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Regularity and Permutability via Transferability of Tolerances *

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Abstract

An algebra A has n-transferable tolerances if for any $a, b, c \in A$ there exist $d_1, \ldots, d_n \in A$ such that $T(a,b) = T(c,d_1) \vee \ldots \vee T(c,d_n)$ in the tolerance lattice Tol A. We prove that a variety $\mathcal{V}$ is regular and permutable if and only if each $A \in \mathcal{V}$ has n-transferable tolerances. Analogously we characterize varieties with 0-regular and permutable congruences.

Key words: Tolerance relation, transferable tolerances, regularity, 0-regularity, permutability of congruences.

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Recall that an algebra $\mathcal{A}$ is *regular* if $\Theta = \Phi$ for $\Theta, \Phi \in \text{Con}\,\mathcal{A}$ whenever they have a congruence class in common, i.e. if $[a]_\Theta = [a]_\Phi$ for some $a \in A$. $\mathcal{A}$ is *permutable* if $\Theta \bullet \Phi = \Phi \bullet \Theta$ for every two $\Theta, \Phi \in \text{Con}\,\mathcal{A}$. A variety $\mathcal{V}$ is *regular* (*permutable*) if each $\mathcal{A} \in \mathcal{V}$ has this property.

Suppose that a variety $\mathcal{V}$ has a *constant* 0, i.e. 0 is either a nullary operation in type of $\mathcal{V}$ or an equationally defined nullary term. An algebra $\mathcal{A} \in \mathcal{V}$ is 0-*regular* if $\Theta = \Phi$ for $\Theta, \Phi \in \text{Con}\,\mathcal{A}$ whenever $[0]_\Theta = [0]_\Phi$; $\mathcal{V}$ is 0-*regular* if each $\mathcal{A} \in \mathcal{V}$ has this property.

Let $\mathcal{A} = (A, F)$ be an algebra. By a *tolerance* on $\mathcal{A}$ is meant a reflexive and symmetric binary relation on A which has the substitution property with respect to every operation of F. Denote by Tol $\mathcal{A}$ the set of all tolerances on $\mathcal{A}$.

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It is well-known (see e.g. [3]) that Tol $\mathcal{A}$ is an algebraic lattice with respect to set inclusion. Hence, for each subset $M \subseteq A \times A$ there exists the least tolerance on $\mathcal{A}$ containing M; denote it by $T(M)$. If $M = \{\langle a,b\rangle\}$, $T(M)$ will be denoted simply by $T(a,b)$. It si easy to check that for $p, q_1, \ldots, q_n$ of $\mathcal{A}$ we have

$$T(\{\langle p,q_1\rangle,\ldots,\langle p,q_n\rangle\}) = T(p,q_1) \vee \ldots \vee T(p,q_n))$$

in Tol $\mathcal{A}$. For $T \in$ Tol $\mathcal{A}$ and $a \in A$ we denote by $[a]_T = \{x \in A; \langle a,x\rangle \in T\}$.

The concept of transferable tolerance was firstly introduced in [2] and used for a characterization of tolerance regular lattices, see also [1] for the general case. Now, we introduce a bit more general concepts:

Definition 1 An algebra $\mathcal{A} = (A, F)$ has *n-transferable tolerances* if for any $a, b, c \in A$ there exist $d_1, \ldots, d_n \in A$ such that

$$T(a,b) = T(c,d_1) \vee \ldots \vee T(c,d_n)$$

in Tol $\mathcal{A}$.

An algebra $\mathcal{A}$ with a constant 0 has *n-transferable tolerances at 0* if for any $a, b \in A$ there exist $d_1, \ldots, d_n \in A$ such that

$$T(a,b) = T(0,d_1) \vee \ldots \vee T(0,d_n)$$

in Tol $\mathcal{A}$.

A variety $\mathcal{V}$ (with 0) has *n-transferable tolerances* (*at* 0, respectively) if every A of $\mathcal{V}$ has this property.

Theorem 1 *For a variety $\mathcal{V}$, the following conditions are equivalent:*

(1) $\mathcal{V}$ has n-transferable tolerances for some integer $n > 0$;

(2) $\mathcal{V}$ is regular and permutable.

Proof (1) $\Rightarrow$ (2): Let $\mathcal{A} \in \mathcal{V}$, $T, S \in$ Tol $\mathcal{A}$ and a be an element of $\mathcal{A}$. Suppose $[a]_T = [a]_S$. Let $\langle x,y\rangle \in T$. By (1), there exist elements $d_1, \ldots, d_n \in A$ such that

$$T(x,y) = T(a,d_1) \vee \ldots \vee T(a,d_n)\,.$$

From $T(x,y) \subseteq T$ we have $\langle a,d_1\rangle, \ldots, \langle a,d_n\rangle \in T$, i.e. $d_1, \ldots, d_n \in [a]_T = [a]_S$, whence $\langle a,d_1\rangle, \ldots, \langle a,d_n\rangle \in S$. It implies

$$T(x,y) = T(a,d_1) \vee \ldots \vee T(a,d_n) \subseteq S\,,$$

thus $\langle x,y\rangle \in S$ and $T \subseteq S$. Analogously it can be shown $S \subseteq T$, i. e. $T = S$. In the terminology of [3], [4], $\mathcal{A}$ and hence also $\mathcal{V}$ is tolerance regular. By [4] it is equivalent with regularity and permutability of $\mathcal{V}$.

(2) $\Rightarrow$ (1): By [5], permutability of $\mathcal{V}$ implies Tol $\mathcal{A}$ = Con $\mathcal{A}$ for any $\mathcal{A}$ of $\mathcal{V}$. Hence, it remains to show that for every a, b, c of $\mathcal{A}$ there exist $d_1, \ldots, d_n$ of $\mathcal{A}$ such that

$$\Theta(a,b) = \Theta(c,d_1) \vee \ldots \vee \Theta(c,d_n)\,. \qquad (*)$$

Put $\Theta = \Theta(a, b)$. Applying regularity of $\mathcal{V}$ we infer

$$\Theta(a, b) = \Theta\left(\{c\} \times [c]_\Theta\right) .$$

Since Con $\mathcal{A}$ is compactly generated, there exists a finite subset $\{d_1, \ldots, d_n\} \subseteq [c]_\Theta$ with

$$\Theta(a, b) = \Theta\left(\{c\} \times \{d_1, \ldots, d_n\}\right)$$

which is equivalent to $(*)$. □

Theorem 2 *Let $\mathcal{V}$ be a variety with 0. The following conditions are equivalent:*

(1) $\mathcal{V}$ has n-transferable tolerances at 0 for some integer $n > 0$;

(2) $\mathcal{V}$ is permutable and 0*-regular.*

Proof (1) $\Rightarrow$ (2): analogously as in the proof of Theorem 1, if $\mathcal{A} \in \mathcal{V}$ and $T, S \in \text{Tol}\,\mathcal{A}$, then $[0]_T = [0]_S$ implies $T = S$ (we only replace the element a by constant 0). Further, let $\mathcal{A} = F_\mathcal{V}(x, y)$ be a free algebra of $\mathcal{V}$ with two free generators x, y. Put $T = T(x, y) \in \text{Tol}\,\mathcal{A}$. By the foregoing property,

$$T(x, y) = T\left(\{0\} \times [0]_T\right) ,$$

i.e. $\langle x, y\rangle \in T\left(\{0\} \times [0]_T\right)$. However, Tol $\mathcal{A}$ is an algebraic lattice, so there exists a finite subset $\{p_1, \ldots, p_n\} \subseteq [0]_T$ such that $\langle x, y\rangle \in T\left(\{0\} \times \{p_1, \ldots, p_n\}\right) = T(0, p_1) \vee \ldots \vee T(0, p_n)$ in Tol $\mathcal{A}$. By Lemmas 1.4 and 1.5 in [3], there exists a $2n$-ary polynomial φ over $\mathcal{A}$ with

$$\begin{aligned} x &= \varphi(p_1, \ldots, p_n, 0, \ldots, 0) \\ y &= \varphi(0, \ldots, 0, p_1, \ldots, p_n) . \end{aligned}$$

However, $p_i \in F_\mathcal{V}(x, y)$, i.e. $p_i = d_i(x, y)$ $(i = 1, \ldots, n)$ for some binary terms d_i. Since

$$d_i(x, y) \in [0]_{T(x,y)}$$

we conclude $d_i(x, x) = 0$ for $i = 1, \ldots, n$. Further, there exists a $(2 + 2n)$-ary term t with $\varphi(v_1, \ldots, v_n, w_1, \ldots, w_n) = t(x, y, v_1, \ldots, v_n, w_1, \ldots, w_n)$ whence

$$\begin{aligned} x &= t(x, y, d_1(x, y), \ldots, d_n(x, y), 0, \ldots, 0) \\ y &= t(x, y, 0, \ldots, 0, d_1(x, y), \ldots, d_n(x, y)) . \end{aligned}$$

Put $m(x, y, z) = t(x, z, d_1(x, y), \ldots, d_n(x, y), d_1(x, z), \ldots, d_n(x, z))$. The foregoing identities clearly imply

$$\begin{aligned} m(x, x, z) &= t(x, z, 0, \ldots, 0, d_1(x, z), \ldots, d_n(x, z)) = z \\ m(x, y, y) &= t(x, y, d_1(x, y), \ldots, d_n(x, y), 0, \ldots, 0) = x \end{aligned}$$

thus $m(x, y, z)$ is a Mal'cev term and, therefore, $\mathcal{V}$ is permutable. By [5], Tol $\mathcal{A}$ = Con $\mathcal{A}$ for each $\mathcal{A} \in \mathcal{V}$ thus, by the first shown property $[0]_T = [0]_S \Rightarrow T = S$, $\mathcal{V}$ is also regular.

The proof of (2) $\Rightarrow$ (1) is completely analogous to that of Theorem 1 and hence omitted (the constants 0 is considered instead of that element c). □

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