y(t)$ defined on J such that

$$|y(t)|_{p,\varphi} := \left(\int_0^{+\infty} |\varphi^{-1}(s)y(s)|^p ds \right)^{\frac{1}{p}} < +\infty.$$

$L_{p,\varphi}(J)$ with the norm $|\cdot|_{p,\varphi}$ is also a Banach space. Let $\varphi(t) \equiv 1$, then $u \in L_p(J, K)$, $K > 0$ will denote the fact that

$$\int_0^{+\infty} |u(s)|^p ds \le K.$$

In the sequel we shall need the following lemmas:

Lemma 2.1 *(Hölder's inequality). Let S be a Lebesgue measurable subset of R, let functions a_k satisfy $a_k(s) \in L_{p_k}(S)$, $k = 1, \ldots, m$ and*

$$\frac{1}{p_1} + \cdots + \frac{1}{p_m} = 1 .$$

Then

$$\left| \int_S a_1(s) \ldots a_n(s) ds \right| \leq \prod_{k=1}^{m} \left(\int_S |a_k(s)|^{p_k} ds \right)^{\frac{1}{p_k}} .$$

Lemma 2.2 *(Ky Fan's fixed point theorem ([6] Corollary 2.8)). Let A be a closed, bounded and convex subset of a locally convex topological vector space B. If $T : A \to cf(A)$ is upper semicontinuous and $T(A)$ is compact, then there exists $x \in A$ such that $x \in T(x)$.*

3 Main Results

Let $Y(t,s)$ be the Cauchy matrix for the system (a) such that $Y(t,t)$ is the identity matrix. Let a, b, p, q be real numbers such that

$$1 < p < q < +\infty, \qquad 0 < b < q, \qquad 1 - \frac{b}{q} < a, \qquad \frac{1}{p} + \frac{1}{a}(1 - \frac{b}{q}) = 1 . \tag{3.1}$$

Denote now

$$\frac{1}{\alpha} := \frac{1}{q}, \qquad \frac{1}{\beta} := \frac{1}{a}(1 - \frac{b}{q}), \qquad \frac{1}{\gamma} := \frac{1}{p} - \frac{1}{q}$$

(it is easy to see that $\alpha, \beta, \gamma \in (1, +\infty)$ and $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 1$).

Further throughout this paper we shall assume that

(H_0) the functions $\psi(t)$ and $\phi(t)$ are positive continuous functions on $J := [0, +\infty)$

and that the mapping $F : J \times R^n \times R^n \to cf(R^n)$ and the operator $S : B_\psi(J) \to B_\psi(J)$ satisfy the following hypotheses:

(H_1) $F(t, u, v)$ is a nonempty, compact and convex subset of R^n for each $(t, u, v) \in J \times R^n \times R^n$;

(H_2) for every fixed $t \in J$, the function $F(t, u, v)$ is upper semicontinuous;

(H_3) for each $x \in B_\psi(J)$ there exists a measurable function $f_x : J \to R^n$ such that

$$f_x(t) \in F(t, x(t), Sx(t)) \quad \text{a.e. on } J;$$

(H_4) there is a constant $k \in (0, +\infty)$ such that

$$|Sz|_\psi \leq k.|z|_\psi .$$

Given a function $x \in B_\psi(J)$ denote by $M(x)$ the set of all measurable functions $y : J \to R^n$ such that

$$y(t) \in F(t, x(t), Sx(t)) \quad \text{a.e. on } J.$$

Further we shall need the following lemmas. The proof of Lemma 3.1 is analogous to that of Theorem 4 (Theorem 5) of [4].

Lemma 3.1 *Let the hypotheses (H_0)-(H_4) be satisfied and moreover suppose that there exists $g : J \times J \times J \to J$ such that*

i) $g(t, u, v)$ is monotone nondecreasing in u for fixed $t \in J, v \in J$ and monotone nondecreasing in v for each fixed $t \in J$, $u \in J$;

ii) $g(t, c, c) \in L_p(J)$ for any constant $c \geq 0$ and some $p \in [1, +\infty)$;

iii) for each $u, v \in R^n$

$$|F(t, u, v)| \leq \varphi(t) g(t, \psi^{-1}(t)|u|, \psi^{-1}(t)|v|) \quad a.e. \text{ on } J.$$

Then the correspondence $x \to M(x)$ defines a bounded and weakly upper semicontinuous mapping of $B_{\psi,\rho}(J)$ into $cf(L^n_{p,\phi}(J))$.

Lemma 3.2 *Let the hypotheses of Lemma 3.1 be satisfied. Moreover suppose that for some $a, b \in R$ $(a, b \in (3, 1))$ the following inequalities hold*

$$\sup_{t \in J} \left(\int_0^t |\psi^{-1}(t) Y(t, s) \varphi(s)|^a ds \right)^{\frac{1}{a}} \leq m_1 < +\infty \,, \tag{3.2}$$

$$\sup_{t \in J} \left(\int_0^t |\psi^{-1}(t) Y(t, s) \varphi(s)|^b g^p(s, c, c) ds \right) \leq m_2 < +\infty \,. \tag{3.3}$$

Then the operator TM defined by the formula

$$TM \; x := \left\{ z \; : \; z = \int_0^t Y(t, s) f_x(s) \; ds \quad and \quad f_x \in M(x) \right\} \tag{3.4}$$

maps $B_{\psi,\rho}(J)$ into 2^{B_ψ} and is upper semicontinuous.

Proof For each $z \in TM \; x$, $x \in B_{\psi,\rho}(J)$ we have

$$|\psi^{-1}(t) z(t)| \leq \int_0^t \left(|\psi^{-1}(t) Y(t, s) \varphi(s)|^{\frac{b}{q}} |\varphi^{-1}(s) f_x(s)|^{\frac{p}{q}} \right)$$

$$\times \left(|\psi^{-1}(t) Y(t, s) \varphi(s)| \right)^{a(\frac{1}{a} - \frac{b}{aq})} \cdot \left(|\varphi^{-1}(s) f_x(s)^{p(\frac{1}{p} - \frac{1}{q}} \right) ds.$$

Using the Hölder's inequality, (3.2), (3.3) and Lemma 3.1 we have

$$|\psi^{-1}(t)z(t)| \le \left(\int_0^t |\psi^{-1}(t)Y(t,s)\varphi(s)|^b |\varphi^{-1}(s)f_x(s)|^p\, ds\right)^{\frac{1}{\alpha}}$$
$$\times \left(\int_0^t |\psi^{-1}(t)Y(t,s)\varphi(s)|^a ds\right)^{\frac{1}{\beta}} \cdot \left(\int_0^t |\varphi^{-1}(s)f_x(s)|^p\, ds\right)^{\frac{1}{\gamma}}$$
$$\le \left(\int_0^t |\psi^{-1}(t)Y(t,s)\varphi(s)|^b g^p(s,c,c)\, ds\right)^{\frac{1}{\alpha}}$$
$$\times \left(\int_0^t |\psi^{-1}(t)Y(t,s)\varphi(s)|^a ds\right)^{\frac{1}{\beta}} \cdot \left(\int_0^t |\varphi^{-1}(s)f_x(s)|^p\, ds\right)^{\frac{1}{\gamma}}$$
$$\le m_2^{\frac{1}{\alpha}}.m_1^{\frac{a}{\beta}}.|f_x|_{p,\varphi}^{\frac{p}{\gamma}} < +\infty\,. \tag{3.5}$$

Thus $z(t)$ is a ψ-bounded function on J. To prove that TM maps $B_{\psi,\rho}(J)$ into $2^{B_\psi(J)}$ it suffices to prove that $z(t)$ is continuous. Let $0 \le t_1$, $t_2 = t_1 + h$, $|h| < 1$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Then

$$|z(t_2) - z(t_1)| \le \left|\int_0^{t_1} |Y(t_2,s) - Y(t_1,s)|\varphi(s) \cdot |\varphi^{-1}(s)f_x(s)|\, ds\right|$$
$$+ \left|\int_{t_2}^{t_1} |Y(t_2,s)\varphi(s)||\varphi^{-1}(s)f_x(s)|\, ds\right|$$
$$\le \left|\int_0^{t_1} |(Y(t_2,s) - Y(t_1,s))\varphi(s)|^{p'}\, ds\right|^{\frac{1}{p'}} \cdot \left|\int_0^{t_1} g^p(s,\rho,k\rho)\, ds\right|^{\frac{1}{p}}$$
$$+ \left|\int_{t_1}^{t_2} |Y(t_2,s)\varphi(s)|^{p'}\right|^{\frac{1}{p'}} \cdot \left|\int_{t_1}^{t_2} g^p(s,\rho,k\rho)\, ds\right|^{\frac{1}{p}}$$
$$\le \left(\left|\int_0^{t_1} |(Y(t_2,s) - Y(t_1,s))\varphi(s)|^{p'}\right| + \left|\int_{t_1}^{t_2} |Y(t_2,s)\varphi(s)|^{p'}\, ds\right|^{\frac{1}{p'}}\right)$$
$$\times \left(\int_0^{+\infty} g^p(s,c,c)\, ds\right)^{\frac{1}{p}}, \qquad c = \max(\rho, k\rho).$$

This inequality implies the continuity of $z(t)$ at t_1.

Now we shall show that TM is an upper semicontinuous operator. To show this it suffices to prove that the operator TM is upper semicompact. Let $x_n \to x$ (in the topology induced by the family of seminorm (2.1)), $x_n, x \in B_{\psi,\rho}(J)$ and

$z_n \in TM\ x_n$. We have to show that there is a subsequence $\{z_{1n}\}$ of $\{z_n\}$ which converges to some $z \in TM\ x$ (in the topology of $B(J)$) as $n \to +\infty$.

Let

$$z_i = \int_0^t Y(t,s)\, y_i(s)\, ds, \quad y_i \in M(x_i), \quad i = 1, 2, \dots .$$

Since $M(x)$ is weakly upper semicompact, there is a subsequence $\{y_{1i}\}$ of the sequence $\{y_i\}$ which converges weakly to some $y \in M(x)$, i.e.

$$z_{1i}(t) := \int_0^t Y(t,s) y_{1i}(s)\, ds \to \int_0^t Y(t,s) y(s)\, ds =: z(t) \in TM\ x$$

on $[0, +\infty)$ as $i \to +\infty$.

Further since $M(x)$ is bounded, there is a constant $K(\rho)$ such that $|y|_{p,\varphi} \le K(\rho)$ for every $y \in B_{\psi,\rho}(J)$. Using this fact and the inequality (3.5) we get

$$|\psi^{-1}(t) z_{1i}(t)| \le m_2^{\frac{1}{\alpha}} \cdot m_1^{\frac{\alpha}{\beta}} \cdot K^{\frac{p}{\gamma}}(\rho).$$

Thus the functions z_{1i}, $i = 1, 2, \dots$ are uniformly ψ-bounded. By virtue of

$$\begin{aligned}
&|\psi^{-1}(t_2) z_{1i}(t_2) - \psi^{-1}(t_1) z_{1i}(t_1)| \le \\
&\le \int_0^{t_1} |\psi^{-1}(t_2) Y(t_2,s) - \psi^{-1}(t_1) Y(t_1,s)| \varphi(s) \cdot |\varphi^{-1}(s) y_{1i}(s)|\, ds \\
&\quad + \int_{t_1}^{t_2} |\psi^{-1}(t_2) Y(t_2,s) \varphi(s)| \cdot |\varphi^{-1}(s) y_{1i}(s)|\, ds \\
&\le \left(\int_0^{t_1} |(\psi^{-1}(t_2) Y(t_2,s) - \psi^{-1}(t_1) Y(t_1,s) \varphi(s)|^{p'} ds \right)^{\frac{1}{p'}} \cdot \left(\int_0^{t_1} |\varphi^{-1}(s) y_{1i}(s)|^p ds \right)^{\frac{1}{p}} \\
&\quad + \left(\int_{t_1}^{t_2} |\psi^{-1}(t_2) Y(t_2,s) \varphi(s)|^{p'} ds \right) \cdot \left(\int_{t_1}^{t_2} |\varphi^{-1}(s) y_{1i}(s)|^p ds \right)^{\frac{1}{p}} \\
&\le \left(\int_0^{t_1} |(\psi^{-1}(t_2) Y(t_2,s) - \psi^{-1}(t_1) Y(t_1,s)) \varphi(s)|^{p'} ds \right)^{\frac{1}{p'}} \cdot \left(\int_0^{+\infty} g^p(s,c,c) ds \right)^{\frac{1}{p}} \\
&\quad + \left(\int_{t_1}^{t_2} |\psi^{-1}(t_2) Y(t_2,s) \varphi(s)|^{p'} ds \right)^{\frac{1}{p'}} \cdot \left(\int_0^{+\infty} g^p(s,c,c) ds \right)^{\frac{1}{p}},
\end{aligned}$$

where $0 \le t_1 < t_2$, $c = \max(\rho, k\rho)$ and $\frac{1}{p} + \frac{1}{p'} = 1$, $p > 1$ the functions $\psi^{-1}(t) z_{1i}(t)$, $i = 1, 2, \dots$ are equicontinuous on every compact subinterval of J.

By the Ascoli theorem as well as by Cantor's diagonalization process the sequence $\{z_{1i}\}$ contains a subsequence $\{z_{2i}\}$ such that $\{\psi^{-1}(t)z_{2i}(t)\}$ is uniformly convergent on every compact subinterval of J. $\square$

Theorem 3.1 *Let the hypotheses of Lemma 3.2 hold and let the function $g(t,u,v)$ be locally integrable on J for every fixed $u,v \in R^n$. Suppose further*

$$g(t,c,c) \in L_p(J,m_3) \tag{3.6}$$

$$\int_0^t |\psi^{-1}(t)Y(t,s)\varphi(s)|^b g^p(s,c,c)ds \in L_{\frac{p}{q}}(J,m_4) \tag{3.7}$$

for any constant $c \geq 0$. Then the systems (a) and (b) are restricted (ψ,p)-integral equivalent (the sets of ψ-bounded solutions of (a) and of (b) are (ψ,p)-integral equivalent).

Proof Let $y(t)$ be a ψ-bounded solution of (a) on J. Then there is $\rho \geq m_1^{\frac{a}{\beta}} \cdot m_2^{\frac{1}{\alpha}} \cdot m_3^{\frac{1}{\gamma}}$ such that $y \in B_{\psi,\rho}(J)$. Define for $x \in B_{\psi,2\rho}(J)$ the operator

$$TM\ x := \{\, z : \ z(t) = y(t) + \int_0^t Y(t,s)f_x(s)\,ds \text{ and } f_x \in M(x)\}.$$

By Lemma 3.1 and Lemma 3.2 the operator TM maps $B_{\psi,2\rho}(J)$ into $cf(B_\psi(J))$ and is upper semicontinuous. Further, for each $z \in TM\ x$, $x \in B_{\psi,2\rho}(J)$ we have

$$\begin{aligned}
|\psi^{-1}(t)z(t)| &\leq \left(\int_0^t |\psi^{-1}(t)Y(t,s)\varphi(s)|^b |\varphi^{-1}(s)f_x(s)|^p ds\right)^{\frac{1}{\alpha}} \\
&\quad \times \left(\int_0^t |\psi^{-1}(t)Y(t,s)\varphi(s)|^a ds\right)^{\frac{1}{\beta}} \cdot \left(\int_0^t |\varphi^{-1}(s)f_x(s)|^p ds\right)^{\frac{1}{\gamma}} \\
&\leq \rho + \left(\int_0^t |\psi^{-1}(t)Y(t,s)\varphi(s)|^b g^p(s,c,c)ds\right)^{\frac{1}{\alpha}} \\
&\quad \times \left(\int_0^t \psi^{-1}(t)Y(t,s)\varphi(s)|^a ds\right)^{\frac{1}{\beta}} \cdot \left(\int_0^t g^p(s,c,c)ds\right)^{\frac{1}{\gamma}} \\
&\leq \rho + m_2^{\frac{1}{\alpha}} \cdot m_1^{\frac{a}{\beta}} \cdot m_3^{\frac{1}{\gamma}} \\
&\leq 2\rho, \qquad c = \max(\rho, k\rho)
\end{aligned}$$

i.e. TM maps $B_{\psi,2\rho}(J)$ into $cf(B_{\psi,2\rho}(J))$. Now we shall show that $\overline{TM(B_{\psi,2\rho}(J))}$ is a compact set. Since $TM(B_{\psi,2\rho}(J)) \subset B_{\psi,2\rho}(J)$, it follows that the functions of $TM(B_{\psi,2\rho}(J))$ are uniformly bounded. Further, for $z \in TM\ x$, $x \in B_{\psi,2\rho}(J)$ there is $f_x \in M(x)$ such that

$$z'(t) = A(t)z(t) + f_x(t) \quad \text{a.e. on } J\,.$$

Thus for $t_1 < t_2$, $t_1, t_2 \in J$ we have

$$\begin{aligned} |z(t_2) - z(t_1)| &\leq \int_{t_1}^{t_2} |A(s)||z(s)|\,ds + \int_{t_1}^{t_2} |f_x(s)|\,ds \\ &\leq 2\rho \int_{t_1}^{t_2} |A(s)|\psi(s)\,ds + \int_{t_1}^{t_2} \varphi(s)g(s,c,c)\,ds, \qquad c = \max(\rho, k\rho). \end{aligned}$$

Since $A(s)$ and $g(s,c,c)$ are locally integrable functions and $\psi(s), \phi(s)$ are continuous it follows that the functions in $TM(B_{\psi,2\rho}(J))$ are equicontinuous on every compact subinterval of J. Thus $\overline{TM(B_{\psi,2\rho}(J))}$ is compact (in the topology of $B(t_0)$). By Lemma 2.2 there is $x \in B_{\psi,2\rho}(J)$ such that $x \in TM\ x$. Clearly this fixed point $x(t)$ is a ψ-bounded solution of (b). It remains to prove that (d) holds. We have

$$\begin{aligned} |\psi^{-1}(t)(x(t) - y(t))|^p &\leq \left\{ \int_0^t |\psi^{-1}(t)Y(t,s)\varphi(s)|^{\frac{b}{q}} \cdot g^{\frac{p}{q}}(s,c,c) \right. \\ &\times \left. |\psi^{-1}(t)Y(t,s)\varphi(s)|^{a(\frac{1}{a} - \frac{1}{aq})} g^{p(\frac{1}{p} - \frac{1}{q})}(s,c,c)ds \right\}^p, \qquad c = \max(2\rho, 2k\rho). \end{aligned}$$

By Hölder's inequality, (3.1) and (3.5) we get following inequality

$$\begin{aligned} |\psi^{-1}(t)(x(t) - y(t))|^p &\leq \left(\int_0^t |\psi^{-1}(t)Y(t,s)\varphi(s)|^b g^p(s,c,c)ds \right) \\ &\times \left(\int_0^t |\psi^{-1}(t)Y(t,s)\varphi(s)|^a ds \right)^{\frac{p}{\beta}} \cdot \left(\int_0^t g^p(s,c,c)ds \right)^{\frac{p}{\gamma}} \\ &\leq m_1^{\frac{pa}{\beta}} \cdot m_3^{\frac{p}{\gamma}} \cdot \left(\int_0^t |\psi^{-1}(t)Y(t,s)\varphi(s)|^b g^p(s,c,c)ds \right)^{\frac{p}{q}}. \end{aligned}$$

by which (since (3.6)) (d) holds. Conversely, let $x(t)$ be a ψ-bounded solution of (b). Define

$$y(t) := x(t) - \int_0^t Y(t,s)f_x(s)\,ds\,,$$

where

$$f_x(t) := x'(t) - A(t)x(t) \in F(t, x(t), Sx(t)) \quad \text{a.e. on } J.$$

Now the proof will be proceeds in the similar way as that of the first part. $\square$

Lemma 3.3 *Assume that the hypotheses of Lemma 3.2 are satisfied except (3.3) and let instead (3.3) holds*

$$\sup_{t\in J}(\sup_{r\in J}\int_0^r |\psi^{-1}(t)Y(t,s)\varphi(s)|^d s) \le m_2''$$
$$\sup_{t\in J} g^p(t,c,c) \le m_5, \quad \textit{for any } c \ge 0. \tag{3.3'}$$

Then the assertion of Lemma 3.2 holds.

Proof The proof of Lemma 3.3 proceeds analogically as that of Lemma 3.2.

Theorem 3.2 *Let the hypotheses of Lemma 3.3 hold. Moreover, let*

$$\lim_{t\to+\infty} g^p(t,c,c) = 0, \quad \textit{for any } c \ge 0$$

and

$$\lim_{t\to+\infty}\int_0^r |\psi^{-1}(t)Y(t,s)\varphi(s)|^b ds = 0,$$

for any $r > 0$ and $0 \le s \le t$. Then the systems (a) and (b) are restricted ψ-asymptotically equivalent (the sets of ψ -bounded solutions of (a) and of (b) are ψ-asymptotically equivalent.

Proof The proof of Theorem 3.2 is essentially the same as that of Theorem 3.1.

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