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Estimation in Multiepoch Regression Models with Different Structures for Studying Recent Crustal Movements

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Abstract

The study of the recent crustal movements on the basis of the replicated terrestrial geodetic measurements from the statistical viewpoint (i.e., from the viewpoint of processing measured data) requires to solve the following two main problems:

(i) to determine the optimum estimators of the parameters of the first and the second order of the epoch models describing the investigated process and

(ii) to test the hypothesis whether the change of the basic parameters between two epochs is significant or not.

From the mathematical viewpoint the solution of the problem depends mainly on the structure of the models considered.

The aim of the contribution is to systematize structures suitable for studying the recent crustal movements and to give optimum estimators and test algorithms.

Key words: Models with variance components, multiepoch model, growth-curve model, multivariate model, recent crustal movement, LBLUE, LMVQUIE, MINQUE.

1991 Mathematics Subject Classification: 62J05

1 Introduction

An investigation of recent crustal movements (RCM) is a complex problem; its solution requires to respect several different aspects. Up to date knowledge on the physical properties of the crust of the earth resulting into an acceptable model of the process which describes the reality, a choice of measuring techniques for the investigation of the RMC which involves a design of the measurement process (number of epochs, number of replications, etc.), estimation procedures for parameters characterizing the RCM and for parameters characterizing the accuracy of the measurement and test procedures for verifying hypothesis on parameters characterizing the RCM have to be mentioned before anything else.

To prepare an experiment respecting all necessary aspects is a task for a group of experts and it is impossible to describe it in this contribution. The aim of the contribution is more modest; the aspects of mathematical statistics, namely on problems of the estimation and the testing are emphasized here.

It will be shown that solutions of the last mentioned problems depend essentially on structures of models used in the course of an evaluating measurement results.

Two fundamental types of models can be characterized as follows:

(i) Repeated measurements are realized in a separate network especially constructed for this purpose. It consists of a group of supporting points whose position is assumed to be stable (this assumption—hypothesis—is verified during the measurement) and a group of points whose movements related to the position of the stable points are investigated (the coordinates of the group of the stable points are a priori unknown). As far as the processing the measured results is concerned this means that in the framework of each epoch and after finishing each epoch both the coordinates of the supporting points and the coordinates of the investigated points are to be determined. The former serve for verifying the above mentioned hypothesis on the stableness of the group of supporting points.

(ii) The network for studying the dynamism of a locality is joint to the stable points of a geodetic network (they represent the stable supporting points of the preceding type of the network, in contradistinction to it, their coordinates are a priori known). In comparison with the preceding procedure of data processing either the coordinates of the group of the points studied from the viewpoint of the dynamism or directly the coefficients of the functional development modelling the time evolution of the changes are being determined (the growth curve model).

Both of these fundamental types may be of special structures, e.g., [1], [14], [2], [3], [4], [5] and others.

The aim of the paper is to give explicite formulas of estimators in basic multiepoch structures linked up with the replicated measurements of recent crustal movements; in addition to it to give a short survey of these structures.

A comparison of usually used estimators and optimum estimators derived in the contribution is given in Example 11.

2 Notations, definitions and basic assertions

In the following Y denotes an observation vector, i.e., a random vector whose realization y is a vector of results of measurement. If necessary, the dimension is indicated by the lower index in the square brackets. The class of the probability distributions connected with the vector Y is $\mathcal{F} = \{F(.,\beta,\vartheta) : \beta \in \mathcal{V},\ \vartheta \in \underline{\vartheta}\}$, where $F(u,\beta,\vartheta),\ u \in \mathcal{R}^n$, is a distribution function parametrized by $\beta \in \mathcal{V}$ and $\vartheta \in \underline{\vartheta}$; β is a k-dimensional parameter of the first order, $\mathcal{V}$ is a linear manifold of the k-dimensional vector space $\mathcal{R}^k$ (it characterizes the set of all values of the vector β which can occur in the experiment), ϑ is an unknown p-dimensional vector of the second order parameter (variance components) and $\underline{\vartheta}$ is an open (in the Euclidean topology of the space $\mathcal{R}^p$) set. The class $\mathcal{F}$ is supposed to have the following two properties

$$\forall\{\beta \in \mathcal{V}, \vartheta \in \underline{\vartheta}\} \quad \int_{\mathcal{R}^n} u\, dF(u,\beta,\vartheta) = f(\beta), \tag{1}$$

(the independence of the parameter ϑ),

$$\forall\{\beta \in \mathcal{V}, \vartheta \in \underline{\vartheta}\} \quad \int_{\mathcal{R}^n} [u - f(\beta)][u - f(\beta)]'\, dF(u,\beta,\vartheta) = \Sigma(\vartheta) \tag{2}$$

(the independence of the parameter β).

Frequently, either $f(\beta) = X\beta$ or $f(\beta_0) + X(\beta - \beta_0)$ (a linearized form of the function $f(\beta)$), where X is a known $n \times k$ matrix independent of the parameter β and β_0 is a known sufficiently good approximation of the actual value of the parameter β.

In the following this model will be denoted as

$$\Big(Y, X\beta, \Sigma(\vartheta)\Big), \quad \beta \in \mathcal{V}, \quad \vartheta \in \underline{\vartheta}. \tag{3}$$

The covariance matrix $\Sigma(\vartheta)$ is assumed to have a linear structure

$$\Sigma(\vartheta) = \sum_{i=1}^{p} \vartheta_i V_i \quad \text{or} \quad V_0 + \sum_{i=1}^{p} (\vartheta_i - \vartheta_{i,0}) V_i \tag{4}$$

(a linearized form), where $V_1, \ldots, V_p$ are known symmetric matrices and $\vartheta_0 = (\vartheta_{0,1}, \ldots, \vartheta_{0,p})'$ is a known sufficiently good approximation of the actual value of the parameter ϑ.

The assumptions (1) till (4) are realistic and in practice enable us to solve relatively difficult problems induced by the non-linearity of the vector/matrix function $f(.)/\Sigma(.)$ if the linearization is possible, i.e., if the vectors β_0/ϑ_0 are known.

Within the framework of the basic structure (3) the ϑ_0-locally best linear estimator of un unbiasedly linear function $h(\beta)$ of the first order parameters and the ϑ_0-minimum norm quadratic unbiased invariant estimator of un unbiasedly and invariantly estimable function $g(\vartheta)$ of the second order parameters will be considered.

Definition 1 The ϑ_0-LBLUE (locally best linear unbiased estimator) of an unbiasedly linear function $h(\beta) = h_0 + h'\beta$, $\beta \in \mathcal{V}$, is a statistic $L'Y + l$ with the properties

(i) (unbiasedness) $\quad \forall\{\beta \in \mathcal{V}\} \quad E(L'Y + l|\beta) = h_0 + h'\beta$

and

(ii) $\quad \forall\{L_1, l_1 \text{ satisfying (i)}\} \quad \mathrm{Var}(L'Y + l|\vartheta_0) \leq \mathrm{Var}(L_1'Y + l_1|\vartheta_0)$.

Definition 2 The ϑ_0-MINQUE (minimum norm quadratic unbiased invariant estimator) of an unbiasedly and invariantly estimable function $g(\vartheta) = g_0 + g'\vartheta$, $\vartheta \in \underline{\vartheta}$, is a statistic $(Y - X\beta_0)'T(Y - X\beta_0)$, where β_0 is any fixed element from $\mathcal{V}$, with the properties

(i) (unbiasedness)

$$\forall\{\beta \in \mathcal{V}, \vartheta \in \underline{\vartheta}\} \quad E[(Y - X\beta_0)'T(Y - X\beta_0)|\beta, \vartheta] = g_0 + g'\vartheta,$$

(ii) (invariance)

$$\forall\{\kappa \in \mathcal{V}\} \quad (Y - X\kappa - X\beta_0)'T(Y - X\kappa - X\beta_0) = (Y - X\beta_0)'T(Y - X\beta_0),$$

(iii) $\quad \forall\{T_1 \; n \times n \text{ symmetric matrix satisfying (i) and (ii)}\}$

$$\mathrm{Tr}[T\Sigma(\vartheta_0)T\Sigma(\vartheta_0)] \leq \mathrm{Tr}[T_1\Sigma(\vartheta_0)T_1\Sigma(\vartheta_0)].$$

Remark 3 If Y is supposed to be normally distributed ($Y \sim N_n[X\beta, \Sigma(\vartheta)]$), then

$$\mathrm{Var}[(Y - X\beta_0)'T(Y - X\beta_0)|\vartheta] = 2\,\mathrm{Tr}[T\Sigma(\vartheta)T\Sigma(\vartheta)],$$

where $(Y - X\beta_0)'T(Y - X\beta_0)$ satisfies (i) and (ii) from Definition 2. It means that ϑ_0-MINQUE in this case is a ϑ_0-locally minimum variance quadratic unbiased estimator. If $\mathcal{V} = \mathcal{R}^k$, then β_0 can be chosen as the zero vector.

Remark 4 Let the linear manifold $\mathcal{V}$ be defined by the relationship

$$\mathcal{V} = \{u : u \in \mathcal{R}^k, \; b + Bu = 0\};$$

the assumptions on the unbiased estimability of the functions $h(.)$ and $g(.)$ from Definitions 1 and 2, respectively, are equivalent to the requirements $h \in \mathcal{M}(X', B')$ and $g \in \mathcal{M}(C^{(I)}_{XK_B})$, where

$$\{C^{(I)}_{XK_B}\}_{i,j} = \mathrm{Tr}(M_{XK_B}V_iM_{XK_B}V_j), \quad i,j = 1,\ldots,p.$$

Here

$$\mathcal{M}(A_{[n,k]}) = \{Au : u \in \mathcal{R}^k\}, \qquad M_{XK_B} = I_{[n,n]} - XK_B(K_B'X'XK_B)^-K_B'X'$$

and K_B is a matrix with the property $\mathcal{M}(B', K_B) = \mathcal{R}^k$, $BK_B = 0$. The symbol A^- denotes a g-inverse of the matrix A (in more detail see [12]).

The more complicated case of the basic model (3) with

$$\beta = (\beta'_{[k_1]}, \beta'_{[k_2]})' \in \mathcal{V},$$
$$\mathcal{V} = \left\{ \begin{pmatrix} u_{[k_1]} \\ u_{[k_2]} \end{pmatrix} : u_{[k_1]} \in \mathcal{R}^{k_1}, u_{[k_2]} \in \mathcal{R}^{k_2}, b + Bu_{[k_1]} + Fu_{[k_2]} = 0 \right\}$$

and $E(Y|\beta) = X\beta_{[k_1]}$, is investigated in [4].

Remark 5 Other kinds of estimators are not considered here because of

(i) the estimators defined in Definitions 1 and 2, respectively, are commonly used since they are simple and reasonable and

(ii) the limited extend of the contribution.

Assertion 6 *Let, within the model* $\left(Y_{[n,1]}, X_{[n,k]}\beta_{[k,1]} \sum_{i=1}^{p} \vartheta_i V_i\right)$, $\beta \in \mathcal{R}^k$, $\vartheta \in \underline{\vartheta} \subset \mathcal{R}^p$, *in which the observation vector* Y *is normally distributed, the rank* $r(X)$ *of the matrix* X *be* $r(X) = k < n$ *and* $\Sigma_0 = \Sigma(\vartheta_0) = \sum_{i=1}^{p} \vartheta_{0,i} V_i$ *be positively definite. Then*

(i) the ϑ_0*-LBLUE of* β *is*

$$\hat{\beta}(Y) = (X'\Sigma_0^{-1}X)^{-1}X'\Sigma_0^{-1}Y$$

and its covariance matrix at Σ_0 *is* $\mathrm{Var}[\hat{\beta}(Y)|\Sigma_0] = (X'\Sigma_0^{-1}X)^{-1}$.

Let, further, the matrix $S_{(M_X\Sigma_0 M_X)^+}$, *whose* i,j*th element is*

$$\{S_{(M_X\Sigma_0 M_X)^+}\}_{i,j} = \mathrm{Tr}[(M_X\Sigma_0 M_X)^+ V_i (M_X\Sigma_0 M_X)^+ V_j],$$

$i,j = 1,\dots,p$, *be regular (+ means the Moore–Penrose inverse of a matrix). Then*

(ii) the ϑ_0*-MINQUE of* ϑ *is*

$$\hat{\vartheta}(Y) = S^{-1}_{(M_X\Sigma_0 M_X)^+}\hat{\gamma}(Y),$$

where $\hat{\gamma}(Y) = \left(\hat{\gamma}_1(Y), \dots, \hat{\gamma}_p(Y)\right)'$,

$$\hat{\gamma}_i(Y) = Y'(M_X\Sigma_0 M_X)^+ V_i (M_X\Sigma_0 M_X)^+ Y, \quad i = 1,\dots,p,$$

and its covariance matrix at Σ_0 *is*

$$\mathrm{Var}\left[\hat{\vartheta}(Y)|\Sigma_0\right] = 2S^{-1}_{(M_X\Sigma_0 M_X)^+}.$$

Proof *Cf. [13], p. 93.* □

3 Estimation in m-epochs model with the same design in each epoch and with the stable and non-stable points

Let the recent crustal movements be investigated in m epochs. Each epoch is of the same design. (This simple version of multiepoch models is considered from didactical reason. The methods demonstrated in the following can be used for models with more complicated structures). The network used in this experiment involves a group of points which are stable and are characterized by a vector parameter β_1 and a group of non-stable points (in the investigated area) which are characterized by a vector parameter $\beta_2^{(i)}$ in the ith epoch.

The model of the described experiment is

$$\left[Y_{[nm,1]}, (1_{[m,1]} \otimes X_{1[n,k_1]}, I_{[m,m]} \otimes X_{2[n,k_2]}) \begin{pmatrix} \beta_1 \\ \beta_2^{(1)} \\ \vdots \\ \beta_2^{(m)} \end{pmatrix}, \sum_{i=1}^{p} \vartheta_i I_{[m,m]} \otimes V_i, \right], \tag{5}$$

where $1_{[m,1]} = (1, \ldots, 1)'$, I is an identity matrix, $Y = (Y_1', \ldots, Y_m')'$, $Y_1, \ldots, Y_m$ are stochastically independent n-dimensional random vectors, X_1, X_2 are given design matrices such that $r(X_1) = k_1$, $r(X_2) = k_2$, $k_1 + k_2 < n$ and $V_1, \ldots, V_p$ are given symmetric matrices. The characteristics of the accuracy of the measurement (the variance components) $\vartheta = (\vartheta_1, \ldots, \vartheta_p)' \in \underline{\vartheta}$ (an open set in the Euclidean topology) $\subset \mathcal{R}^p$, are supposed to be unknown. The matrix $\sum_{i=1}^{p} \vartheta_i V_i$ is supposed to be positively definite.

Assertion 7 *The ϑ_0-LBLUE of the vector $(\beta_1', \beta_2^{(1)'}, \ldots, \beta_2^{(m)'})' = (\beta_1', \beta_2^{(\cdot)'})'$ is*

$$\begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2^{(\cdot)} \end{pmatrix} = \begin{pmatrix} [X_1'(M_{X_2}\Sigma_0 M_{X_2})^+ X_1]^{-1} X_1'(M_{X_2}\Sigma_0 M_{X_2})^+ \overline{Y} \\ a + b \end{pmatrix},$$

where

$$\begin{aligned} a &= I \otimes (X_2'\Sigma_0^{-1}X_2)^{-1}X_2'\Sigma_0^{-1} \begin{pmatrix} Y_1 - \overline{Y} \\ \vdots \\ Y_m - \overline{Y} \end{pmatrix}, \\ b &= 1 \otimes [X_2'(M_{X_1}\Sigma_0 M_{X_1})^+ X_2]^{-1} X_2'(M_{X_1}\Sigma_0 M_{X_1})^+ \overline{Y}, \\ \hat{\beta}_2^{(\cdot)} &= (\hat{\beta}_2^{(1)'}, \ldots, \hat{\beta}_2^{(m)'})', \\ M_{X_2} &= I_{[n,n]} - X_2(X_2'X_2)^{-1}X_2', \\ \Sigma_0 &= \sum_{i=1}^{p} \vartheta_{0,i} V_i, \ \vartheta_0 = (\vartheta_{0,1}, \ldots, \vartheta_{0,p})', \\ \overline{Y} &= (1/m) \sum_{i=1}^{m} Y_i \end{aligned}$$

and

$$\operatorname{Var}\left[\begin{pmatrix}\hat{\beta}_1\\ \hat{\beta}_2^{(\cdot)}\end{pmatrix}\Big|\Sigma_0\right]=\begin{pmatrix}\boxed{11}, & \boxed{12}\\ \boxed{21}, & \boxed{22}\end{pmatrix};$$

here

$$\begin{aligned}
\boxed{11} &= (1/m)[X_1'(M_{X_2}\Sigma_0 M_{X_2})^{+}X_1]^{-1},\\
\boxed{12} &= -\frac{1'}{m}\otimes(X_1'\Sigma_0^{-1}X_1)^{-1}X_1'\Sigma_0^{-1}X_2[X_2'(M_{X_1}\Sigma_0 M_{X_1})^{+}X_2]^{-1}\\
&= -\frac{1'}{m}\otimes[X_1'(M_{X_2}\Sigma_0 M_{X_2})^{+}X_1]^{-1}X_1'\Sigma_0^{-1}X_2(X_2'\Sigma_0^{-1}X_2)^{-1},\\
\boxed{21} &= \boxed{12}\,'\\
\boxed{22} &= M_m\otimes(X_2'\Sigma_0^{-1}X_2)^{-1}+P_m\otimes[X_2'(M_{X_1}\Sigma_0 M_{X_1})^{+}X_2]^{-1},
\end{aligned}$$

where $P_m = 1_{[m,1]}1'_{[m,1]}/m,\ M_m = I_{[m,m]} - P_m$.

Proof With respect to Assertion 6,

$$\begin{pmatrix}\hat{\beta}_1\\ \hat{\beta}_2^{(\cdot)}\end{pmatrix}=\left[(1\otimes X_1, I\otimes X_2)'(I\otimes\Sigma_0)^{-1}(1\otimes X_1, I\otimes X_2)\right]^{-1}\times$$

$$\times(1\otimes X_1, I\otimes X_2)'(I\otimes\Sigma_0^{-1})\begin{pmatrix}Y_1\\ \vdots\\ Y_m\end{pmatrix}=$$

$$=\begin{pmatrix}m\otimes(X_1'\Sigma_0^{-1}X_1), & 1'\otimes(X_1'\Sigma_0^{-1}X_2)\\ 1\otimes(X_2'\Sigma_0^{-1}X_1), & I\otimes(X_2'\Sigma_0^{-1}X_2)\end{pmatrix}^{-1}\begin{pmatrix}1'\otimes(X_1'\Sigma_0^{-1})\\ I\otimes(X_2'\Sigma_0^{-1})\end{pmatrix}\begin{pmatrix}Y_1\\ \vdots\\ Y_m\end{pmatrix}.$$

It can be easily verified that

$$\begin{pmatrix}m\otimes(X_1'\Sigma_0^{-1}X_1), & 1'\otimes(X_1'\Sigma_0^{-1}X_2)\\ 1\otimes(X_2'\Sigma_0^{-1}X_1), & I\otimes(X_2'\Sigma_0^{-1}X_2)\end{pmatrix}^{-1}=\begin{pmatrix}\boxed{11}, & \boxed{12}\\ \boxed{21}, & \boxed{22}\end{pmatrix}.$$

Taking into account the formula

$$(M_m\otimes A_{[t,n]}+P_m\otimes B_{[t,n]})\begin{pmatrix}Y_1\\ \vdots\\ Y_m\end{pmatrix}=(I\otimes A)\begin{pmatrix}Y_1-\overline{Y}\\ \vdots\\ Y_m-\overline{Y}\end{pmatrix}+1\otimes(B\overline{Y})$$

(which can be easily verified) we can finish the proof in a straightforward way. □

Remark 8 Assertion 7 implies the following conclusions:

(i) The ϑ_0–LBLUE $\hat{\beta}_1(Y_1,\dots,Y_j)$ of the parameter β_1 (the coordinates of the stable points) based on the results of measurement after j epochs equals the arithmetic mean of the corresponding estimators within each separate epoch

$$\hat{\beta}_1(Y_1,\dots,Y_j)=(1/j)\sum_{i=1}^{j}\hat{\beta}_1(Y_i),$$

where

$$\hat{\beta}_1(Y_i)=[X_1'(M_{X_2}\Sigma_0M_{X_2})^+X_1]^{-1}X_1'(M_{X_2}\Sigma_0M_{X_2})^+Y_i,$$

$i=1,\dots,j$, is the ϑ_0–LBLUE of β_1 within the ith epoch, i.e., within the model

$$\left[Y_i,\ (X_1,X_2)\begin{pmatrix}\beta_1\\ \beta_2^{(\cdot)}\end{pmatrix},\Sigma_0\right].$$

Furthermore, if

$$Y\sim N_{[mn]}\left[(1\otimes X_1,I\otimes X_2)\begin{pmatrix}\beta_1\\ \beta_2^{(\cdot)}\end{pmatrix},I\otimes\Sigma_0\right]$$

and the parameter β_1 can be assumed to be stable

$$\hat{\beta}_1(Y_j)-\hat{\beta}_1(Y_1,\dots,Y_{j-1})\sim N_{k_1}\Big(0,[j/(j-1)][X_1'(M_{X_2}\Sigma_0M_{X_2})^+X_1]^{-1}\Big).$$

Thus the stability of these points in each epoch can be tested. The test of the null hypothesis $H_0: E[\hat{\beta}_1(Y_j)]=E[\hat{\beta}_1(Y_1,\dots,Y_{j-1})]$ is based on the statistic

$$\begin{aligned}&\frac{j-1}{j}\Big[\hat{\beta}_1(Y_j)-\hat{\beta}_1(Y_1,\dots,Y_{j-1})\Big]'\times\\ &\qquad\times\Big[X_1'(M_{X_2}\Sigma_0M_{X_2})^+X_1\Big]\Big(\hat{\beta}_1(Y_j)-\hat{\beta}_1(Y_1,\dots,Y_{j-1})\Big)\sim\chi^2_{k_1}(\delta);\end{aligned}$$

here $\chi^2_{k_1}(\delta)$ denotes the random variable with non-central chi-square probability distribution and k_1 degrees of freedom.

If the points are stable, then $\delta=0$; if not, then the parameter of noncentrality is

$$\begin{aligned}\delta=\frac{j-1}{j}&\left(\beta_1^{(j)}-\frac{1}{j-1}\sum_{i=1}^{j-1}\beta_1^{(i)}\right)'\times\\ \times\Big[X_1'\,(M_{X_2}\Sigma_0M_{X_2})^+\,X_1\Big]&\left(\beta_1^{(j)}-\frac{1}{j-1}\sum_{i=1}^{j-1}\beta_1^{(i)}\right);\end{aligned}$$

$\beta_1^{(i)}$, $i=1,\dots,j-1$, is the coordinate vector of an actual position of the stable points in the ith epoch. Here Σ_0 is supposed to be an actual covariance matrix.

(ii) Any linear hypothesis on the m-tuple of the non-stable points $\beta_2^{(1)}, \ldots, \beta_2^{(m)}$ (i.e., the hypothesis concerning the recent crustal movements)

$$H_0: \; H_{[q,k_2m]} \begin{pmatrix} \beta_2^{(1)} \\ \vdots \\ \beta_2^{(m)} \end{pmatrix} + h_{[q,1]} = 0_{[q,1]}, \; r(H) = q,$$

can be tested using the statistic

$$\left[h + H \begin{pmatrix} \hat{\beta}_2^{(1)}(Y_1, \ldots, Y_m) \\ \vdots \\ \hat{\beta}_2^{(m)}(Y_1, \ldots, Y_m) \end{pmatrix} \right]' (H \boxed{22} H')^{-1} \left[h + H \begin{pmatrix} \hat{\beta}_2^{(1)}(Y_1, \ldots, Y_m) \\ \vdots \\ \hat{\beta}_2^{(m)}(Y_1, \ldots, Y_m) \end{pmatrix} \right]$$

which (under the condition that the null hypothesis H_0 is true) possesses the central chi-square distribution with q degrees of freedom. Here Σ_0 in the matrix $\boxed{22}$ is supposed to be an actual covariance matrix.

(iii) As

$$\hat{\beta}_2^{(j)}(Y_j) = [X_2'(M_{X_1}\Sigma_0 M_{X_1})^+ X_2]^{-1} X_2'(M_{X_1}\Sigma_0 M_{X_1})^+ Y_j$$

and simultaneously

$$\begin{aligned} \hat{\beta}_2^{(j)}(Y_1, \ldots, Y_m) &= [X_2'(M_{X_1}\Sigma_0 M_{X_1})^+ X_2]^{-1} X_2'(M_{X_1}\Sigma_0 M_{X_1})^+ \overline{Y} \\ &\quad + (X_2'\Sigma_0^{-1}X_2)^{-1} X_2'\Sigma_0^{-1}(Y_j - \overline{Y}), \end{aligned}$$

there holds

$$\begin{aligned} \hat{\beta}_2^{(j)}(Y_1, \ldots, Y_m) &= \hat{\beta}_2^{(j)}(Y_j) + \{(X_2'\Sigma_0^{-1}X_2)^{-1} X_2'\Sigma_0^{-1} - \\ &\quad - [X_2'(M_{X_1}\Sigma_0 M_{X_1})^+ X_2]^{-1} X_2'(M_{X_1}\Sigma_0 M_{X_1})^+\} \times \\ &\quad \times (Y_j - \overline{Y}) = \hat{\beta}_2^{(j)}(Y_j) + \text{a correction term}. \end{aligned}$$

In order to analyse the correction term, let us consider the stochastical model

$$[Y_j - \overline{Y}, \; X_2(\beta_2^{(j)} - (1/m)\sum_{i=1}^{m} \beta_2^{(i)}), \; (1 - 1/m)\Sigma(\vartheta_0)]$$

within which the ϑ_0–LBLUE of the unknown parameter $\beta_2^{(j)} - \frac{1}{m}\sum_{i=1}^{m} \beta_2^{(i)}$ is

$$\widehat{\left(\beta_2^{(j)} - \frac{1}{m}\sum_{i=1}^{m} \beta_2^{(i)}\right)}(Y_j - \overline{Y}) = (X_2'\Sigma_0^{-1}X_2)^{-1} X_2'\Sigma_0^{-1}(Y_j - \overline{Y});$$

furthermore

$$\frac{1}{m}\sum_{i=1}^{m} \hat{\beta}_2^{(i)}(Y_i) = [X_2'(M_{X_1}\Sigma_0 M_{X_1})^+ X_2]^{-1} X_2'(M_{X_1}\Sigma_0 M_{X_1})^+ \overline{Y},$$

therefore the correction term is of the form

$$\left(\widehat{\beta_2^{(j)} - \frac{1}{m}\sum_{j=1}^{m}\beta_2^{(i)}}\right)(Y_j - \overline{Y}) - \left[\hat{\beta}_2^{(j)}(Y_j) - \frac{1}{m}\sum_{j-1}^{m}\hat{\beta}_2^{(i)}(Y_i)\right].$$

(iv) If the null-hypothesis from (ii) reads

$$H_0 : \left(I_{[m,m]} \otimes H_{2[q,k_2]}\right)\begin{pmatrix}\beta_2^{(1)}\\ \vdots \\ \beta_2^{(m)}\end{pmatrix} + 1_{[m,1]} \otimes h_{2[q,1]} = 0_{[mq,1]},$$

where $r(h_2) = q$, the test statistic is

$$\mathrm{Tr}\left\{H_2'[H_2(X_2'\Sigma_0X_2)^{-1}H_2']^{-1}H_2A\right\} +$$

$$+m(h_2 + H_2\overline{\overline{\widehat{\beta_2^{(\cdot)}}}})'\left\{H_2[X_2'(M_{X_1}\Sigma_0M_{X_1})^{-1}X_2]^{-1}H_2\right\}^{-1}(h_2 + H_2'\overline{\overline{\widehat{\beta_2^{(\cdot)}}}}) \sim \chi^2_{mq},$$

where

$$\widehat{\widehat{\beta}}_2^{(j)} = \widehat{\beta}_2^{(j)}(Y_1,\dots,Y_m),\quad \overline{\overline{\widehat{\beta_2^{(\cdot)}}}} = \frac{1}{m}\sum_{j=1}^{m}\widehat{\beta}_2^{(j)}(Y_1,\dots,Y_m)$$

and $A = \sum_{j=1}^{m}(\widehat{\widehat{\beta}}_2^{(j)} - \overline{\overline{\widehat{\beta_2^{(\cdot)}}}})(\widehat{\widehat{\beta}}_2^{(j)} - \overline{\overline{\widehat{\beta_2^{(\cdot)}}}})'$.

Remark 8 demonstrates that all inferences concerning the recent crustal movements can be drawn from calculations performed within the framework of the separate epochs only, where the matrices of the types $k_1 \times k_1$ and $k_2 \times k_2$ instead of the "large matrices" $(mk_1) \times (mk_1)$ and $(mk_2) \times (mk_2)$, respectively, are used. This has at least two advantages: within each epoch it is possible to compare easily a result from this separate epoch with results from all the preceding epochs, which is important from the point of view of the statistical analysis of measurement results and furthermore it creates a basis for equivalent algorithms of estimation. The last fact enables us to check the numerical stability of the calculation and to verify the correctness of numerical results.

In addition to this an analysis of estimation procedures represents a basis for understanding the influence of the results of the ith epoch on the estimation based on the results of the jth $(i \neq j)$ epoch. The epistemological aspect of this analysis leads to an inside of the m-epochs model which is not a simple sum of single epochs but a qualitative new entity.

In what follows we shall deal with variances of the estimators applied including the estimators of the second order parameters.

Assertion 9 *It holds*

$$\mathrm{Var}\left[\hat{\beta}_2^{(j)}(Y_j)|\Sigma_0\right] = \left[X_2'(M_{X_1}\Sigma_0M_{X_1})^+X_2\right]^{-1} =$$

$$= (X_2'\Sigma_0^{-1}X_2)^{-1} + (X_2'\Sigma_0^{-1}X_2)^{-1}X_2'\Sigma_0^{-1}X_1 \times$$
$$\times [X_1'\Sigma_0^{-1}X_1 - X_1'\Sigma_0^{-1}X_2(X_2'\Sigma_0^{-1}X_2)^{-1}X_2'\Sigma_0^{-1}X_1]^{-1} \times$$
$$\times X_1'\Sigma_0^{-1}X_2(X_2'\Sigma_0^{-1}X_2)^{-1}$$

and

$$\mathrm{Var}\left[\hat\beta_2^{(j)}(Y_1,\dots,Y_m)|\Sigma_0\right] =$$
$$= (1-\frac{1}{m})(X_2'\Sigma_0^{-1}X_2)^{-1} + \frac{1}{m}\left[X_2'(M_{X_1}\Sigma_0 M_{X_1})^+X_2\right]^{-1} =$$
$$= (X_2'\Sigma_0^{-1}X_2)^{-1} + \frac{1}{m}(X_2'\Sigma_0^{-1}X_2)^{-1}X_2'\Sigma_0^{-1}X_1 \times$$
$$\times \left[X_1'\Sigma_0^{-1}X_1 - X_1'\Sigma_0^{-1}X_2(X_2'\Sigma_0^{-1}X_2)^{-1}X_2'\Sigma_0^{-1}X_1\right]^{-1} \times$$
$$\times X_1'\Sigma_0^{-1}X_2(X_2'\Sigma_0^{-1}X_2)^{-1}.$$

Proof If the relationships

$$(M_{X_1}\Sigma_0 M_{X_1})^+ = \Sigma_0^{-1} - \Sigma_0^{-1}X_1(X_1'\Sigma_0^{-1}X_1)^{-1}X_1'\Sigma_0^{-1}$$

and

$$[X_2'\Sigma_0^{-1}X_2 - X_2'\Sigma_0^{-1}X_1(X_1'\Sigma_0^{-1}X_1)^{-1}X_1'\Sigma_0^{-1}X_2]^{-1} =$$
$$= (X_2'\Sigma_0^{-1}X_2)^{-1} + (X_2'\Sigma_0^{-1}X_2)^{-1}X_2'\Sigma_0^{-1}X_1 \times$$
$$\times \left[X_1'\Sigma_0^{-1}X_1 - X_1'\Sigma_0^{-1}X_2(X_2'\Sigma_0^{-1}X_2)^{-1}X_2'\Sigma_0^{-1}X_1\right]^{-1} \times$$
$$\times X_1'\Sigma_0^{-1}X_2(X_2'\Sigma_0^{-1}X_2)^{-1}$$

are taken into account, the first part of the assertion is proved.

As

$$\begin{pmatrix} \hat\beta_2^{(1)}(Y_1,\dots,Y_m) \\ \vdots \\ \hat\beta_2^{(m)}(Y_1,\dots,Y_m) \end{pmatrix} = \Big\{M_m \otimes (X_2'\Sigma_0^{-1}X_2)^{-1}X_2'\Sigma_0^{-1} +$$
$$+ \; P_m \otimes [X_2'(M_{X_1}\Sigma_0 M_{X_1})^+X_2]^{-1}X_2'(M_{X_1}\Sigma_0 M_{X_1})^+\Big\} \begin{pmatrix} Y_1 \\ \vdots \\ Y_m \end{pmatrix},$$

we can write

$$\mathrm{Var}\left[\hat\beta_2^{(j)}(Y_1,\dots,Y_m)|\Sigma_0\right] =$$
$$= (1-\frac{1}{m})(X_2'\Sigma_0^{-1}X_2)^{-1} + \frac{1}{m}[X_2'(M_{X_1}\Sigma_0 M_{X_1})^+X_2]^{-1}$$

(here the equality

$$(M_{X_1}\Sigma_0 M_{X_1})^+\Sigma_0(M_{X_1}\Sigma_0 M_{X_1})^+ = (M_{X_1}\Sigma_0 M_{X_1})^+$$

was taken into account).

The proof can be easily finished by applying the first part of the assertion. □

Assertion 10 *(i) In the model (5) the ϑ_0-MINQUE of the vector ϑ is*

$$\hat{\vartheta}(Y_1,\dots,Y_m)=\left[(m-1)S_{(M_{X_2}\Sigma_0 M_{X_2})^+}+S_{(M_{(X_1,X_2)}\Sigma_0 M_{(X_1,X_2)})^+}\right]^{-1}\hat{\gamma}(Y),$$

where

$$\left\{S_{(M_{X_2}\Sigma_0 M_{X_2})^+}\right\}_{i,j}=\operatorname{Tr}\left[(M_{X_2}\Sigma_0 M_{X_2})^+V_i(M_{X_2}\Sigma_0 M_{X_2})^+V_j\right],$$

$i,j=1,\dots,p,$

$$\begin{aligned}&\left\{S_{(M_{(X_1,X_2)}\Sigma_0 M_{(X_1,X_2)})^+}\right\}_{i,j}=\\ &\quad=\operatorname{Tr}\left[(M_{(X_1,X_2)}\Sigma_0 M_{(X_1,X_2)})^+V_i(M_{(X_1,X_2)}\Sigma_0 M_{(X_1,X_2)})^+V_j\right],\end{aligned}$$

$i,j=1,\dots,p,$

$$\begin{aligned}\hat{\gamma}(Y)&=(\hat{\gamma}_1(Y),\dots,\hat{\gamma}_p(Y))',\\ \hat{\gamma}_i(Y)&=\operatorname{Tr}\left[(M_{X_2}\Sigma_0 M_{X_2})^+V_i(M_{X_2}\Sigma_0 M_{X_2})^+\sum_{j=1}^m(Y_j-\overline{Y})(Y_j-\overline{Y})'\right]+\\ &\quad+m\overline{Y}'(M_{(X_1,X_2)}\Sigma_0 M_{(X_1,X_2)})^+V_i(M_{(X_1,X_2)}\Sigma_0 M_{(X_1,X_2)})^+\overline{Y},\end{aligned}$$

$i=1,\dots,p,$ *and*

$$M_{(X_1,X_2)}=I-X_1(X_1'M_{X_2}X_1)^{-1}X_1'M_{X_2}-X_2(X_2'M_{X_1}X_2)^{-1}X_2'M_{X_1}.$$

(ii) In the jth epoch we have

$$\begin{aligned}\hat{\vartheta}(Y_j)&=S^{-1}_{(M_{(X_1,X_2)}\Sigma_0 M_{(X_1,X_2)})^+}\hat{\kappa}(Y_j),\\ \hat{\kappa}(Y_j)&=(\hat{\kappa}_1(Y_j),\dots,\hat{\kappa}_p(Y_j))',\\ \hat{\kappa}_i(Y_j)&=Y_j'(M_{(X_1,X_2)}\Sigma_0 M_{(X_1,X_2)})^+V_i(M_{(X_1,X_2)}\Sigma_0 M_{(X_1,X_2)})^+Y_j,\end{aligned}$$

$i=1,\dots,p.$

Proof With respect to Assertion 6, the ϑ_0-MINQUE of the vector $\vartheta=(\vartheta_1,\dots,\vartheta_p)'$ of variance components is

$$\hat{\vartheta}(Y_1,\dots,Y_m)=S_{[M_{(1\otimes X_1,I\otimes X_2)}(I\otimes\Sigma_0)M_{(1\otimes X_1,I\otimes X_2)}]^+}\hat{\gamma}(Y),$$

where

$$\begin{aligned}\hat{\gamma}(Y)&=(\hat{\gamma}_1(Y),\dots,\hat{\gamma}_p(Y))',\\ \hat{\gamma}_i(Y)&=(Y_1',\dots,Y_m')[M_{(1\otimes X_1,I\otimes X_2)}(I\otimes\Sigma_0)M_{(1\otimes X_1,I\otimes X_2)}]^+(I\otimes V_i)\times\\ &\quad\times[M_{(1\otimes X_1,I\otimes X_2)}(I\otimes\Sigma_0)M_{(1\otimes X_1,I\otimes X_2)}]^+\begin{pmatrix}Y_1\\ \vdots\\ Y_m\end{pmatrix}.\end{aligned}$$

It can be verified that

$$\begin{aligned}&[M_{(1\otimes X_1, I\otimes X_2)}(I\otimes\Sigma_0)M_{(1\otimes X_1,I\otimes X_2)}]^+ =\\ &= M_m\otimes(M_{X_2}\Sigma_0 M_{X_2})^+ + P_m\otimes(M_{(X_1,X_2)}\Sigma_0 M_{(X_1,X_2)})^+.\end{aligned}$$

Further

$$\begin{aligned}&\{S_{[M_{(1\otimes X_1,I\otimes X_2)}(I\otimes\Sigma_0)M_{(1\otimes X_1,I\otimes X_2)}]^+}\}_{i,j} =\\ &= \mathrm{Tr}\{[M_m\otimes(M_{X_2}\Sigma_0 M_{X_2})^+ +\\ &\quad + P_m\otimes(M_{(X_1,X_2)}\Sigma_0 M_{(X_1,X_2)})^+](I\otimes V_i)[M_m\otimes(M_{X_2}\Sigma_0 M_{X_2})^+ +\\ &\quad + P_m\otimes(M_{(X_1,X_2)}\Sigma_0 M_{(X_1,X_2)})^+](I\otimes V_j)\} =\\ &= (m-1)\,\mathrm{Tr}[(M_{X_2}\Sigma_0 M_{X_2})^+ V_i (M_{X_2}\Sigma_0 M_{X_2})^+ V_j] +\\ &\quad + \mathrm{Tr}[(M_{(X_1,X_2)}\Sigma_0 M_{(X_1,X_2)})^+ V_i (M_{(X_1,X_2)}\Sigma_0 M_{(X_1,X_2)})^+ V_j].\end{aligned}$$

Taking into account the relationship (valid for any $n\times n$ matrices A and B)

$$\begin{aligned}(Y_1',\ldots,Y_m')(M_m\otimes A_{[n,n]} + P_m\otimes B_{[n,n]})\begin{pmatrix}Y_1\\ \vdots\\ Y_m\end{pmatrix} =\\ = \mathrm{Tr}\left[A\sum_{i=1}^m (Y_i-\overline{Y})(Y_i-\overline{Y})'\right] + m\overline{Y}'B\overline{Y}\end{aligned}$$

we can easily finish the proof. □

Example 11 Let us compare variances (i.e., efficiences) of estimators of the unit dispersion within the special case of the stochastical model (5) for $p=1$, i.e., $\Sigma = \sigma^2 I$, $\sigma^2\in(0,\infty)$ within separate epochs and after all epochs. The normality of the observation vector is assumed.

In accordance with Assertion 10, the σ^2-MINQUE of σ^2 (it represents in our case the uniformly minimum variance unbiased estimator of the unit dispersion) is

$$\hat{\sigma}_1^2 = \hat{\sigma}^2(Y_1,\ldots,Y_m) = [(m-1)S_{M_{X_2}} + S_{M_{(X_1,X_2)}}]^{-1}\hat{\gamma}(Y),$$

where

$$\begin{aligned}\hat{\gamma}(Y_1,\ldots,Y_m) &= \mathrm{Tr}[M_{X_2}\sum_{j=1}^m (Y_j-\overline{Y})(Y_j-\overline{Y})' + m\overline{Y}'M_{(X_1,X_2)}\overline{Y}],\\ S_{M_{X_2}} &= \mathrm{Tr}(M_{X_2}) = n-k_2,\\ S_{M_{(X_1,X_2)}} &= \mathrm{Tr}(M_{(X_1,X_2)}) = n-k_1-k_2,\end{aligned}$$

thus, with respect to Assertion 6,

$$\mathrm{Var}[\hat{\sigma}_1^2(Y_1,\ldots,Y_m)|\sigma^2] = \frac{2\sigma^4}{(m-1)(n-k_2)+n-k_1-k_2}.$$

Another unbiased estimator of σ^2 is of the form

$$\hat{\sigma}_2^2 = \frac{\mathrm{Tr}\left[M_{X_2}\sum_{j=1}^m (Y_j - \overline{Y})(Y_j - \overline{Y})'\right]}{(m-1)(n-k_2)}$$

since

$$E\left\{\mathrm{Tr}\left[M_{X_2}\sum_{j=1}^m (Y_j - \overline{Y})(Y_j - \overline{Y})'\right]\Big|\sigma^2\right\} =$$
$$= (m-1)\,\mathrm{Tr}(\sigma^2 M_{X_2}) = (m-1)(n-k_2)\sigma^2,$$

(here the relation

$$E\left\{\mathrm{Tr}\left[A\sum_{j=1}^m (Y_j - \overline{Y})(Y_j - \overline{Y})'\right]\Big|\Sigma\right\} =$$
$$= (m-1)\,\mathrm{Tr}(A\Sigma) + \sum_{j=1}^m (\beta_2^{(j)} - \overline{\beta_2^{(\cdot)}})' X_2' A X_2 (\beta_2^{(j)} - \overline{\beta_2^{(\cdot)}})$$

was utilized). The dispersion of this estimator is

$$\mathrm{Var}(\hat{\sigma}_2^2|\sigma^2) = 2\sigma^4/[(m-1)(n-k_2)].$$

Further let us consider the unbiased estimator $\hat{\sigma}_3^2$ of the form

$$\hat{\sigma}_3^2 = \frac{1}{m}\sum_{j=1}^m \hat{\sigma}_{3,j}^2(Y_j),$$

where

$$\hat{\sigma}_{3,j}^2 = \hat{\sigma}^2(Y_j) = S^{-1}_{M_{(X_1,X_2)}} Y_j' M_{(X_1,X_2)} Y_j = \frac{1}{n-k_1-k_2} Y_j' M_{(X_1,X_2)} Y_j,$$

$$\mathrm{Var}[\hat{\sigma}_3^2(Y_j)|\sigma^2] = \frac{2\sigma^2}{n-k_1-k_2}, \ j = 1,\dots,m;$$

its dispersion is

$$\mathrm{Var}(\hat{\sigma}_3^2|\sigma^2) = \frac{2\sigma^4}{m(n-k_1-k_2)}.$$

For comparing the given estimators following ratios are used:

$$\frac{\mathrm{Var}(\hat{\sigma}_1^2|\sigma^2)}{\mathrm{Var}(\hat{\sigma}_2^2|\sigma^2)} = \frac{1}{1 + \frac{n-k_1-k_2}{(m-1)(n-k_2)}},$$

$$\frac{\mathrm{Var}(\hat{\sigma}_2^2|\sigma^2)}{\mathrm{Var}(\hat{\sigma}_3^2|\sigma^2)} = \frac{m(n-k_1-k_2)}{(m-1)(n-k_1)}$$

and

$$\frac{\mathrm{Var}(\hat{\sigma}_1^2|\sigma^2)}{\mathrm{Var}(\hat{\sigma}_3^2|\sigma^2)} = \frac{1}{1+\frac{(m-1)k_1}{m(n-k_1-k_2)}}.$$

For different m, n, k_1, k_2 the values of these ratios are in the following Table

m	n	k_1	k_2	$\frac{\mathrm{Var}(\hat{\sigma}_1^2\|\sigma^2)}{\mathrm{Var}(\hat{\sigma}_2^2\|\sigma^2)}$	$\frac{\mathrm{Var}(\hat{\sigma}_2^2\|\sigma^2)}{\mathrm{Var}(\hat{\sigma}_3^2\|\sigma^2)}$	$\frac{\mathrm{Var}(\hat{\sigma}_1^2\|\sigma^2)}{\mathrm{Var}(\hat{\sigma}_3^2\|\sigma^2)}$
5	25	6	8	68/79	55/68	55/79
10	30	6	8	198/214	160/198	160/214
50	30	6	8	1078/1094	800/1078	800/1094
10	300	60	80	1980/2140	1600/1980	1600/2140
20	300	60	80	4180/4340	3200/4180	3200/4340

It is quite clear that the efficiency of the unit dispersion estimator $\hat{\sigma}_1^2$ based on the results from the whole experiment is substantially greater than the efficiency of the unit dispersion estimator $\hat{\sigma}_3^2$ obtained by averaging the estimators based on the separate epochs. Moreover, the efficiency of the unit dispersion estimator $\hat{\sigma}_2^2$ which is based on the matrix $\sum_{i=1}^{m}(Y_i - \overline{Y})(Y_i - \overline{Y})'$ gets near the efficiency of the estimator $\hat{\sigma}_1^2$ based on the hypervector $Y = (Y_1', \dots, Y_m')'$. Asymptotically, for $m \to \infty$, the efficiences of the estimators $\hat{\sigma}_1^2$ and $\hat{\sigma}_2^2$ become equivalent. The estimator $\hat{\sigma}_3^2$, commonly used in practice, is significantly worse.

4 A systemization of basic structures and their combination

Two basic classes of structures will be considered:

(i) The A-structure: a structure with a group of stable and with a group of non-stable points.

(ii) The B-structure: a structure with non-stable points only.

A typical representant of the A-structure is the model investigated partly in the preceding section.

A typical representant of the B-structure is the growth-curve model.

Let $(\beta_{1,1}, \beta_{2,1}, \dots, \beta_{k,1})'$ be the vector of the unknown values of the network parameters (e.g. horizontal coordinates of points of the geodetic network) at the time $t = t_1$, (the beginning of the investigation of the RCM) and let

$$\beta_l(t_i) = \beta_{l,1} + \sum_{j=2}^{s} \beta_{l,j}^* \phi_j(t_i), \quad l = 1, \dots, k; \tag{6}$$

$i = 1, \dots, m$, be the value of the lth parameter at the time t_i.

The functions $\phi_1(.)(\equiv 1)$, $\phi_2(.), \dots, \phi_s(.)$ suitably chosen for a good approximation of the time course of variations of the parameters $\beta_l(t)$, $t \geq t_1$, $l = 1, \dots, k$, fulfil the conditions $\phi_j(t_1) = 0$, $j = 2, \dots, s$. If none explanatory model of the character of the RCM in the investigated region is available, then we can choose $\phi_j(t) = (t - t_1)^{j-1}$, $j = 2, \dots, s$. If the observation vector of the rth epoch is $Y_{[n,1]}(t_r)$, then

$$E\left[Y(t_r)|\beta(t_r)\right] = XB\phi(t_r),$$

where X is a design matrix (in each epoch it is the same),

$$B = \begin{pmatrix} \beta_{1,1}, & \beta^*_{1,2}, & \dots, & \beta^*_{1,s} \\ \vdots & \vdots & \vdots & \vdots \\ \beta_{k,1}, & \beta^*_{k,2}, & \dots, & \beta^*_{k,s} \end{pmatrix}$$

and

$$\phi(t_r) = \begin{pmatrix} \phi_1(t_r) \\ \phi_2(t_r) \\ \vdots \\ \phi_s(t_r) \end{pmatrix}.$$

Thus

$$E(\underline{Y}|B) = XBZ,$$

where $\underline{Y} = (Y(t_1), \dots, Y(t_m)), Z = (\phi(t_1), \dots, \phi(t_m))$. In the case with the same design in each epoch the covariance matrix of the random vector $\mathrm{vec}(\underline{Y}) = (Y'_{t_1}, \dots, Y'_{t_m})'$ is

$$\mathrm{Var}\left[\mathrm{vec}(\underline{Y})|\Sigma(\vartheta)\right] = I_{[m,m]} \otimes \sum_{i=1}^{p} \vartheta_i V_i.$$

Some natural conditions on s and m must be fulfilled in order the matrix B be unbiasedly estimable (for further detail cf. [5]).

Let X be the design matrix of the first epoch (i.e., in the A-structure $X = (X_1, X_2)$). Several typical situations are to be distinguished in the model of the first epoch

(i) $(Y, X\beta, \Sigma(\vartheta))$, $\beta \in \mathcal{V} = \mathcal{R}^k$,

(ii) $\Big(Y, \beta, \Sigma(\vartheta)\Big)$, $\beta \in \mathcal{V} = \{u \in \mathcal{R}^n : d_{[q,1]} + D_{[q,n]}u = 0_{[q,1]}\}$,

(iii) $\Big(Y, \beta_1, \Sigma(\vartheta)\Big)$, $\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \in \mathcal{V} =$

$$= \left\{ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} : u_1 \in \mathcal{R}^n, u_2 \in \mathcal{R}^k,\ d_{[q,1]} + D_{[q,n]}u_1 + F_{[q,k]}u_2 = 0_{[q,1]} \right\},$$

(iv) $\Big(Y, X\beta, \Sigma(\vartheta)\Big)$, $\beta \in \mathcal{V} = \{u \in \mathcal{R}^k, d_{[q,1]} + D_{[q,k]}u = 0_{[q,1]}\}$,

(v) $\Big(Y, X\beta_1, \Sigma(\vartheta)\Big)$, $\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \in \mathcal{V} =$

$$= \left\{ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} : u_1 \in \mathcal{R}^{k_1}, u_2 \in \mathcal{R}^{k_2}, d_{[q,1]} + D_{[q,k_1]}u_1 + F_{[q,k_2]}u_2 = 0_{[q,1]} \right\}$$

(for more detail cf. [9]).

It is to be remarked that a decomposition of the vector β to β_1 and β_2 in the A-structure need not correspond to a decomposition of the vector β in the cases (iii) and (v). If an A-structure is considered, then in the case (v) the following structure can occur in the first epoch

$$\left(Y, (X_1, X_2)\begin{pmatrix} \beta_1 \\ \beta_2^{(1)} \end{pmatrix}, \Sigma(\vartheta)\right),$$

where the stable/non-stable points are connected with the parameter

$$\beta_1 = \begin{pmatrix} \beta_{1,1} \\ \beta_{2,1} \end{pmatrix} \Big/ \beta_2^{(1)} = \begin{pmatrix} \beta_{1,2}^{(1)} \\ \beta_{2,2}^{(1)} \end{pmatrix}$$

and

$$\begin{pmatrix} \beta_1 \\ \beta_2^{(1)} \end{pmatrix} \in \mathcal{V} = \left\{ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} : u_1 \in \mathcal{R}^{k_1}, u_2 \in \mathcal{R}^{k_2}, \right.$$

$$\left. d_{[q,1]} + (D_1, D_2)\begin{pmatrix} u_{1,1} \\ u_{2,1} \end{pmatrix} + (F_1, F_2)\begin{pmatrix} u_{1,2} \\ u_{2,2} \end{pmatrix} = 0_{[q,1]} \right\}.$$

The structures (i) till (v) are basic. Sometimes these basic structures must be generalized at least in the three following directions:

(a) Instead of the parameter β a parameter $\binom{\beta}{\kappa}$ must be considered. Here β is a useful vector parameter connecter with the RCM and κ is a nuisance vector parameter which has no importance for the researcher; e.g., the vector κ is a vector of coefficients of a polynomial which models a drift of a measurement device. Then $X = (A, S)$ and the problem of an optimum elimination of the parameter κ has to be solved (in more detail cf. [8] and [2]).

(b) Sometimes a time varying position of an investigated point cannot be adequately expressed by the deterministic part of the model: a process noise must be involved in the model. In the simplest case th observation vector Y can be expressed as (in more detail cf. [9])

$$Y = X\beta + \nu + \varepsilon$$

where ν is a process noise vector and ε is a measurement noise vector. It leads to the model (ν and ε are independent)

$$\left(Y, X\beta, \Sigma(\vartheta_1, \vartheta_2) = \sum_{i=1}^{p_1} \vartheta_{1,i} V_{1,i} + \sum_{i=1}^{p_2} \vartheta_{2,i} V_{2,i}\right)$$

which is a generalization of the basic model, which enable us to solve new problems, e.g., to find the best approximation of the random variable $\{X\}_{i,.}\beta + \nu_i$, where $\{X\}_{i.} = e_i' X$, e_i is the ith row of the matrix $I_{[n,n]}$ (a collocation problem of Moritz [11], see also [7]).

(c) In many cases the measurement in a geodetic network is performed by a group of measurement devices (e.g., gravimeters) simultaneously. Then the basic model (ii) can be established, e.g., in the form

$$\left[\begin{pmatrix} Y_1 \\ \vdots \\ Y_g \end{pmatrix}, (1 \otimes X)\beta, Diag(\sigma_1^2, \dots, \sigma_g^2) \otimes I_{[n,n]}\right],$$

where $Diag(\sigma_1^2, \dots, \sigma_g^2)$ is a $g \times g$ diagonal matrix and σ_i^2 is a dispersion of the ith device (in more detail cf. [10]).

Till now the first epoch was considered only; if different designs are used in different epochs a new class of models occurs. For example in the case of the A-structure

$$\left[\begin{pmatrix} Y_1 \\ \vdots \\ Y_m \end{pmatrix}, \begin{pmatrix} X_1^{(1)}, & X_2^{(1)}, & 0, & \dots, & 0 \\ X_1^{(2)}, & 0, & X_2^{(2)}, & \dots, & 0 \\ \vdots & & & & \\ X_1^{(m)}, & 0, & 0, & \dots, & X_2^{(m)} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2^{(1)} \\ \vdots \\ \beta_2^{(m)} \end{pmatrix}, \Sigma(\vartheta)\right],$$

where $\Sigma(\vartheta)$ is, e.g., if the same measurement devices are used in each epoch, of the form

$$\Sigma(\vartheta) = \sum_{i=1}^{p} \vartheta_i \begin{pmatrix} V_{1,i}, & 0, & \dots, & 0 \\ \dots & & & \\ 0, & 0, & \dots, & V_{m,i} \end{pmatrix}.$$

Another case occurs if e.g., $\beta_2^{(i)}$ in model (5) is of the form

$$\beta_2^{(i)} = (\beta_1^*, \dots, \beta_s^*)\Phi(t_i), \ \Phi(t_i) = [\Phi_1(t_i), \dots, \Phi_s(t_i)]',$$

(i.e., the time courses of changes of the nonstable points are modelled using the known functions $\Phi_1(\cdot), \dots, \Phi_s(\cdot)$), then the corresponding model is

$$\left[\begin{pmatrix} Y_1 \\ \vdots \\ Y_m \end{pmatrix}, \left[1 \otimes X_1, \begin{pmatrix} \Phi'(t_1) \\ \vdots \\ \Phi'(t_m) \end{pmatrix} \otimes X_2\right] \begin{pmatrix} \beta_1 \\ \beta_1^* \\ \beta_2^* \\ \vdots \\ \beta_s^* \end{pmatrix}, \sum_{i=1}^{p} \vartheta_i (I \otimes V_i)\right].$$

Now it is quite clear that a combination of structures A, B, cases (i), (ii), (iii), (iv), (v) and generalizations (a), (b), (c) creates a huge class of models for the RCM; only a few of them are already analyzed.

5 Conclusion

Estimation problems in linear regression models suitable for studying RCM do not stay always and sufficiently in centre of attention of experts. Usually the phase of collecting data is emphasized, where measurements are performed using expensive devices and exacting procedures to the detriment of an application of optimum statistical procedures for processing them. However an analysis of adequate models (mainly multiepoch models with variance components) in more detail shows the amount of the work expended in the first phase deteriorated when efficient estimators are not used (cf Example 11); the percentage of the deteriorated work is given by the ratio (difference between the actual dispersion of the estimator and the dispersion of the efficient estimator)/(the actual dispersion of the estimator). This simple and well known fact represents sometimes a nonnegligible loss of finance and labour and still it is frequently not taken into account in practice.

The main aim of the paper was to attract attention of experts to statistical problems of processing data obtained by measurements of RCM and to emphasize the possibilities they offer.

From didactical reasons the models, within which the algorithms for optimum estimation of the first and second order parameters are developed, are relatively simple. It is no problem to apply ideas explained for them in more complicated models more adequately describing the concrete situation (e.g. in multiepoch models with different design matrices in each epoch, in a combinations of epoch and multistage models, etc). Authors tried to apply the described approach to other models summed up in the monograph [6]. Nevertheless, results obtained till now represent only the first steps in developing the estimation theory in models linked up to the RCM.

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