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Two-point Functional Boundary Value Problems without Growth Restrictions *

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Abstract

We establish existence results for second order functional differential equations with fully nonlinear two point boundary conditions. Sufficient conditions are formulated only in terms of sign conditions. Results are proved by the topological degree theory.

Key words: Two point boundary value problem, functional differential equation, existence, Carathéodory conditions, Leray–Schauder degree, Borsuk theorem, homotopy.

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1 Introduction

Let $J = [0, T]$, $g : \mathbb{R} \to \mathbb{R}$ be an increasing homeomorphism with inverse g^{-1} $g(0) = 0$ and

$$F : C^0(J) \times C^0(J) \times \mathbb{R} \to L_1(J), \quad (x, y, a) \longmapsto (F(x, y, a))(t),$$

be an operator with the following properties (see $[S_1]$):

(a) $(F(x, y, z(t)))(t) \in L_1(J)$ for $x, y, z \in C^0(J)$,

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(b) $\lim_{t\to\infty}(x_n, y_n, z_n) = (x, y, z)$ in $C^0(J) \times C^0(J) \times C^0(J)$
$\Rightarrow \lim_{n\to\infty}(F(x_n, y_n, z_n(t)))(t) = (F(x, y, z(t)))(t)$ in $L_1(J)$,

(c) for each $b \in (0, \infty)$, there exists $k_b \in L_1(J)$ such that $x, y \in C^0(J)$, $a \in \mathbb{R}$, $||x|| + ||y|| + |a| \le b \Rightarrow |(F(x, y, a))(t)| \le k_b(t)$ for a.e. $t \in J$.

Here $||x|| = \max\{|x(t)|; t \in J\}$ is the norm in the Banach space $C^0(J)$.

Consider the boundary value problem (BVP for short)

$$(g(x'(t)))' = (F(x, x', x'(t)))(t), \tag{1}$$

$$p_1(x(0), x'(0), x(T), x'(T)) = 0, \qquad p_2(x(0), x'(0), x(T), x'(T)) = 0 \tag{2}$$

where $p_1, p_2 \in C^0(\mathbb{R}^4)$.

We say that $x \in C^1(J)$ is a *solution of BVP* (1), (2) if $g(x'(t))$ is absolutely continuous on J, x satisfies boundary conditions (2) and (1) is satisfied for a.e. $t \in J$.

The special cases of the operator equation (1) are the equations

$$(g(x'(t)))' = (Q_1(x, x'))(t)f(t, x(t), x'(t), x'(t)) + (Q_2(x, x'))(t),$$

$$(g(x'(t)))' = f\Big(t, \int_0^t (Q_1(x, x'))(s)\, ds, x(t), x'(t)\Big) + (Q_2(x, x'))(t),$$

where $f : J \times \mathbb{R}^3 \to \mathbb{R}$ satisfies the Carathéodory conditions on $J \times \mathbb{R}^3$, $Q_i : C^0(J) \times C^0(J) \to L_1(J)$ $(i = 1, 2)$ are continuous and for each $b \in (0, \infty)$ there exists $l_b \in L_1(J)$ such that $x, y \in C^0(J)$, $||x|| + ||y|| \le b \Rightarrow |(Q_i(x, y))(t)| \le l_b(t)$ for a.e. $t \in J$.

There are many papers devoted to the consideration of existence results for BVP (1), (2) where (1) is ordinary differential equation $x'' = f(t, x, x')$ with f either continuous or satisfying the Carathéodory conditions on $J \times \mathbb{R}^2$ and (2) are fully nonlinear two-point boundary conditions (see, e.g., [BL], [GM], [GGL], [LL], [Ki], [T_1], [T_2] and the references cited therein). Existence results are proved by the method of lower and upper solutions. By this method one can proved existence results if compatibilities conditions between boundary conditions (2) and the lower and upper solutions hold and, in addition, f satisfies assumptions guaranteeing a priori bounds on the derivatives of solutions. Compatibility conditions were in detail studied in [T_1] and [T_2]. A priori bounds on x' follow, for example, if f satisfies either Bernstein–Nagumo growth condition with respect to x' (see [B], [N]) or its one sided generalizations (see [GK]).

Gelashvili and Kiguradze [GK] considered a system of first-order functional differential equations with the boundary conditions $x_k(t_k) = \varphi_k(x_1, x_2, \dots, x_n)$ $(k = 1, 2, \dots, n)$. Here $\varphi_k : \underbrace{C^0(J) \times \dots \times C^0(J)}_{n} \to \mathbb{R}$ are continuous functionals. They gave sufficient conditions for the existence (and uniqueness) of this BVP. These conditions are of the type of one sided growth restrictions on the right members of the system and both sided on the functionals φ_k $(k = 1, 2, \dots, n)$.

We observe that Thompson [T₁] and [T₂] and Kiguradze [Ki] considered as well BVPs with boundary conditions $(x(0), x'(0)) \in \Omega_0$, $(x(T), x'(T)) \in \Omega_1$ where Ω_0, Ω_1 are closed connected subsets of $\mathbb{R}^2$.

The second group of papers formulates sufficient conditions for the existence results only in the terms of sign conditions, that is without growth restrictions (see, e.g., [Ke], [RS₁], [RS₂], [RT], [S₂]). But two-point boundary conditions have usually the linear form. Solutions of the equation $x'' = f(t, x, x')$ with the Neumann, Dirichlet or mixed boundary conditions were considered in [Ke] and [RT]. Solutions of a second order functional differential equation with the above boundary conditions were studied in [RS₁], [RS₂] and with functional boundary conditions in [RS₁] and [S₂].

The aim of this paper is to consider BVP (1), (2) with fully nonlinear boundary conditions (2). Sufficient conditions for the existence of solutions are formulated only in terms of sign conditions. Results are proved by the topological degree method (see, e.g., [D]).

Throughout the paper we will need the following assumptions:

(H_1) There exist constants $L_1 < 0 < L_2$, $M_1 < M_2$, $\varepsilon \in \{-1, 1\}$ and $\delta \in \{-1, 1\}$ such that
$$(F(x, y, L_2))(t) \le 0 \le (F(x, y, L_1))(t) \tag{3}$$
for a.e. $t \in J$ and each $x, y \in C^0(J)$, $M_1 - L_2T \le x(t) \le M_2 - L_1T$, $L_1 \le y(t) \le L_2$ for $t \in J$ and
$$\varepsilon p_1(u, L_1, w, z) \le 0 \le \varepsilon p_1(u, L_2, w, z), \tag{4}$$
$$\delta p_2(u, v, M_1, z) \le 0 \le \delta p_2(u, v, M_2, z) \tag{5}$$
for $u \in [M_1 - L_2T, M_2 - L_1T]$, $v, z \in [L_1, L_2]$ and $w \in [M_1, M_2]$.

(H_2) There exist constants $L_1 < 0 < L_2$, $M_1 < M_2$, $\varepsilon \in \{-1, 1\}$ and $\delta \in \{-1, 1\}$ such that
$$(F(x, y, L_1))(t) \le 0 \le (F(x, y, L_2))(t) \tag{6}$$
for a.e. $t \in J$ and each $x, y \in C^0(J)$, $M_1 + L_1T \le x(t) \le M_2 + L_2T$, $L_1 \le y(t) \le L_2$ for $t \in J$ and
$$\varepsilon p_1(M_1, v, w, z) \le 0 \le \varepsilon p_1(M_2, v, w, z), \tag{7}$$
$$\delta p_2(u, v, w, L_1) \le 0 \le \delta p_2(u, v, w, L_2) \tag{8}$$
for $u \in [M_1, M_2]$, $v, z \in [L_1, L_2]$ and $w \in [M_1 + L_1T, M_2 + L_2T]$.

Remark 1 The special case of boundary conditions (2) are boundary conditions
$$x(0) = \phi_1(x'(0)), \quad x(T) = \psi_1(x'(T)), \tag{2'}$$
$$x'(0) = \phi_2(x(0)), \quad x'(T) = \psi_2(x(T)), \tag{2''}$$
$$x(0) = \phi_3(x'(0)), \quad x'(T) = \psi_3(x(T)) \tag{2'''}$$

and

$$x'(0) = \phi_4(x(0)), \quad x(T) = \psi_4(x'(T)), \tag{2''''}$$

where $\phi_i, \psi_i \in C^0(\mathbb{R})$ $(i = 1, 2, 3, 4)$. It is easy to check that inequalities (4) and (5), e.g., for (2') are equivalent to $M_1 \le \psi_1(v) \le M_2$ for $v \in [L_1, L_2]$ and either $\phi_1(L_1) \ge M_2 - L_1T$, $\phi_1(L_2) \le M_1 - L_2T$ or $\phi_1(L_1) \le M_1 - L_2T$, $\phi_1(L_2) \ge M_2 - L_1T$, and inequalities (7) and (8), e.g., for (2'') are equivalent to $L_1 \le \psi_2(v) \le L_2$ for $v \in [M_1 + L_1T, M_2 + L_2T]$ end either $\phi_2(M_1) \le L_1$, $\phi_2(M_2) \ge L_2$ or $\phi_2(M_1) \ge L_2$, $\phi_2(M_2) \le L_1$.

The paper is organized as follows. First, we define two auxiliary BVPs depending on the parameters $\lambda \in [0,1]$ and $n \in \mathbb{N}$. These BVPs are constructed accordingly if assumption (H_1) or (H_2) is satisfied. We next prove a priori estimates for solutions of our auxiliary BVPs (Lemma 1 and Lemma 2). Applying the topological degree theory (see, e.g., [D]), the existence of a sequence $\{u_n(t)\}$ of solutions for BVPs with $\lambda = 1$ is proved in Lemma 3 and Lemma 4. Finally, by the Arzelà–Ascoli theorem, the existence of a subsequence $\{u_{k_n}(t)\}$ converging to a solution of BVP (1), (2) is proved (Theorem 1 and Theorem 2).

2 Notation, lemmas

Let $M_1, M_2, L_1, L_2, \varepsilon, \delta$ be constants in assumption (H_1) (and (H_2)). For each $x \in C^0(J)$, $a, U, V \in \mathbb{R}$, $U < V$ and $n \in \mathbb{N}$, define $\bar{x}, x^*, \tilde{x} \in C^0(J)$, $a\Big|_U^V \in \mathbb{R}$ and $q_n(\cdot; U, V) : \mathbb{R} \to \mathbb{R}$ continuous by the formulas

$$\bar{x}(t) = \begin{cases} M_2 - L_1T & \text{for } x(t) > M_2 - L_1T \\ x(t) & \text{for } M_1 - L_2T \le x(t) \le M_2 - L_1T \\ M_1 - L_2T & \text{for } x(t) < M_1 - L_2T, \end{cases}$$

$$x^*(t) = \begin{cases} M_2 + L_2T & \text{for } x(t) > M_2 + L_2T \\ x(t) & \text{for } M_1 + L_1T \le x(t) \le M_2 + L_2T \\ M_1 + L_1T & \text{for } x(t) < M_1 + L_1T, \end{cases} \tag{9}$$

$$\tilde{x}(t) = \begin{cases} L_2 & \text{for } x(t) > L_2 \\ x(t) & \text{for } L_1 \le x(t) \le L_2 \\ L_1 & \text{for } x(t) < L_1, \end{cases}$$

$$a\Big|_U^V = \begin{cases} V & \text{for } a > V \\ a & \text{for } U \le a \le V \\ U & \text{for } a < U \end{cases} \tag{10}$$

and

$$q_n(v;U,V) = \begin{cases} -\frac{1}{n} & \text{for } v > V + \frac{1}{n} \\ V - v & \text{for } V < v \le V + \frac{1}{n} \\ 0 & \text{for } U \le v \le V \\ U - v & \text{for } U - \frac{1}{n} \le v < U \\ \frac{1}{n} & \text{for } v < U - \frac{1}{n}. \end{cases} \tag{11}$$

Let $F_n^i : C^0(J) \times C^0(J) \times \mathbb{R} \to L_1(J)$, $p_{1n}^i, p_{2n}^i : \mathbb{R}^4 \to \mathbb{R}$ ($i = 1,2$; $n \in \mathbb{N}$) be defined by

$$(F_n^1(x,y,a))(t) = \left(F\left(\bar{x}, \tilde{y}, a\Big|_{L_1}^{L_2}\right)\right)(t) + q_n(a; L_1, L_2), \tag{12}$$

$$(F_n^2(x,y,a))(t) = \left(F\left(x^*, \tilde{y}, a\Big|_{L_1}^{L_2}\right)\right)(t) - q_n(a; L_1, L_2), \tag{13}$$

$$\begin{aligned} p_{1n}^1(u,v,w,z) &= \varepsilon p_1\left(u\Big|_{M_1-L_2T}^{M_2-L_1T}, v\Big|_{L_1}^{L_2}, w\Big|_{M_1}^{M_2}, z\Big|_{L_1}^{L_2}\right) - q_n(v; L_1, L_2), \\ p_{2n}^1(u,v,w,z) &= \delta p_2\left(u\Big|_{M_1-L_2T}^{M_2-L_1T}, v\Big|_{L_1}^{L_2}, w\Big|_{M_1}^{M_2}, z\Big|_{L_1}^{L_2}\right) - q_n(w; M_1, M_2), \end{aligned} \tag{14}$$

$$\begin{aligned} p_{1n}^2(u,v,w,z) &= \varepsilon p_1\left(u\Big|_{M_1}^{M_2}, v\Big|_{L_1}^{L_2}, w\Big|_{M_1+L_1T}^{M_2+L_2T}, z\Big|_{L_1}^{L_2}\right) - q_n(u; M_1, M_2), \\ p_{2n}^2(u,v,w,z) &= \delta p_2\left(u\Big|_{M_1}^{M_2}, v\Big|_{L_1}^{L_2}, w\Big|_{M_1+L_1T}^{M_2+L_2T}, z\Big|_{L_1}^{L_2}\right) - q_n(z; L_1, L_2). \end{aligned} \tag{15}$$

Consider BVPs (for $n \in \mathbb{N}$)

$$(g(x'(t)))' = \lambda(F_n^1(x,x',x'(t)))(t), \quad \lambda \in [0,1], \tag{16_n)$_\lambda$}$$

$$p_{1n}^1(x(0),x'(0),x(T),x'(T)) = 0, \qquad p_{2n}^1(x(0),x'(0),x(T),x'(T)) = 0 \tag{17_n}$$

and

$$(g(x'(t)))' = \lambda(F_n^2(x,x',x'(t)))(t), \quad \lambda \in [0,1], \tag{18_n)$_\lambda$}$$

$$p_{1n}^2(x(0),x'(0),x(T),x'(T)) = 0, \qquad p_{2n}^2(x(0),x'(0),x(T),x'(T)) = 0 \tag{19_n}$$

depending on the parameters λ and n.

Lemma 1 (A priori estimates) *Let assumption* (H_1) *be satisfied and* u *be a solution of BVP* $(16_n)_\lambda, (17_n)$ *for some* $\lambda \in [0,1]$ *and* $n \in \mathbb{N}$. *Then for* $t \in J$,

$$\begin{aligned} M_1 - (L_2 + \tfrac{1}{n})T < u(t) < M_2 - (L_1 - \tfrac{1}{n})T, \\ L_1 - \tfrac{1}{n} < u'(t) < L_2 + \tfrac{1}{n}, \\ M_1 \le u(T) \le M_2. \end{aligned} \tag{20}$$

Proof Assume $u'(0) > L_2$. Then

$$\begin{aligned} p_{1n}^1(u(0), u'(0), u(T), u'(T)) &= \varepsilon p_1 \left(u(0)\Big|_{M_1 - L_2T}^{M_2 - L_1T}, L_2, u(T)\Big|_{M_1}^{M_2}, u'(T)\Big|_{L_1}^{L_2} \right) \\ - q_n(u'(0); L_1, L_2) &\ge -q_n(u'(0); L_1, L_2) = -\max\{L_2 - u'(0), -\tfrac{1}{n}\} > 0, \end{aligned}$$

which contradicts $p_{1n}^1(u(0), u'(0), u(T), u'(T)) = 0$. If $u'(0) < L_1$, then

$$\begin{aligned} p_{1n}^1(u(0), u'(0), u(T), u'(T)) &= \varepsilon p_1 \left(u(0)\Big|_{M_1 - L_2T}^{M_2 - L_1T}, L_1, u(T)\Big|_{M_1}^{M_2}, u'(T)\Big|_{L_1}^{L_2} \right) \\ - q_n(u'(0); L_1, L_2) &\ge -q_n(u'(0); L_1, L_2) = -\min\{L_1 - u'(0), \tfrac{1}{n}\} < 0, \end{aligned}$$

which contradicts $p_{1n}^1(u(0), u'(0), u(T), u'(T)) = 0$. Hence

$$L_1 \le u'(0) \le L_2. \tag{21}$$

If $\lambda = 0$ then $(g(u'(t)))' = 0$, and so $u'(t) = S$ for $t \in J$, where S is a constant, $L_1 \le S \le L_2$ (see (21)).

Let $\lambda \in (0,1]$. Assume $u'(\xi) = \max\{u'(t); t \in J\} \ge L_2 + \frac{1}{n}$ for a $\xi \in J$. Then (cf. (21)) $\xi \in (0,T]$ and there exist $t_0 \in (0,T)$ and $\nu_0 > 0$ such that $u'(t_0) = L_2$, $u'(t_0 + \nu_0) = L_2 + \frac{1}{n}$ and $L_2 < u'(t) < L_2 + \frac{1}{n}$ for $t \in (t_0, t_0 + \nu_0)$. Integrating the equality

$$(g(u'(t)))' = \lambda (F_n^1(u, u', u'(t)))(t) \quad \text{for a.e. } t \in J \tag{22}$$

from t_0 to $t_0 + \nu_0$ we obtain (cf. (3), (11) and (12))

$$\begin{aligned} g(u'(t_0 + \nu_0)) - g(u'(t_0)) &= \lambda \int_{t_0}^{t_0+\nu_0} (F_n^1(u, u', u'(t)))(t)\, dt \\ = \lambda \int_{t_0}^{t_0+\nu_0} (F(\bar{u}, \widetilde{u'}, L_2))(t)\, dt &+ \lambda \int_{t_0}^{t_0+\nu_0} q_n(u'(t); L_1, L_2)\, dt \\ \le \lambda \int_{t_0}^{t_0+\nu_0} q_n(u'(t); L_1, L_2)\, dt &= \lambda \int_{t_0}^{t_0+\nu_0} (L_2 - u'(t))\, dt < 0, \end{aligned}$$

contrary to $g(u'(t_0 + \nu_0)) - g(u'(t_0)) = g(L_2 + \frac{1}{n}) - g(L_2) > 0$.

Assume $u'(\varrho) = \min\{u'(t);\, t \in J\} \le L_1 - \frac{1}{n}$ for a $\varrho \in J$. Then (cf. (21)) $\varrho \in (0,T]$ and there exist $t_1 \in (0,T)$ and $\nu_1 > 0$ such that $u'(t_1) = L_1$, $u'(t_0 + \nu_1) = L_1 - \frac{1}{n}$ and $L_1 - \frac{1}{n} < u'(t) < L_1$ for $t \in (t_1, t_1 + \nu_1)$. Integrating (22) from t_1 to $t_1 + \nu_1$ we have (cf. (3), (11) and (12))

$$g(u'(t_1+\nu_1)) - g(u'(t_1)) = \lambda \int_{t_1}^{t_1+\nu_1} (F_n^1(u, u', u'(t)))(t)\, dt$$

$$= \lambda \int_{t_1}^{t_1+\nu_1} (F(\bar{u}, \widetilde{u'}, L_1))(t)\, dt + \lambda \int_{t_1}^{t_1+\nu_1} q_n(u'(t); L_1, L_2)\, dt$$

$$\ge \lambda \int_{t_1}^{t_1+\nu_1} q_n(u'(t); L_1, L_2)\, dt = \lambda \int_{t_0}^{t_0+\nu_0} (L_1 - u'(t))\, dt > 0,$$

contrary to $g(u'(t_1+\nu_1)) - g(u'(t_1)) = g(L_1 - \frac{1}{n}) - g(L_1) < 0$.

We have proved

$$L_1 - \tfrac{1}{n} < u'(t) < L_2 + \tfrac{1}{n}, \quad t \in J. \tag{23}$$

Assume $u(t) > M_2$. Then (cf. (5), (14) and (21))

$$p_{2n}^1(u(0), u'(0), u(T), u'(T)) = \delta p_2 \left(u(0)\Big|_{M_1 - L_2 T}^{M_2 - L_1 T}, u'(0), M_2, u'(T)\Big|_{L_1}^{L_2} \right)$$

$$- q_n(u(T); M_1, M_2) \ge -q_n(u(T); M_1, M_2) = -\max\{M_2 - u(T), -\tfrac{1}{n}\} > 0,$$

which contradicts $p_{2n}^1(u(0), u'(0), u(T), u'(T)) = 0$.

If $u(T) < M_1$, then

$$p_{2n}^1(u(0), u'(0), u(T), u'(T)) = \delta p_2 \left(u(0)\Big|_{M_1 - L_2 T}^{M_2 - L_1 T}, u'(0), M_1, u'(T)\Big|_{L_1}^{L_2} \right)$$

$$- q_n(u(T); M_1, M_2) \le -q_n(u(T); M_1, M_2) = -\min\{M_1 - u(T), \tfrac{1}{n}\} < 0,$$

which is impossible. Hence

$$M_1 \le u(T) \le M_2. \tag{24}$$

From (23) and (24) it follows

$$\begin{aligned} u(t) &= u(T) - \textstyle\int_t^T u'(s)\, ds \le M_2 - (L_1 - \tfrac{1}{n})(T-t) \le M_2 - (L_1 - \tfrac{1}{n})T, \\ u(t) &= u(T) - \textstyle\int_t^T u'(s)\, ds \ge M_1 - (L_2 + \tfrac{1}{n})(T-t) \ge M_1 - (L_2 + \tfrac{1}{n})T \end{aligned} \tag{25}$$

for $t \in J$. Thus (cf. (23)–(25)) inequalities (20) are satisfied. □

Lemma 2 (A priori estimates) *Let assumption (H_2) be satisfied and u be a solution of BVP $(18_n)_\lambda$, (19_n) for some $\lambda \in [0,1]$ and $n \in \mathbb{N}$. Then for $t \in J$,*

$$\begin{gathered} M_1 + (L_1 - \tfrac{1}{n})T < u(t) < M_2 + (L_2 + \tfrac{1}{n})T, \\ L_1 - \tfrac{1}{n} < u'(t) < L_2 + \tfrac{1}{n}, \\ M_1 \le u(0) \le M_2. \end{gathered} \tag{26}$$

Proof Assume $u'(T) > L_2$. Then

$$p_{2n}^2(u(0), u'(0), u(T), u'(T)) = \delta p_2 \left(u(0)\Big|_{M_1}^{M_2}, u'(0)\Big|_{L_1}^{L_2}, u(T)\Big|_{M_1+L_1T}^{M_2+L_2T}, L_2 \right)$$
$$-q_n(u'(T); L_1, L_2) \ge -q_n(u'(T); L_1, L_2) = -\max\{L_2 - u'(T), -\tfrac{1}{n}\} > 0,$$

which contradicts $p_{2n}^2(u(0), u'(0), u(T), u'(T)) = 0$. Let $u'(T) < L_1$. Then

$$p_{2n}^2(u(0), u'(0), u(T), u'(T)) = \delta p_2 \left(u(0)\Big|_{M_1}^{M_2}, u'(0)\Big|_{L_1}^{L_2}, u(T)\Big|_{M_1+L_1T}^{M_2+L_2T}, L_1 \right)$$
$$-q_n(u'(T); L_1, L_2) \le -q_n(u'(T); L_1, L_2) = -\min\{L_1 - u'(T), \tfrac{1}{n}\} < 0,$$

which is impossible. Thus

$$L_1 \le u'(T) \le L_2. \tag{27}$$

If $\lambda = 0$ then $u'(t) = V$ for $t \in J$, where V is a constant, $V \in [L_1, L_2]$ (see (27)). Let $\lambda \in (0,1]$. Assume $u'(\xi) = \max\{u'(t); t \in J\} \ge L_2 + \frac{1}{n}$ for some $\xi \in J$. Then (cf. (27)) $\xi \in [0,T)$ and there exist $t_0 \in [\xi, T)$ and $\varepsilon_0 > 0$ such that $u'(t_0) = L_2 + \frac{1}{n}$, $u'(t_0 + \varepsilon_0) = L_2$ and $L_2 < u'(t) < L_2 + \frac{1}{n}$ for $t \in (t_0, t_0 + \varepsilon_0)$. Integrating the equality

$$(g(u'(t)))' = \lambda(F_n^2(u, u', u'(t)))(t) \quad \text{for a.e. } t \in J \tag{28}$$

over $[t_0, t_0 + \varepsilon_0]$ we obtain (cf. (6), (11) and (13))

$$\begin{aligned} g(u'(t_0 + \varepsilon_0)) - g(u'(t_0)) &= \lambda \int_{t_0}^{t_0+\nu_0} (F_n^2(u, u', u'(t)))(t)\, dt \\ = \lambda \int_{t_0}^{t_0+\nu_0} (F(u^*, \widetilde{u'}, L_2))(t)\, dt &- \lambda \int_{t_0}^{t_0+\varepsilon_0} q_n(u'(t); L_1, L_2)\, dt \\ \ge -\lambda \int_{t_0}^{t_0+\varepsilon_0} q_n(u'(t); L_1, L_2)\, dt &= -\lambda \int_{t_0}^{t_0+\varepsilon_0} (L_2 - u'(t))\, dt > 0, \end{aligned}$$

contrary to $g(u'(t_0 + \varepsilon_0)) - g(u'(t_0)) = g(L_2) - g(L_2 + \frac{1}{n}) < 0$.

Assume $u'(\nu) = \min\{u'(t); t \in J\} \le L_1 - \frac{1}{n}$ for a $\nu \in J$. Then (cf. (27)) $\nu \in [0,T)$ and there exist $t_1 \in [\nu, T)$ and $\varepsilon_1 > 0$ such that $u'(t_1) = L_1 - \frac{1}{n}$, $u'(t_1 + \varepsilon_1) = L_1$ and $L_1 - \frac{1}{n} < u'(t) < L_1$ for $t \in (t_1, t_1 + \varepsilon_1)$. Integrating (28) from t_1 to $t_1+\varepsilon_1$ we have (cf. (6), (11) and (13))

$$g(u'(t_1+\varepsilon_1))-g(u'(t_1))=\lambda\int_{t_1}^{t_1+\varepsilon_1}(F_n^2(u,u',u'(t)))(t)\,dt$$

$$=\lambda\int_{t_1}^{t_1+\varepsilon_1}(F(u^*,\widetilde{u'},L_1))(t)\,dt-\lambda\int_{t_1}^{t_1+\varepsilon_1}q_n(u'(t);L_1,L_2)\,dt$$

$$\le-\lambda\int_{t_1}^{t_1+\varepsilon_1}q_n(u'(t);L_1,L_2)\,dt=-\lambda\int_{t_0}^{t_0+\varepsilon_0}(L_1-u'(t))\,dt<0,$$

contrary to $g(u'(t_1+\varepsilon_1))-g(u'(t_1))=g(L_1)-g(L_1-\frac{1}{n})>0$.

Summarizing, (23) is satisfied.

Assume $u(0)>M_2$. Then (cf. (7), (15) and (27))

$$p_{1n}^2(u(0),u'(0),u(T),u'(T))=\varepsilon p_1\left(M_2,u'(0)\Big|_{L_1}^{L_2},u(T)\Big|_{M_1+L_1T}^{M_2+L_2T},u'(T)\right)$$

$$-q_n(u(0);M_1,M_2)\ge-q_n(u(0);M_1,M_2)=-\max\{M_2-u(0),-\tfrac{1}{n}\}>0,$$

which is impossible. If $u(0)<M_1$ then (cf. (7), (15) and (27))

$$p_{1n}^2(u(0),u'(0),u(T),u'(T))=\varepsilon p_1\left(M_1,u'(0)\Big|_{L_1}^{L_2},u(T)\Big|_{M_1+L_1T}^{M_2+L_2T},u'(T)\right)$$

$$-q_n(u(0);M_1,M_2)\le-q_n(u(0);M_1,M_2)=-\min\{M_1-u(T),\tfrac{1}{n}\}<0,$$

a contradiction. Hence

$$M_1\le u(0)\le M_2, \tag{29}$$

and so (cf. (23))

$$\begin{aligned}u(t)&=u(0)+\textstyle\int_0^t u'(s)\,ds\le M_2+(L_2+\frac{1}{n})t\le M_2+(L_2+\frac{1}{n})T,\\ u(t)&=u(0)+\textstyle\int_0^t u'(s)\,ds\ge M_1+(L_1-\frac{1}{n})t\ge M_1+(L_1-\frac{1}{n})T\end{aligned} \tag{30}$$

for $t\in J$. Inequalities (26) follow from (23), (29) and (30). □

Lemma 3 *Let assumption* (H_1) *be satisfied and* $n\in\mathbb{N}$. *Then BVP* $(16_n)_1$, (17_n) *has a solution* $u(t)$ *satisfying the inequalities* (20).

Proof Let

$$\begin{aligned}L=\max\{-L_1,\,L_2\},\quad K=(L+1)T+\max\{|M_1|,\,|M_2|\},\\ G(v)=\max\{-g(-v),\,g(v)\}\quad\text{for } v\in[0,\infty).\end{aligned} \tag{31}$$

Then K,L are positive constants and

$$|g(v)|\le G(|v|),\quad v\in\mathbb{R}. \tag{32}$$

Set

$$\begin{aligned}\Omega = \{(x,y,z,a,b);\ &(x,y,z,a,b) \in C^0(J) \times C^0(J) \times C^0(J) \times \mathbb{R}^2,\\ \|x\| < K+1,\ \|y\| < L+1,\ &\|z\| < L+1,\ |a| < K+1,\ |b| < G(L+1)\}\end{aligned} \tag{33}$$

and define the operator $W : [0,3] \times \bar{\Omega} \to C^0(J) \times C^0(J) \times C^0(J) \times \mathbb{R}^2$ by

$$W(\lambda,x,y,z,a,b) = \begin{cases} \lambda\Big(a + g^{-1}(b)t, g^{-1}(b), g^{-1}(b), 0, 0\Big) & \text{for } \lambda \in [0,1] \\ \Big(a + g^{-1}(b)t, g^{-1}(b), g^{-1}(b), \\ (\lambda-1)(a - p^1_{2n}(x(0), y(0), x(T), y(T))), 0\Big) & \text{for } \lambda \in (1,2] \\ \Big(a + g^{-1}(b)t, g^{-1}(b), g^{-1}(b), \\ a - p^1_{2n}(x(0), y(0), x(T), y(T)), \\ (\lambda-2)(b - p^1_{1n}(x(0), y(0), x(T), y(T)))\Big) & \text{for } \lambda \in (2,3]. \end{cases}$$

We now show that

$$\mathrm{D}(I - W(3,\cdot,\cdot,\cdot,\cdot,\cdot), \Omega, 0) = 1, \tag{34}$$

where "D" denotes the Leray–Schauder degree and I is the identity operator on $C^0(J) \times C^0(J) \times C^0(J) \times \mathbb{R}^2$. Since $W(0,x,y,z,a,b) = (0,0,0,0,0)$ for $(x,y,z,a,b) \in \bar{\Omega}$, and so $\mathrm{D}(I - W(0,\cdot,\cdot,\cdot,\cdot,\cdot),\Omega,0) = \mathrm{D}(I,\Omega,0) = 1$, to prove (34) is suffices to verify, by the degree theory, that

(i) W is a compact operator, and

(ii) $W(\lambda,x,y,z,a,b) \neq (x,y,z,a,b)$ for $(\lambda,x,y,z,a,b) \in [0,3] \times \partial\Omega$.

From the continuity of $g^{-1}, p^1_{1n}, p^1_{2n}$ and the Bolzano–Weierstrass theorem it may be concluded that W is a compact operator. We verify the property (ii) of W. Assume

$$W(\lambda_0, x_0, y_0, z_0, a_0, b_0) = (x_0, y_0, z_0, a_0, b_0)$$

for some $(\lambda_0, x_0, y_0, z_0, a_0, b_0) \in [0,1] \times \partial\Omega$. Then

$$\begin{gathered} x_0(t) = \lambda_0(a_0 + g^{-1}(b_0)t), \\ (x_0'(t) =)\ y_0(t) = z_0(t) = \lambda_0 g^{-1}(b_0), \quad a_0 = b_0 = 0, \end{gathered}$$

and consequently $x_0(t) = y_0(t) = z_0(t) = 0$ for $t \in J$. Thus $(x_0, y_0, z_0, a_0, b_0) = (0,0,0,0,0) \notin \partial\Omega$, a contradiction.

Let

$$W(\lambda_1, x_1, y_1, z_1, a_1, b_1) = (x_1, y_1, z_1, a_1, b_1)$$

for some $(\lambda_1, x_1, y_1, z_1, a_1, b_1) \in (1,2] \times \partial\Omega$. Then

$$\begin{gathered} x_1(t) = a_1 + g^{-1}(b_1)t, \quad (x_1'(t) =)\ y_1(t) = z_1(t) = g^{-1}(b_1), \\ a_1 = (\lambda_1 - 1)(a_1 - p^1_{2n}(x_1(0), y_1(0), x_1(T), y_1(T))), \quad b_1 = 0, \end{gathered}$$

and so

$$x_1(t) = a_1, \qquad y_1(t) = z_1(t) = 0$$

for $t \in J$, $a_1(2-\lambda_1) = (1-\lambda_1)p_{2n}^1(a_1, 0, a_1, 0)$. Hence (cf. (14))

$$a_1(2-\lambda_1) = (1-\lambda_1)\left[\delta p_2\Big(a_1\Big|_{M_1-L_2T}^{M_2-L_1T}, 0, a_1\Big|_{M_1}^{M_2}, 0\Big) - q_n(a_1; M_1, M_2)\right].$$

If $a_1 > \max\{0, M_2\}$ then $a_1(2-\lambda_1) \geq 0$, which contradicts (cf. (5))

$$(1-\lambda_1)\left[\delta p_2\Big(a_1\Big|_{M_1-L_2T}^{M_2-L_1T}, 0, a_1\Big|_{M_1}^{M_2}, 0\Big) - q_n(a_1; M_1, M_2)\right]$$

$$= (1-\lambda_1)\left[\delta p_2\Big(a_1\Big|_{M_1-L_2T}^{M_2-L_1T}, 0, M_2, 0\Big) - \max\{M_2 - a_1, -\tfrac{1}{n}\}\right] < 0.$$

If $a_1 < \min\{0, M_1\}$ then $a_1(2-\lambda_1) \leq 0$, which contradicts (cf. (5))

$$(1-\lambda_1)\left[\delta p_2\Big(a_1\Big|_{M_1-L_2T}^{M_2-L_1T}, 0, a_1\Big|_{M_1}^{M_2}, 0\Big) - q_n(a_1; M_1, M_2)\right]$$

$$= (1-\lambda_1)\left[\delta p_2\Big(a_1\Big|_{M_1-L_2T}^{M_2-L_1T}, 0, M_1, 0\Big) - \min\{M_1 - a_1, \tfrac{1}{n}\}\right] > 0.$$

Thus $\min\{0, M_1\} \leq a_1 \leq \max\{0, M_2\}$, and consequently

$$|a_1| \leq \max\{|M_1|, |M_2|\} < K$$

which yields $(x_1, y_1, z_1, a_1, b_1) = (a_1, 0, 0, a_1, 0) \notin \partial\Omega$, a contradiction.

Let

$$W(\lambda_2, x_2, y_2, z_2, a_2, b_2) = (x_2, y_2, z_2, a_2, b_2)$$

for some $(\lambda_2, x_2, y_2, z_2, a_2, b_2) \in (2, 3] \times \partial\Omega$. Then

$$x_2(t) = a_2 + g^{-1}(b_2)t, \tag{35}$$

$$(x_2'(t) =)\ y_2(t) = z_2(t) = g^{-1}(b_2), \tag{36}$$

$$p_{2n}^1(x_2(0), y_2(0), x_2(T), y_2(T)) = 0, \tag{37}$$

$$b_2 = (\lambda_2 - 2)(b_2 - p_{1n}^1(x_2(0), y_2(0), x_2(T), y_2(T))). \tag{38}$$

From (14), (35), (36) and (38) we deduce that

$$b_2(3-\lambda_2) = (2-\lambda_2)p_{1n}^1(a_2, g^{-1}(b_2), a_2 + g^{-1}(b_2)T, g^{-1}(b_2))$$

$$= (2-\lambda_2)\Big[\varepsilon p_1\Big(a_2\Big|_{M_1-L_2T}^{M_2-L_1T}, g^{-1}(b_2)\Big|_{L_1}^{L_2}, (a_2 + g^{-1}(b_2)T)\Big|_{M_1}^{M_2}, g^{-1}(b_2)\Big|_{L_1}^{L_2}\Big)$$

$$- q_n(g^{-1}(b_2); L_1, L_2)\Big].$$

Assume $b_2 > g(L_2)$, resp. $b_2 < g(L_1)$. Then $b_2(3-\lambda_2) \geq 0$ (resp. $b_2(3-\lambda_2) \leq 0$), which contradicts (cf. (4) and (14))

$$\varepsilon p_1\left(a_2\Big|_{M_1-L_2T}^{M_2-L_1T}, g^{-1}(b_2)\Big|_{L_1}^{L_2}, (a_2+g^{-1}(b_2)T)\Big|_{M_1}^{M_2}, g^{-1}(b_2)\Big|_{L_1}^{L_2}\right) - q_n(g^{-1}(b_2); L_1, L_2)$$

$$= \varepsilon p_1\left(a_2\Big|_{M_1-L_2T}^{M_2-L_1T}, L_2, (a_2+g^{-1}(b_2)T)\Big|_{M_1}^{M_2}, L_2\right) - \max\{L_2 - g^{-1}(b_2), -\tfrac{1}{n}\} > 0\,,$$

resp.

$$\varepsilon p_1\left(a_2\Big|_{M_1-L_2T}^{M_2-L_1T}, g^{-1}(b_2)\Big|_{L_1}^{L_2}, (a_2+g^{-1}(b_2)T)\Big|_{M_1}^{M_2}, g^{-1}(b_2)\Big|_{L_1}^{L_2}\right) - q_n(g^{-1}(b_2); L_1, L_2)$$

$$= \varepsilon p_1\left(a_2\Big|_{M_1-L_2T}^{M_2-L_1T}, L_1, (a_2+g^{-1}(b_2)T)\Big|_{M_1}^{M_2}, L_1\right) - \min\{L_1 - g^{-1}(b_2), \tfrac{1}{n}\} < 0.$$

Hence

$$L_1 \leq g^{-1}(b_2) \leq L_2. \tag{39}$$

If $a_2 + g^{-1}(b_2)T > M_2$ (resp. $a_2 + g^{-1}(b_2)T < M_1$) then (5), (11), (14), (37) and (39) imply

$$0 = p_{2n}^1(x_2(0), y_2(0), x_2(T), y_2(T))$$

$$= \delta p_2\left(a_2\Big|_{M_1-L_2T}^{M_2-L_1T}, g^{-1}(b_2)\Big|_{L_1}^{L_2}, (a_2+g^{-1}(b_2)T)\Big|_{M_1}^{M_2}, g^{-1}(b_2)\Big|_{L_1}^{L_2}\right) - q_n(a_2 + g^{-1}(b_2)T; M_1, M_2)$$

$$= \delta p_2\left(a_2\Big|_{M_1-L_2T}^{M_2-L_1T}, g^{-1}(b_2), M_2, g^{-1}(b_2)\right) - \max\{M_2 - a_2 - g^{-1}(b_2)T, -\tfrac{1}{n}\} > 0\,,$$

resp.

$$0 = p_{2n}^1(x_2(0), y_2(0), x_2(T), y_2(T))$$

$$= \delta p_2\left(a_2\Big|_{M_1-L_2T}^{M_2-L_1T}, g^{-1}(b_2), M_1, g^{-1}(b_2)\right) - \min\{M_1 - a_2 - g^{-1}(b_2)T, \tfrac{1}{n}\} < 0,$$

which is impossible. Thus

$$M_1 \leq a_2 + g^{-1}(b_2)T \leq M_2, \tag{40}$$

and so (cf. (32), (35), (36), (39) and (40))

$$|a_2| \leq K, \qquad |b_2| \leq G(L),$$

$$|x_2(t)| \le \max\{|M_2 - g^{-1}(b_2)(T-t)|, |M_1 - g^{-1}(b_2)(T-t)|\}$$
$$\le \max\{|M_2| + LT, |M_1| + LT\} < K,$$
$$|y_2(t)| = |z_2(t)| \le L, \quad t \in J,$$

which contradicts $(x_2, y_2, z_2, a_2, b_2) \in \partial\Omega$.

We have verified that the properties (i) and (ii) of the operator W are satisfied, and so (34) holds.

Let now the operator $Z : [0,1] \times \bar{\Omega} \to C^0(J) \times C^0(J) \times C^0(J) \times \mathbb{R}^2$ be given by the formula

$$Z(\lambda, x, y, z, a, b) = \Bigg(a + \int_0^t g^{-1}\Big(b + \lambda \int_0^s (F_n^1(x, y, z(\nu)))(\nu)\, d\nu\Big)\, ds,$$

$$g^{-1}\Big(b + \lambda \int_0^t (F_n^1(x, y, z(s)))(s)\, ds\Big),\ g^{-1}\Big(b + \lambda \int_0^t (F_n^1(x, y, z(s)))(s)\, ds\Big),$$

$$a - p_{2n}^1(x(0), y(0), x(T), y(T)),\ b - p_{1n}^1(x(0), y(0), x(T), y(T))\Bigg).$$

We see that $Z(0, \cdot, \cdot, \cdot, \cdot, \cdot) = W(3, \cdot, \cdot, \cdot, \cdot, \cdot)$. Moreover, if $(x, y, z, a, b) \in \bar{\Omega}$ is a fixed point of the operator $Z(\lambda, \cdot, \cdot, \cdot, \cdot, \cdot)$ with a $\lambda \in [0,1]$, then x is a solution of BVP $(16_n)_\lambda$, (17_n) and $y = z = x'$, $a = x(0)$, $b = g(x'(0))$, and conversely, if x is a solution of BVP $(16_n)_\lambda$, (17_n) with a $\lambda \in [0,1]$ and $(x, x', x', x(0), g(x'(0))) \in \bar{\Omega}$, then $(x, x', x', x(0), g(x'(0)))$ is a fixed point of $Z(\lambda, \cdot, \cdot, \cdot, \cdot, \cdot)$. By Lemma 1, the inequalities (20) are satisfied for any solution u of BVP $(16_n)_\lambda$, (17_n) with $\lambda \in [0,1]$, and so $Z(\lambda, x, y, z, a, b) \neq (x, y, z, a, b)$ for any $(\lambda, x, y, z, a, b) \in [0,1] \times \partial\Omega$.

The proof is completed by showing that Z is a compact operator. In this case we have

$$\mathrm{D}(I - Z(1, \cdot, \cdot, \cdot, \cdot, \cdot), \Omega, 0) = \mathrm{D}(I - W(3, \cdot, \cdot, \cdot, \cdot, \cdot), \Omega, 0) \quad (= 1 \text{ by } (34)).$$

From the assumptions imposed on F, g, p_1 and p_2 it follows that Z is a continuous operator. Let $\{(\lambda_j, x_j, y_j, z_j, a_j, b_j)\} \subset [0,1] \times \bar{\Omega}$ and set

$$(u_j, v_j, w_j, A_j, B_j) = Z(\lambda_j, x_j, y_j, z_j, a_j, b_j), \quad j \in \mathbb{N}.$$

Then (for $j \in \mathbb{N}$)

$$\begin{aligned} u_j(t) &= a_j + \textstyle\int_0^t g^{-1}(b_j + \lambda_j \int_0^s (F_n^1(x_j, y_j, z_j(\nu)))(\nu)\, d\nu)\, ds, \\ (u_j'(t) =)\ v_j(t) &= w_j(t) = g^{-1}(b_j + \lambda_j \textstyle\int_0^t (F_n^1(x_j, y_j, z_j(s)))(s)\, ds, \\ A_j &= a_j - p_{2n}^1(x_j(0), y_j(0), x_j(T), y_j(T)), \\ B_j &= b_j - p_{1n}^1(x_j(0), y_j(0), x_j(T), y_j(T)). \end{aligned} \tag{41}$$

Let

$$P(v) = \max\{-g^{-1}(-v),\, g^{-1}(v)\}, \quad v \in [0,\infty). \tag{42}$$

Then $P \in C^0([0,\infty))$ and

$$|g^{-1}(v)| \le P(|v|), \quad v \in \mathbb{R}. \tag{43}$$

By the property (c) of the operator F, there exists $k \in L_1(J)$ such that (cf. (11) and (12))

$$|(F_n^1(x_j, y_j, z_j(t)))(t) \le k(t) \quad \text{for a.e. } t \in J \text{ and each } j \in \mathbb{N}, \tag{44}$$

which yields (cf. (14), (32), (41)–(44) and the definition of the set Ω)

$$|u_j(t)| \le K + 1 + TP\Big(G(L+1) + \int_0^T k(t)\,dt\Big),$$

$$|u_j'(t)| = |v_j(t)| = |w_j(t)| \le P\Big(G(L+1) + \int_0^T k(t)\,dt\Big),$$

$$|g(v_j(t_1)) - g(v_j(t_2))| = |g(w_j(t_1)) - g(w_j(t_2))| \le \Big|\int_{t_1}^{t_2} k(t)\,dt\Big|,$$

$$|A_j| \le K + 1 + \max\{|p_2(\alpha,\beta,\gamma,\delta)|;\ |\alpha| \le K, |\beta| \le L, |\gamma| \le K, |\delta)| \le L\} + 1,$$

$$|B_j| \le G(L+1) + \max\{|p_1(\alpha,\beta,\gamma,\delta|;\ |\alpha| \le K, |\beta| \le L, |\gamma| \le K, |\delta| \le L\} + 1$$

for $t, t_1, t_2 \in J$ and $j \in \mathbb{N}$. Thus $\{u_j(t)\}$, $\{v_j(t)\}$, $\{w_j(t)\}$ are uniformly bounded and equicontinuous on J and $\{A_j\}$, $\{B_j\}$ are bounded. By the Arzelà–Ascoli theorem and the Bolzano–Weierstrass theorem, there exists a subsequence $\{(u_{j_n}, v_{j_n}, w_{j_n}, A_{j_n}, B_{j_n})\}$ converging in $C^0(J) \times C^0(J) \times C^0(J) \times \mathbb{R}^2$. Hence Z is a compact operator. This completes the proof. □

Lemma 4 *Let assumption* (H_2) *be satisfied and* $n \in \mathbb{N}$. *Then BVP* $(18_n)_1$, (19_n) *has a solution* $u(t)$ *satisfying the inequalities* (26).

Proof Let the constants K, L and the function G be defined by (31) and let the set Ω be given by (33). Define $W^* : [0,3] \times \bar\Omega \to C^0(J) \times C^0(J) \times C^0(J) \times \mathbb{R}^2$ by

$$W^*(\lambda, x, y, z, a, b) = \begin{cases} \lambda\Big(a + g^{-1}(b)t, g^{-1}(b), g^{-1}(b), 0, 0\Big) & \text{for } \lambda \in [0,1] \\ \Big(a + g^{-1}(b)t, g^{-1}(b), g^{-1}(b), \\ (\lambda - 1)(a - p_{1n}^2(x(0), y(0), x(T), y(T))), 0\Big) & \text{for } \lambda \in (1,2] \\ \Big(a + g^{-1}(b)t, g^{-1}(b), g^{-1}(b), \\ a - p_{1n}^2(x(0), y(0), x(T), y(T)), \\ (\lambda - 2)(b - p_{2n}^2(x(0), y(0), x(T), y(T)))\Big) & \text{for } \lambda \in (2,3]. \end{cases}$$

To prove that $D(I - W^*(3,\cdot,\cdot,\cdot,\cdot,\cdot),\Omega,0) = 1$ we have to verify:

(j) W^* is a compact operator, and

(jj) $W^*(\lambda,x,y,z,a,b) \neq (x,y,z,a,b)$ for $(\lambda,x,y,z,a,b) \in [0,3] \times \partial\Omega$.

It is easily seen that W^* is a compact operator and $W^*(\lambda,x,y,z,a,b) \neq (x,y,z,a,b)$ for $(\lambda,x,y,z,a,b) \in [0,1] \times \partial\Omega$ (see the proof of Lemma 3).

Let

$$W^*(\lambda_1,x_1,y_1,z_1,a_1,b_1) = (x_1,y_1,z_1,a_1,b_1)$$

for some $(\lambda_1,x_1,y_1,z_1,a_1,b_1) \in (1,2] \times \partial\Omega$. Then

$$x_1(t) = a_1(t) + g^{-1}(b_1)t, \quad (x_1'(t) =)\ y_1(t) = z_1(t) = g^{-1}(b_1),$$
$$a_1 = (\lambda_1 - 1)(a_1 - p_{1n}^2(x_1(0),y_1(0),x_1(T),y_1(T))), \quad b_1 = 0,$$

and consequently $x_1(t) = a_1$, $y_1(t) = z_1(t) = 0$ for $t \in J$,

$$a_1(2-\lambda_1) = (1-\lambda_1)p_{1n}^2(a_1,0,a_1,0).$$

From the last equality we see that (cf. (15))

$$a_1(2-\lambda_1) = (1-\lambda_1)\left[\varepsilon p_1\left(a_1\Big|_{M_1}^{M_2},0,a_1\Big|_{M_1+L_1T}^{M_2+L_2T},0\right) - q_n(a_1;M_1,M_2)\right].$$

If $a_1 > \max\{0,M_2\}$ then $a_1(2-\lambda_1) \geq 0$, which contradicts (cf. (7))

$$(1-\lambda_1)\left[\varepsilon p_1\left(a_1\Big|_{M_1}^{M_2},0,a_1\Big|_{M_1+L_1T}^{M_2+L_2T},0\right) - q_n(a_1;M_1,M_2)\right]$$
$$= (1-\lambda_1)\left[\varepsilon p_1\left(M_2,0,a_1\Big|_{M_1+L_1T}^{M_2+L_2T},0\right) - \max\{M_2 - a_1, -\tfrac{1}{n}\}\right] < 0.$$

If $a_1 < \min\{0,M_1\}$ then $a_1(2-\lambda_1) \leq 0$, which contradicts (cf. (7))

$$(1-\lambda_1)\left[\varepsilon p_1\left(a_1\Big|_{M_1}^{M_2},0,a_1\Big|_{M_1+L_1T}^{M_2+L_2T},0\right) - q_n(a_1;M_1,M_2)\right]$$
$$= (1-\lambda_1)\left[\varepsilon p_1\left(M_1,0,a_1\Big|_{M_1+L_1T}^{M_2+L_2T},0\right) - \min\{M_1 - a_1, \tfrac{1}{n}\}\right] > 0.$$

Hence $\min\{0,M_1\} \leq a_1 \leq \max\{0,M_2\}$, and consequently

$$|a_1| \leq \max\{|M_1|,|M_2|\} < K$$

which yields $(x_1,y_1,z_1,a_1,b_1) = (a_1,0,0,a_1,0) \notin \partial\Omega$, a contradiction.

Let

$$W^*(\lambda_2,x_2,y_2,z_2,a_2,b_2) = (x_2,y_2,z_2,a_2,b_2)$$

for some $(\lambda_2,x_2,y_2,z_2,a_2,b_2) \in (2,3] \times \partial\Omega$. Then

$$x_2(t) = a_2 + g^{-1}(b_2)t, \tag{45}$$

$$(x_2'(t) =)\ y_2(t) = z_2(t) = g^{-1}(b_2), \tag{46}$$

$$p_{1n}^2(x_2(0), y_2(0), x_2(T), y_2(T)) = 0, \tag{47}$$

$$b_2 = (\lambda_2 - 2)(b_2 - p_{2n}^2(x_2(0), y_2(0), x_2(T), y_2(T))) \tag{48}$$

and from (15), (45), (46) and (48) we conclude that

$$\begin{aligned} b_2(3-\lambda_2) &= (2-\lambda_2)p_{2n}^2(a_2, g^{-1}(b_2), a_2 + g^{-1}(b_2)T, g^{-1}(b_2)) \\ &= (2-\lambda_2)\Big[\delta p_2\Big(a_2\Big|_{M_1}^{M_2}, g^{-1}(b_2)\Big|_{L_1}^{L_2}, (a_2 + g^{-1}(b_2)T)\Big|_{M_1+L_1T}^{M_2+L_2T}, g^{-1}(b_2)\Big|_{L_1}^{L_2}\Big) \\ &\qquad - q_n(g^{-1}(b_2); L_1, L_2)\Big]. \end{aligned} \tag{49}$$

If $b_2 > g(L_2)$, resp. $b_2 < g(L_1)$. Then $b_2(3-\lambda_2) \geq 0$, resp. $b_2(3-\lambda_2) \leq 0$, which contradicts (cf. (8), (11) and (49))

$$\begin{aligned} &(2-\lambda_2)p_{2n}^2(a_2, g^{-1}(b_2), a_2 + g^{-1}(b_2)T, g^{-1}(b_2)) \\ &= (2-\lambda_2)\Big[\delta p_2\Big(a_2\Big|_{M_1}^{M_2}, L_2, (a_2 + g^{-1}(b_2)T)\Big|_{M_1+L_1T}^{M_2+L_2T}, L_2\Big) \\ &\qquad - \max\{L_2 - g^{-1}(b_2), -\tfrac{1}{n}\}\Big] < 0, \end{aligned}$$

resp.

$$\begin{aligned} &(2-\lambda_2)p_{2n}^2\Big(a_2, g^{-1}(b_2), a_2 + g^{-1}(b_2)T, g^{-1}(b_2)\Big) \\ &= (2-\lambda_2)\Big[\delta p_2\Big(a_2\Big|_{M_1}^{M_2}, L_1, (a_2 + g^{-1}(b_2)T)\Big|_{M_1+L_1T}^{M_2+L_2T}, L_1\Big) \\ &\qquad - \min\{L_1 - g^{-1}(b_2), \tfrac{1}{n}\}\Big] > 0. \end{aligned}$$

Hence the inequalities (39) are satisfied.

Assume $a_2 > M_2$, resp. $a_2 < M_1$. From (7), (11), (15), (39) and (47) we deduce

$$\begin{aligned} 0 &= p_{1n}^2(x_2(0), y_2(0), x_2(T), y_2(t)) \\ &= \varepsilon p_1\left(M_2, g^{-1}(b_2), (a_2 + g^{-1}(b_2)T)\Big|_{M_1+L_1T}^{M_2+L_2T}, g^{-1}(b_2)\right) - q_n(a_2; M_1, M_2) \\ &\geq -\max\{M_2 - a_2, -\tfrac{1}{n}\} > 0, \end{aligned}$$

a contradiction, resp.

$$0 = p_{1n}^2(x_2(0), y_2(0), x_2(T), y_2(t))$$
$$= \varepsilon p_1\left(M_1, g^{-1}(b_2), (a_2 + g^{-1}(b_2)T)\Big|_{M_1+L_1T}^{M_2+L_2T}, g^{-1}(b_2)\right) - q_n(a_2; M_1, M_2)$$
$$\le -\min\{M_1 - a_2, \tfrac{1}{n}\} < 0,$$

a contradiction.

Therefore

$$M_1 \le a_2 \le M_2, \tag{50}$$

and consequently (cf. (32), (39), (45), (46) and (50))

$$|a_2| \le \max\{|M_1|, |M_2|\} < K, \ |b_2| \le G(L), \quad x_2(t)| \le K, \ |y_2(t)| = |z_2(t)| \le L$$

for $t \in J$, which contradicts $(x_2, y_2, z_2, a_2, b_2) \in \partial\Omega$. We have verified that W^* has properties (j) and (jj), and so $\mathrm{D}(I - W^*(3, \cdot, \cdot, \cdot, \cdot, \cdot), \Omega, 0) = 1$.

Let $Z^* : [0,1] \times \bar{\Omega} \to C^0(J) \times C^0(J) \times C^0(J) \times \mathbb{R}^2$ be given by the formula

$$Z^*(\lambda, x, y, z, a, b) = \left(a + \int_0^t g^{-1}\Big(b + \lambda \int_0^s (F_n^2(x, y, z(\nu)))(\nu)\, d\nu\Big)\, ds,\right.$$
$$g^{-1}\Big(b + \lambda \int_0^t (F_n^2(x, y, z(s)))(s)\, ds\Big),\ g^{-1}\Big(b + \lambda \int_0^t (F_n^2(x, y, z(s)))(s)\, ds\Big),$$
$$\left. a - p_{1n}^2(x(0), y(0), x(T), y(T)),\ b - p_{2n}^2(x(0), y(0), x(T), y(T))\right).$$

Then $Z^*(0, \cdot, \cdot, \cdot, \cdot, \cdot) = W^*(3, \cdot, \cdot, \cdot, \cdot, \cdot)$ and applying Lemma 2 we can proceed analogously to the proof of Lemma 3 to show that Z^* is a compact operator and

$$Z^*(\lambda, x, y, z, a, b) \ne (x, y, z, a, b), \quad (\lambda, x, y, z, a, b) \in [0,1] \times \partial\Omega.$$

Hence $\mathrm{D}(I - Z^*(1, \cdot, \cdot, \cdot, \cdot, \cdot), \Omega, 0) = \mathrm{D}(I - W^*(3, \cdot, \cdot, \cdot, \cdot, \cdot), \Omega, 0) = 1$ and, in consequence, there exists a fixed point of $Z^*(1, \cdot, \cdot, \cdot, \cdot, \cdot)$, say (u, x_*, y_*, a_*, b_*). Then $u(t)$ is a solution of BVP $(18_n)_1$, (19_n) satisfying the inequalities (26). □

3 Existence results

Theorem 1 *Let assumption (H_1) be satisfied. Then BPV (1), (2) has a solution $u(t)$ satisfying the inequalities*

$$M_1 - L_2T \le u(t) \le M_2 - L_1T, \quad L_1 \le u'(t) \le L_2, \quad M_1 \le u(T) \le M_2 \tag{51}$$

for $t \in J$.

Proof By Lemma 3, for each $n \in \mathbb{N}$ there exists a solution $u_n(t)$ of BVP $(16_n)_1, (17_n)$ satisfying the inequalities (20) (with $u = u_n$). By the properties (a) and (c) of F, there is $l \in L_1(J)$ such that

$$|(F_n^1(u_n, u_n', u_n(t)))(t)| \leq l(t) \quad \text{for a.e. } t \in J \text{ and each } n \in \mathbb{N}.$$

From (20) (with $u = u_n$) and the equalities

$$u_n(t) = u_n(0) + \int_0^t g^{-1}\Big(u_n'(0) + \int_0^s (F_n^1(u_n, u_n', u_n'(\nu)))(\nu)\, d\nu\Big)\, ds \tag{52}$$

where $t \in J$ and $n \in \mathbb{N}$ we deduce that

$$|u_n(t)| \leq S + TP\Big(L + 1 + \int_0^T l(t)\, dt\Big),$$

$$|u_n'(t)| \leq P\Big(L + 1 + \int_0^T l(t)\, dt\Big),$$

$$|g(u_n'(t_1)) - g(u_n'(t_2))| \leq \Big|\int_{t_1}^{t_2} l(t)\, dt\Big|$$

for $t, t_1, t_2 \in J$ and $n \in \mathbb{N}$, where $S = \max\{|M_1| + (L_2+1)T, |M_2| + (1-L_1)T\}$, $L = \max\{-L_1, L_2\}$ and the function P is defined by (42). By the Arzelà–Ascoli theorem, there exists a subsequence $\{u_{k_n}(t)\}$ converging in $C^1(J)$, say $\lim_{n\to\infty} u_{k_n} = u$. Clearly, $u \in C^1(J)$, u satisfies the inequalities (51) and taking the limit in (52) (with k_n instead of n) as $n \to \infty$, it follows that (cf. the property (b) of F, (12) and (51))

$$u(t) = u(0) + \int_0^t g^{-1}\Big(u'(0) + \int_0^s (F(u, u', u'(\nu)))(\nu)\, d\nu\Big)\, ds, \quad t \in J. \tag{53}$$

Then u is a solution of (1). Since (cf. (14) and (51))

$$\lim_{n\to\infty} p_{1n}^1(u_{k_n}(0), u_{k_n}'(0), u_{k_n}(T), u_{k_n}'(T)) = \varepsilon p_1(u(0), u'(0), u(T), u'(T)),$$

$$\lim_{n\to\infty} p_{2n}^1(u_{k_n}(0), u_{k_n}'(0), u_{k_n}(T), u_{k_n}'(T)) = \delta p_2(u(0), u'(0), u(T), u'(T)),$$

we have $p_1(u(0), u'(0), u(T), u'(T)) = 0$, $p_2(u(0), u'(0), u(T), u'(T)) = 0$. Hence u satisfies the boundary conditions (2). □

Theorem 2 *Let assumption (H_2) be satisfied. Then BPV (1), (2) has a solution $u(t)$ satisfying the inequalities*

$$M_1 + L_1 T \leq u(t) \leq M_2 + L_2 T, \quad L_1 \leq u'(t) \leq L_2, \quad M_1 \leq u(0) \leq M_2 \tag{54}$$

for $t \in J$.

Proof By Lemma 4, for each $n \in \mathbb{N}$ there exists a solution $u_n(t)$ of BVP $(18_n)_1, (19_n)$ satisfying the inequalities (26) (with $u = u_n$). Let

$$|(F_n^2(u_n, u_n', u_n(t)))(t)| \le k(t) \quad \text{for a.e. } t \in J \text{ and each } n \in \mathbb{N},$$

where $k \in L_1(J)$. The existence of k is guaranteed by the properties (a) and (c) of F. We conclude from (26) (with $u = u_n$) and the equalities

$$u_n(t) = u_n(0) + \int_0^t g^{-1}\Big(u_n'(0) + \int_0^s (F_n^2(u_n, u_n', u_n'(\nu)))(\nu)\, d\nu\Big)\, ds \tag{55}$$

where $t \in J$ and $n \in \mathbb{N}$, that

$$|u_n(t)| \le S^* + TP\Big(L + 1 + \int_0^T k(t)\, dt\Big),$$

$$|u_n'(t)| \le P\Big(L + 1 + \int_0^T k(t)\, dt\Big),$$

$$|g(u_n'(t_1) - g(u_n'(t_2))| \le \Big|\int_{t_1}^{t_2} k(t)\, dt\Big|$$

for t, t_1, $t_2 \in J$ and $n \in \mathbb{N}$, where $S^* = \max\{|M_1|, |M_2|\}$, $L = \max\{-L_1, L_2\}$ and the function P is defined by (42). Going if necessary to a subsequence, we can assume, by the Arzelà–Ascoli theorem, that $\{u_n\}$ converges in $C^1(J)$ to a $u \in C^1(J)$. Then u satisfies the inequalities (54) and taking the limit in (55) as $n \to \infty$ we obtain the equality (53). Then u is a solution of (1). Since (cf. (15) and (54))

$$\lim_{n\to\infty} p_{1n}^2(u_n(0), u_n'(0), u_n(T), u_n'(T)) = \varepsilon p_1(u(0), u'(0), u(T), u'(T)),$$

$$\lim_{n\to\infty} p_{2n}^2(u_n(0), u_n'(0), u_n(T), u_n'(T)) = \delta p_2(u(0), u'(0), u(T), u'(T)),$$

we see that $p_1(u(0), u'(0), u(T), u'(T)) = 0$, $p_2(u(0), u'(0), u(T), u'(T)) = 0$, and consequently u satisfies the boundary conditions (2). □

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