Viktor Witkovský

On Seifert's ANOVA-like test for variance components

Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 39 (2000), No. 1, 249--261

Persistent URL: http://dml.cz/dmlcz/120413

Acta Univ. Palacki. Olomuc., Fac. rer. nat.,
Mathematica **39** (2000) 249–261

On Seifert's ANOVA-Like Test for Variance Components *

VIKTOR WITKOVSKÝ

Institute of Measurement Science, Slovak Academy of Sciences,
Dúbravská cesta 9, 842 19 Bratislava, Slovak Republic
e-mail: umerwitk@savba.sk

(Received January 28, 2000)

Abstract

We modify the ANOVA-like test for variance components, originally proposed by Seifert [14], by using different critical region. It is shown that the modified test has some desired optimality properties. In particular, in the mixed linear model with two variance components the modified ANOVA-like test is the optimum test, i.e. it coincides with the UMPI (Uniformly Most Powerful Test) and/or LBI (Locally Best Invariant) test for variance components derived in [2, 8, 9, 16].

Key words: Mixed model, variance components, ANOVA-like test, UMPI test, LBI test, MINQUE.

1991 Mathematics Subject Classification: 62F03, 62E15

1 Introduction

Seifert [14] suggested the so-called ANOVA-like test for variance components. As the author noticed the test is heuristically motivated and leads to the optimal ANOVA-test or to Satterthwaite's approximate test in balanced situations and is asymptotically correct and optimal.

As our experience shows the ANOVA-like test behaves very well in many different situations. However, we have had to change the critical region of the

*Supported by Grant VEGA 1/7295/20 from the Science Grant Agency of the Slovak Republic.

test. The modified test has then superior local optimality properties. In particular, in general mixed model with two variance components the test coincides with the optimum test, i.e. with the uniformly most powerful invariant test (UMPI test) or with the locally best invariant test (LBI test) of variance components, derived by Das and Sinha [2] for one-way unbalanced classification, by Mathew and Sinha [9] for unbalanced two-way model without interactions, and by Mathew and Westfall [8, 16] under the general model involving two variance components. In the general model with more than two variance components the test is no more exact (it depends on nuisance parameters), nor unbiased and optimal in the sense of LBIU (locally best invariant unbiased), but its properties are reasonably good if compared with other available tests, as is illustrated in Section 5.

2 MINQUE and Seifert's ANOVA-like test for variance components

The ANOVA-like test statistic is based on MINQUE, precisely on MINQE(U,I), of variance components. MINQE(U,I), the minimum norm quadratic estimator (unbiased and invariant) of the linear function of the variance components, was proposed by Rao [11] and developed by many others, for more details see [12].

In the following we will consider the general mixed linear model:

$$y = X\beta + U\alpha + \varepsilon, \tag{1}$$

where y is an n-vector of observations of response variable, X and $U = (U_1 : \cdots : U_r)$ are fixed and known $n\times k$ and $n\times m$ matrices, $m = \sum_{i=1}^{r} m_i$, $\mathcal{R}(U_i) \not\subseteq \mathcal{R}(X)$, where $\mathcal{R}(A)$ denotes the linear space spanned by columns of the matrix A, β is a k-vector of unknown fixed effects, and α and ε are uncorrelated random m- and n-vectors. Here $\alpha = (\alpha_1', \dots, \alpha_r')'$ represents the joint vector of r random effects (all levels) and ε represents unexplained random error. We will consider the natural ordering of random effects, i.e. $i \leq j$ whenever $\mathcal{R}(U_i) \subseteq \mathcal{R}(U_j)$. Throughout this paper we will assume normal distribution of random vectors, i.e. we assume $\alpha_i \sim N(0, \sigma_i^2 I_{m_i})$, $i = 1, \dots, r$, and $\varepsilon \sim N(0, \sigma_{r+1}^2 I_n)$. Then

$$E(y) = X\beta, \qquad Var(y) = \sum_{i=1}^{r+1} \sigma_i^2 V_i, \tag{2}$$

where $V_i = U_i U_i'$, $i = 1, \dots, r$, and $V_{r+1} = I$.

Let $\vartheta = (\sigma_1^2, \dots, \sigma_r^2, \sigma_{r+1}^2)' \in \Theta$, where Θ represents the parameter space, denotes the vector of variance components and $\theta = (\theta_1, \dots, \theta_r, 1)'$ denotes the vector of ratios $\theta_i = \sigma_i^2/\sigma_{r+1}^2$, $i = 1, \dots, r+1$. For fixed prior choice ϑ_0 of ϑ the MINQE(U,I) of the linear function $g'\vartheta$, (where g is a fixed known vector such that $g \in \mathcal{R}(K_{UI})$, i.e. exists a vector λ that $g = K_{UI}\lambda$), is given by

$$\widehat{g'\vartheta} = g' K_{UI}^- q = \lambda' q, \tag{3}$$

where K_{UI}^{-} is a g-inverse of the MINQE(U,I) criterial matrix K_{UI} given by elements

$$\{K_{UI}\}_{ij} = tr\left((MV_0M)^{+}V_i(MV_0M)^{+}V_j\right), \qquad i,j = 1,\ldots,r+1, \tag{4}$$

and $q = (q_1, \ldots, q_{r+1})'$ is the MINQE(U,I) vector of quadratics with

$$q_i = y'(MV_0M)^{+}V_i(MV_0M)^{+}y, \qquad i = 1,\ldots,r+1. \tag{5}$$

Here $tr(A)$ denotes the trace of the matrix A. A^{+} denotes Moore–Penrose g-inverse of A, and $M = I - XX^{+}$ is an orthogonal projector on $\mathcal{R}(X)^{\perp}$, the space orthogonal to the space spanned by columns of the matrix X. $V_0 = V(\vartheta_0) = \sum_{i=1}^{r+1}\vartheta_{i0}V_i$ and $(MV_0M)^{+} = V_0^{-1} - V_0^{-1}X(X'V_0^{-1}X)^{-}X'V_0^{-1}$ if V_0^{-1} exists.

Notice that MINQE(U,I) of ϑ is any solution of $K_{UI}\vartheta = q$ and MINQE(U,I) of $g'\vartheta$ is unique. $\widehat{g'\vartheta}$ is a quadratic estimator of $g'\vartheta$ which is invariant under the group of transformations $y \mapsto c(y + X\gamma)$, for all γ and $c > 0$. Moreover, MINQE(U,I) $\widehat{g'\vartheta}$ is an unbiased estimator of $g'\vartheta$, it does not depend on $c > 0$ if $\vartheta_0 = c\theta_0$, and under normality assumptions

$$Var(\widehat{g'\vartheta}) = 2g'K_{UI}^{-}g = 2\lambda'K_{UI}\lambda, \tag{6}$$

locally at $\vartheta = \vartheta_0$. If K_{UI}^{-1} exits then $\hat{\vartheta} = K_{UI}^{-1}q$ is unique and $E(\hat{\vartheta}) = \vartheta$ and $Var(\hat{\vartheta}) = 2K_{UI}^{-1}$ locally at ϑ_0.

For testing the hypothesis $H_0 : \sigma_i^2 = 0$ against the alternative $H_1 : \sigma_i^2 > 0$ (equivalently $H_0 : \theta_i = 0$ against $H_1 : \theta_i > 0$), $i = 1,\ldots,r$, Seifert in [14] proposed the so-called ANOVA-like test statistic based on the ratio of locally uncorrelated MINQE(U,I) estimators:

Let $\vartheta_0 \in H_0$ be a fixed vector of priors, i.e. $\vartheta_{i0} = 0$. Assume for simplicity that the inverse matrix K_{UI}^{-1} exists. Then, let $z = L\hat{\vartheta}$ denotes the vector of locally uncorrelated linear combinations of $\hat{\vartheta}$ — the MINQE(U,I) of ϑ. Here L denotes an upper triangular matrix with all diagonal elements equal to ones and such that locally at ϑ_0

$$Var(z) = L\,Var(\hat{\vartheta})L' = 2LK_{UI}^{-1}L' = D, \tag{7}$$

where $D = Diag(D_{ii})$, $i = 1,\ldots,r+1$, is a diagonal matrix. We note that L can be obtained by Cholesky decomposition of K_{UI}.

The test statistic for testing $H_0 : \vartheta_i = 0$ proposed by Seifert is

$$T = \frac{z_i}{z_i - \hat{\vartheta}_i} = \frac{\hat{\vartheta}_i + \sum_{j=i+1}^{r+1} L_{ij}\hat{\vartheta}_j}{\sum_{j=i+1}^{r+1} L_{ij}\hat{\vartheta}_j} = \frac{\hat{\vartheta}_i + \ell'\hat{\vartheta}}{\ell'\hat{\vartheta}}, \tag{8}$$

where $\ell = (0,\ldots,0,L_{i\,i+1},\ldots,L_{i\,r+1})'$. By construction the local covariance of numerator and denominator is zero, i.e. $Cov(z_i, z_i - \hat{\vartheta}_i) = 0$ locally at ϑ_0, (in fact, $Cov(z_i, z_i - \hat{\vartheta}_i) = 0$ locally at $\vartheta_0 = c\theta_0$ for all $c > 0$), and under H_0 we get

$$E(z_i) = E(\hat{\vartheta}_i + \ell'\hat{\vartheta}) = E\Big(\hat{\vartheta}_i + \sum_{j=i+1}^{r+1} L_{ij}\hat{\vartheta}_j\Big) = \sum_{j=i+1}^{r+1} L_{ij}\vartheta_j = \ell'\vartheta,$$

and

$$E(z_i - \hat{\vartheta}_i) = E(\ell'\hat{\vartheta}) = E\Big(\sum_{j=i+1}^{r+1} L_{ij}\hat{\vartheta}_j\Big) = \sum_{j=i+1}^{r+1} L_{ij}\vartheta_j = \ell'\vartheta.$$

Seifert [14] suggested to reject H_0 for large values of T.

In balanced ANOVA models, if the standard ANOVA-test of H_0 exists, the above ANOVA-like test coincides with the usual ANOVA test of H_0 based on the ratio of two independent MSE's with distributions proportional to χ^2-distributions with f_1 and f_2 degrees of freedom (i.e. under H_0 the test statistic T does not depend on the priors ϑ_0 and has a central F-distribution with appropriate f_1 and f_2 degrees of freedom). In the other situations and in unbalanced ANOVA models Seifert suggested to approximate the distribution of T by the well-known Sattertwaite's approximation: The critical region of an approximate test of level α is given by $T > F_{1-\alpha,f_1,f_2}$, where $F_{1-\alpha,f_1,f_2}$ represents the $(1-\alpha)$ critical value of F-distribution with f_1 and f_2 degrees of freedom, where

$$f_1 = \frac{2(\ell'\vartheta_0)^2}{D_{ii}}, \quad f_2 = \frac{(\ell'\vartheta_0)^2}{\ell' K_{UI}^{-1}\ell}. \tag{9}$$

Kleffe and Seifert [6] checked the performance of this approximate ANOVA-like test with critical region $T > F_{1-\alpha,f_1,f_2}$ in unbalanced two-way random model. It was shown by simulations that under H_0 the true level of significance was uniformly below the nominal value α for different $\vartheta \in H_0$. If compared with the test with critical region $T > c_\alpha$, where c_α was the critical value found by simulations, it was shown that for different $\vartheta \in H_0$ the estimated probabilities $P(T > c_\alpha)$ "do not differ much from each other and are all nearly correct".

3 Modified ANOVA-like test

In general, for given $\vartheta_0 \in H_0$, the ANOVA-like test statistic T given by (8) for testing $H_0 : \vartheta_i = 0$ against $H_1 : \vartheta_i > 0$ has, under the usual normality assumptions, known distribution for all $\vartheta \in \Theta$. The T statistic is given as a ratio of two quadratics in normal variables.

For testing $H_0 : \vartheta_i = 0$ against $H_1 : \vartheta_i > 0$ we suggest to use the modified ANOVA-like test, which depends on the choice of $\vartheta_0 \in H_0$, with the critical region defined by

$$z_i - c_\alpha(z_i - \hat{\vartheta}_i) > 0, \tag{10}$$

where c_α is the critical value of the distribution such that $P(z_i - c_\alpha(z_i - \hat{\vartheta}_i) > 0) = \alpha$, under the assumption that true $\vartheta = \vartheta_0 \in H_0$. The critical value c_α can be find by the help of Imhof's algorithm, see section 6.

Notice that the critical region $T > c_\alpha^*$ with c_α^* such that under $\vartheta = \vartheta_0 \in H_0$ we get $P(T > c_\alpha^*) = \alpha$, is not equivalent with the critical region $z_i - c_\alpha(z_i - \hat{\vartheta}_i) > 0$ with c_α defined by the equation $P(z_i - c_\alpha(z_i - \hat{\vartheta}_i) > 0) = \alpha$ for $\vartheta = \vartheta_0 \in H_0$. In particular, if z_i is nonnegative definite quadratic, then under $\vartheta = \vartheta_0$, we get

$$P\Big(z_i - c_\alpha(z_i - \hat{\vartheta}_i) \le 0\Big) = P\left(0 \le T \le c_\alpha\right) = 1 - \alpha. \tag{11}$$

The modified ANOVA-like test with critical region given in (10) is equivalent with the original Seifert's ANOVA-like test only if the test statistic T is a ratio of two non-negative definite quadratics. However, it is not true in general, and so, the newly proposed test is different from the original Seifert's ANOVA-like test.

4 Model with two variance components

Consider a mixed linear model with two variance components

$$y = X\beta + U\alpha + \varepsilon, \tag{12}$$

where y is an n-vector of observations of response variable, X and U are fixed and known $n \times k$ and $n \times m$ matrices, β is a k-vector of unknown fixed effects, and α and ε are uncorrelated random m- and n-vectors, $\alpha \sim N(0, \sigma_1^2 I_m)$, and $\varepsilon \sim N(0, \sigma^2 I_n)$. Then

$$E(y) = X\beta, \qquad Var(y) = \sigma_1^2 V_1 + \sigma^2 I, \tag{13}$$

where $V_1 = UU'$. Let $M = I - X(X'X)^- X'$ and B be a $(n-k) \times n$ matrix such that $M = B'B$ and $BB' = I_{n-k}$. Then $t = By$ is a maximal invariant with respect to the group of transformations $y \mapsto y + X\beta$ and $E(t) = 0$, and $Var(t) = \sigma_1^2 W + \sigma^2 I_{n-k}$, where $W = BUU'B'$.

Let $\lambda_1 > \lambda_2 > \cdots > \lambda_h \geq 0$ be h distinct eigenvalues of W with their respective multiplicities $\nu_1, \ldots, \nu_h$. Then the spectral decomposition of W is given by $W = \sum_{i=1}^h \lambda_i Q_i$, where $Q_i = E_i E_i'$ and E_i is a matrix built from orthonormal eigenvectors corresponding to the eigenvalue λ_i. In [10] is described a minimal sufficient statistic for the family of distributions of a maximal invariant statistic: It is a set of h independent quadratics in t, say $Z_i = t'Q_i t/\nu_i$, and such that $\nu_i Z_i/(\sigma_1^2 \lambda_i + \sigma^2) \sim \chi^2_{\nu_i}$.

Mathew and Westfall [8, 16], derived an optimum test for testing $H_0 : \theta = 0$ against $H_1 : \theta > 0$, where $\theta = \sigma_1^2/\sigma^2$, what is equivalent with testing $H_0 : \sigma_1^2 = 0$ against $H_1 : \sigma_1^2 > 0$, in general model with two variance components (12). He noticed that the testing problem is invariant under the group of transformations $y \mapsto c(y + X\beta)$, for arbitrary $c > 0$ and β, and the maximal invariant is $t/||t||$. Applying Wijsman's representation theorem, see [17], he derived the ratio R of nonnull to null densities of the maximal invariant:

$$R = |I + \theta V|^{-\frac{1}{2}} |X'X|^{\frac{1}{2}} |X'(I + \theta V)X|^{-\frac{1}{2}} \left(\frac{t'(I + \theta W)^{-1} t}{t't} \right)^{-n-\frac{k}{2}}. \tag{14}$$

There are two important cases to distinguish: i) the nonzero eigenvalues of W are all equal (i.e. $h \leq 2$), and b) the nonzero eigenvalues of W are not equal. The optimum test is given in [5], Theorem 6.2.2:

Theorem 1 *Consider model (12) and let $\nu = rank(W)$, assume $\nu > 0$.*

1. *Suppose $0 < \nu < n-k$ and the nonzero eigenvalues of W are equal to λ, where $\lambda > 0$, with associated orthonormal eigenvectors given by E_1, and let $Q_1 = E_1E_1'$. Then for testing $H_0 : \theta = 0$ versus $H_1 : \theta > 0$, the UMPI test (uniformly most powerful invariant test) rejects H_0 for large values of*

$$F_0 = \frac{n-k-\nu}{\nu}\frac{t'Q_1t}{t'(I-Q_1)t}. \tag{15}$$

Under H_0, $F_0 \sim F_{\nu,n-k-\nu}$.

2. *If the nonzero eigenvalues of W are not all equal, then for testing $H_0 : \theta = 0$ versus $H_1 : \theta > 0$, the LBI test (locally best invariant test) rejects H_0 for large values of*

$$F_* = \frac{t'Wt}{t't} = \frac{t'(\sum_{i=1}^h \lambda_i Q_i)t}{t'(\sum_{i=1}^h Q_i)t} = \frac{\sum_{i=1}^h \lambda_i \nu_i Z_i}{\sum_{i=1}^h \nu_i Z_i}. \tag{16}$$

We will show that the modified ANOVA-like test with critical region $z_1 - c_\alpha(z_1 - \hat{\vartheta}_1) > 0$, where c_α is such that $P(z_1 - c_\alpha(z_1 - \hat{\vartheta}_1) > 0) = \alpha$ under the null hypothesis, is equivalent with the optimum test for testing $H_0 : \sigma_1^2 = 0$ versus $H_1 : \sigma_1^2 > 0$.

Lemma 1 *Consider model (12) and the testing problem $H_0 : \sigma_1^2 = 0$ versus $H_1 : \sigma_1^2 > 0$. Let $\vartheta = (\sigma_1^2, \sigma^2)'$ and let $\vartheta_0 \in H_0$. Then, under given assumptions, we get*

$$F_* = \frac{q_1}{q_2}, \tag{17}$$

i.e. the LBI test statistic coincides with the ratio of MINQE(U,I) quadratics, where q_i, $i = 1,2$, are given by (5).

Proof If $\vartheta_0 \in H_0$ then $\vartheta_0 = (0, \sigma_0^2)'$, for fixed $\sigma_0^2 > 0$ and $V_0 = V(\vartheta_0) = \sigma_0^2 I$. Let $M = I - X(X'X)^- X' = B'B$, where B is such $(n-k) \times n$ matrix that $BB' = I_{n-k}$, and denote $W = BVB'$.

Then according to (5), $q_i = y'(MV_0M)^+ V_i (MV_0M)^+ y$, $i = 1,2$, where $V_1 = V$ and $V_2 = I$, and $MV_0M = \sigma_0^2 B'BB'B = \sigma_0^2 B'B = \sigma_0^2 M$ and so $(MV_0M)^+ = \frac{1}{\sigma_0^2}M$. Finally we get

$$\begin{aligned} q_1 &= y'(MV_0M)^+ V (MV_0M)^+ y = \frac{1}{\sigma_0^4} y'B'BVB'By = \frac{1}{\sigma_0^4} t'Wt \\ q_2 &= y'(MV_0M)^+ I (MV_0M)^+ y = \frac{1}{\sigma_0^4} y'B'BB'By = \frac{1}{\sigma_0^4} t't. \end{aligned} \tag{18}$$

To derive the ANOVA-like test statistic we need to express the criterial matrix K_{UI} given by (4). It is easy to show that

$$K_{UI} = \frac{1}{\sigma_0^4} \begin{pmatrix} tr(W^2) & tr(W) \\ tr(W) & n-k \end{pmatrix}, \tag{19}$$

where $tr(W^2) = tr(\sum_{i=1}^h \lambda_i^2 Q_i) = \sum_{i=1}^h \lambda_i^2 \nu_i$, and $tr(W) = \sum_{i=1}^h \lambda_i \nu_i$. Then the matrix

$$L = \begin{pmatrix} 1 & \frac{tr(W)}{tr(W^2)} \\ 0 & 1 \end{pmatrix}, \tag{20}$$

fulfills the required condition $2LK_{UI}^{-1}L' = D$, where D is a diagonal matrix. Notice that L does not depend on ϑ_0. Then, considering MINQE(U,I)

$$\hat{\vartheta} = K_{UI}^{-1} q = \frac{1}{Det} \begin{pmatrix} n-k & -tr(W) \\ -tr(W) & tr(W^2) \end{pmatrix} \begin{pmatrix} t'Wt \\ t't \end{pmatrix}, \tag{21}$$

where $Det = (n-k)tr(W^2) - tr(W)^2$, and by solving $z = L\hat{\vartheta}$ we get

$$\begin{aligned} z_1 &= \frac{1}{tr(W^2)} t'Wt, \\ z_2 &= \frac{tr(W^2)}{Det} t't - \frac{tr(W)}{Det} t'Wt = \hat{\vartheta}_2, \\ z_1 - \hat{\vartheta}_1 &= \frac{1}{Det} \left(tr(W)t't - \frac{tr(W)^2}{tr(W^2)} t'Wt \right). \end{aligned} \tag{22}$$

Note that z_1 is nonnegative definite quadratic form in t.

The ANOVA-like test statistic for testing $H_0 : \sigma_1^2 = 0$ versus $H_1 : \sigma_1^2 > 0$ is then given as

$$T = \frac{z_1}{z_1 - \hat{\vartheta}_1} = \frac{tr(W^2)}{tr(W)} \frac{z_1}{\hat{\vartheta}_2} = \frac{(Det)t'Wt}{tr(W)tr(W^2)t't - tr(W)^2 t'Wt}. \tag{23}$$

Theorem 2 *The modified ANOVA-like test with critical region $z_1 - c_\alpha(z_1 - \hat{\vartheta}_1) > 0$, where z_1 and $z_1 - \hat{\vartheta}_1$ are given by (22), and $c_\alpha > 0$ is such that under the null hypothesis $P(z_1 - c_\alpha(z_1 - \hat{\vartheta}_1) > 0) = \alpha$, is the optimum test (UMPI or LBI test) for testing $H_0 : \sigma_1^2 = 0$ versus $H_1 : \sigma_1^2 > 0$ in the mixed linear model (12) with two variance components.*

Proof First, let us assume that $\lambda > 0$ is the only positive eigenvalue of W with its multiplicity ν, $0 < \nu < n-k$. Then $W = \lambda Q_1$, where $Q_1 = E_1 E_1'$, and E_1 is a matrix which columns are the orthonormal eigenvectors associated with λ and further, $tr(W) = \lambda\nu$ and $tr(W^2) = \lambda^2 \nu$. Let E_2 denotes the matrix of the orthonormal eigenvectors associated with the eigenvalue 0 and $Q_2 = E_2 E_2'$, then $I_{n-k} = Q_1 + Q_2$. Using (23) it is easy to see that

$$T = \frac{z_1}{z_1 - \hat{\vartheta}_1} = \frac{n-k-\nu}{\nu} \frac{t'Q_1 t}{t'(I - Q_1)t}, \tag{24}$$

what is exactly the test statistic F_0 of UMPI test given in (15). The quadratic forms $t'Q_1 t$ and $t'(I - Q_1)t$ are nonnegative definite, so the critical region $T > c_\alpha$ is equivalent with the critical region $z_1 - c_\alpha(z_1 - \hat{\vartheta}_1) > 0$. Moreover, based on the results in [10], under the null hypothesis $T \sim F_{\nu, n-k-\nu}$.

On the other hand, in general we get

$$T^{-1} = \frac{tr(W)tr(W^2)}{Det}\frac{t't}{t'Wt} - \frac{tr(W)^2}{Det} = k_1\frac{q_2}{q_1} - k_2, \tag{25}$$

where T^{-1} denotes the reciprocal value of T, and q_1, q_2 are MINQE(U,I) quadratics given by (5), and $k_1 = \frac{tr(W)tr(W^2)}{Det}$ and $k_2 = \frac{tr(W)^2}{Det}$ are positive constants. From that we get

$$F_*^{-1} = \frac{q_2}{q_1} = \frac{T^{-1} + k_2}{k_1}, \tag{26}$$

where, according to Lemma 1, F_*^{-1} is the reciprocal value of the LBI test statistic F_*.

Let $z_1 - c_\alpha(z_1 - \hat{\vartheta}_1) > 0$ is the critical region of the modified ANOVA-like test, with the critical value $c_\alpha > 0$, such that under H_0 $P(z_1 - c_\alpha(z_1 - \hat{\vartheta}_1) > 0) = \alpha$. Then, using that q_1, q_2, and z_1 are nonnegative definite, we get

$$\begin{aligned} P(z_1 - c_\alpha(z_1 - \hat{\vartheta}_1) > 0) &= \\ &= P(T < 0) + P(T > c_\alpha) \\ &= P(T^{-1} < 0) + P(0 < T^{-1} < 1/c_\alpha) \\ &= P\left(T^{-1} < \frac{1}{c_\alpha}\right) = P\left(\frac{T^{-1} + k_2}{k_1} < \frac{1/c_\alpha + k_2}{k_1}\right) \\ &= P\left(F_*^{-1} < \frac{1/c_\alpha + k_2}{k_1}\right) = P\left(F_* > \frac{k_1}{1/c_\alpha + k_2}\right) \\ &= P\left(F_* > c_\alpha^*\right), \end{aligned} \tag{27}$$

where c_α^* is the critical value of the LBI test, see (16).

5 General mixed linear model

Consider the general mixed linear model (1) with variance components $\vartheta = (\vartheta_1, \ldots, \vartheta_{r+1})'$. Only little is known about the optimality of tests for variance components, $H_0 : \vartheta_i = 0$ versus $H_1 : \vartheta_i > 0$, $i = 1, \ldots, r$, in the general model (1). In the recent book [5], there are remarkable developments in the area of exact (unbiased) tests and local optimality of unbiased invariant tests for special designs. However, such tests can be constructed only in a limited class of problems.

In general, the modified ANOVA-like test given by (10) is no more unbiased and locally optimal invariant test for testing $H_0 : \vartheta_i = 0$ against $H_1 : \vartheta_i > 0$. Typically, the test statistic and its distribution under H_0 depends on some nuisance parameters. On the other hand, the ANOVA-like test is based on the MINQE(U,I) quadratics, $q = (q_1, \ldots, q_{r+1})'$, given in (5). Seifert [15] proved the following important result:

Theorem 3 *The MINQE(U,I) vector of quadratics q is locally sufficient for the class of invariant tests.*

By definition, a statistic q is called locally sufficient at $\vartheta_0 \in H_0$ for a testing problem, if for any test $\phi(y)$ there is a test $\phi_*(q)$ such that

$$E_\vartheta(\phi(y)) = E_\vartheta(\phi_*(q)) + o(\vartheta - \vartheta_0), \tag{28}$$

where $o(\vartheta - \vartheta_0)$ is such function that $o(\vartheta - \vartheta_0)/||\vartheta - \vartheta_0|| \mapsto 0$ for $\vartheta \mapsto \vartheta_0$.

In [6] was considered the LBI tests, under the model (1), for given nuisance variance components $\vartheta_0 = (\vartheta_{01}, \ldots, \vartheta_{0r+1})'$, $\vartheta_0 \in H_0$, without assumptions about unbiasedness.

They showed that for given $\vartheta_0 \in H_0$ the locally best invariant test for testing $H_0 : \vartheta_i = 0$ versus $H_1 : \vartheta_i > 0$ rejects the null hypothesis for large values of

$$R_* = \frac{q_i}{\sum_{j=1}^{r+1} \vartheta_{0j} q_j}. \tag{29}$$

Unfortunately, the significance levels (under H_0) of these tests strongly depends on nuisance variances.

Seifert [15] suggested new optimality criteria for testing hypotheses on variance components. In particular, he introduced and characterized LBLUIT—the locally best locally unbiased invariant test. The critical region of LBLUIT is of the form $q_i > \sum_{j \neq i} k_j q_j$, and such that the test, say $\phi(q)$, on significance level α, fulfils the following restrictions: $E_{\vartheta_0}(\phi(q)q_j) - \alpha E_{\vartheta_0}(q_j) = 0$, for all $j \neq i$. Unfortunately, the derivation of such test still remains a big problem.

On the other hand, as our experience shows, the modified ANOVA-like test with critical region given by (10) behaves better then the above the LBI test and also better then the originally proposed ANOVA-like test. The new test gives a good approximation of the nominal significance level at H_0 and a high power at H_1.

We shall illustrate properties of the modified ANOVA-like test in the setup of a random two-way classification model

$$y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \varepsilon_{ijk}, \tag{30}$$

where μ is an unknown constant, and $\alpha \sim N(0, \sigma_\alpha^2 I)$, $\beta \sim N(0, \sigma_\beta^2 I)$, $\gamma \sim N(0, \sigma_\gamma^2 I)$, $\varepsilon \sim N(0, \sigma_\varepsilon^2 I)$ are independent random vectors. The incidence matrix of the model is given in the table bellow:

		j			
		1	2	3	4
	1	4	0	0	0
i	2	5	5	4	0
	3	6	5	4	3

The model was taken from [6] where similar illustration, based on simulations, was given for the originally proposed ANOVA-like test.

Let $\vartheta = (\vartheta_1, \ldots, \vartheta_4)'$ denotes the vector of variance components. The hypothesis of interest is $H_0 : \vartheta_1 = 0$ versus $H_1 : \vartheta_1 > 0$. Let $\vartheta = (0, 1, 1, 1)'$ be the true parameter from H_0. Then for chosen prior value $\vartheta_0 = (0, 1, 1, 1)'$ the modified ANOVA-like test rejects the null hypothesis for $z_1 - c_{0.05}(z_1 - \hat{\vartheta}_1) > 0$, where the critical value $c_{0.05} = 15.5146$ was calculated by Imhof's procedure, and is such that locally at $\vartheta = \vartheta_0$ we get $P(z_1 - c_{0.05}(z_1 - \hat{\vartheta}_1) > 0) = 0.05$. We notice that Kleffe and Seifert [6] derived the critical region of the original ANOVA-like test, under the same conditions, such that the test rejects the null hypothesis for $T > 8.1047$, where $T = z_1/(z_1 - \hat{\vartheta}_1)$. The critical value 8.1047 was derived by simulations, in fact, the true critical value computed by Imhof's algorithm should be 7.9850. Notice that the modified ANOVA-like test rejects H_0 for $T \notin \langle 0, 15.5146 \rangle$.

Similarly, we have derived the LBI test, see (29), for testing $H_0 : \vartheta_1 = 0$ versus $H_1 : \vartheta_1 > 0$ locally at ϑ_0. The test rejects the null hypothesis for $R_* > 0.3178$, and $P(R_* > 0.3178) = 0.05$ locally at ϑ_0.

To compare the properties of the tests we report Table 1, Table 2, and Table 3, with significance levels of the tests under different values of the true parameter $\vartheta = (0, \vartheta_2, \vartheta_3, 1)'$, $\vartheta \in H_0$. Last row in each table reports the power of the test for alternatives $\vartheta \in H_1$, where $\vartheta = (\vartheta_1, 1, 1, 1)'$ and $\vartheta_1 = 0, 0.5, 1, 5, 100$. All calculations are based on the Imhof's algorithm.

		ϑ_2				
		0	0.5	1	5	100
	0	0.0472	0.0506	0.0523	0.0507	0.0332
	0.5	0.0488	0.0496	0.0506	0.0537	0.0389
ϑ_3	1	0.0491	0.0495	0.0500	0.0530	0.0425
	5	0.0494	0.0495	0.0495	0.0504	0.0522
	10	0.0495	0.0495	0.0495	0.0495	0.0505
	Power	0.0500	0.1040	0.1366	0.1991	0.1796

Table 1: Seifert's ANOVA-like test. The levels of significance $P(T > 7.9850)$ for different values of the true parameter $\vartheta \in H_0$, where $\vartheta = (0, \vartheta_2, \vartheta_3, 1)'$. The last row reports the power of the test for different alternatives $\vartheta \in H_1$, where $\vartheta = (\vartheta_1, 1, 1, 1)'$ and $\vartheta_1 = 0, 0.5, 1, 5, 100$.

		ϑ_2				
		0	0.5	1	5	100
	0	0.0426	0.0554	0.0640	0.0977	0.1685
	0.5	0.0436	0.0489	0.0531	0.0730	0.1469
ϑ_3	1	0.0437	0.0471	0.0500	0.0650	0.1336
	5	0.0438	0.0447	0.0455	0.0512	0.0948
	100	0.0438	0.0439	0.0439	0.0443	0.0515
	Power	0.0500	0.1195	0.1732	0.3630	0.5731

Table 2: Modified ANOVA-like test. The levels of significance $P(z_1 - 15.5146 (z_1 - \hat{\vartheta}_1) > 0)$ for different values of the true parameter $\vartheta \in H_0$, where $\vartheta = (0, \vartheta_2, \vartheta_3, 1)'$. The last row reports the power of the test for different alternatives $\vartheta \in H_1$, where $\vartheta = (\vartheta_1, 1, 1, 1)'$ and $\vartheta_1 = 0, 0.5, 1, 5, 100$.

		ϑ_2				
		0	0.5	1	5	100
	0	0.0001	0.0001	0.0001	0.0001	0.0000
	0.5	0.0139	0.0138	0.0137	0.0129	0.0040
ϑ_3	1	0.0522	0.0510	0.0500	0.0437	0.0132
	5	0.3160	0.3116	0.3073	0.2786	0.1051
	100	0.6411	0.6404	0.6397	0.6342	0.5332
	Power	0.0500	0.1970	0.3089	0.6554	0.9716

Table 3: LBI test. The levels of significance $P(R_* > 0.3178)$ for different values of the true parameter $\vartheta \in H_0$, where $\vartheta = (0, \vartheta_2, \vartheta_3, 1)'$. The last row reports the power of the test for different alternatives $\vartheta \in H_1$, where $\vartheta = (\vartheta_1, 1, 1, 1)'$ and $\vartheta_1 = 0, 0.5, 1, 5, 100$.

6 Computing the distribution of a linear combination of independent chi-squared variables

Imhof [4] gave an expression for computing the cumulative distribution function of the quadratic form in normal variables. Let $y \sim N(\mu, V)$ and let $Q = y'Ay$. If V is non-singular then $Q = \sum_{r=1}^{m} \lambda_r \chi^2_{\nu_r, \delta_r^2}$, where λ_r are the distinct non-zero eigenvalues of AV, the ν_r their respective orders of multiplicity, the δ_r are certain linear combinations of μ and the $\chi^2_{\nu_r, \delta_r^2}$ are independent χ^2-variables with ν_r degrees of freedom and non-centrality parameter δ_r^2. Then

$$P(Q > x) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{sin\theta(u)}{u\varrho(u)}\, du, \tag{31}$$

where

$$\begin{aligned} \theta(u) &= \frac{1}{2} \sum_{r=1}^{m} \left(\nu_r \arctan(\lambda_r u) + \delta_r^2 \lambda_r u (1 + \lambda_r^2 u^2)^{-1}\right) - \frac{1}{2} x u, \\ \varrho(u) &= \prod_{r=1}^{m} (1 + \lambda_r^2 u^2)^{\frac{1}{4}\nu_r} \exp\left\{ \frac{\sum_{r=1}^{m} (\delta_r \lambda_r u)^2}{2(1 + \lambda_r^2 u^2)} \right\}. \end{aligned} \tag{32}$$

The probability $P(Q > x)$ can be obtained quite easily by straightforward numerical integration. The exact value of $P(Q > x)$ can be obtained by use of the computer algorithm proposed by Davies [3]. The algorithm is publicly available as Algorithm AS155 at STATLIB.

Alternatively, we could use a very fast and efficient method for approximate calculation of $P(Q > x)$ based on the saddlepoint approximation, as suggested by Kuonen [7]. If $Q = \sum_{r=1}^{m} \lambda_r \chi^2_{\nu_r, \delta_r^2}$ then the cumulant generating function of Q is

$$K(\zeta) = \sum_{r=1}^{m} \left(-\frac{1}{2} h_r \log(1 - 2\zeta\lambda_r) + \frac{\zeta \delta_r^2 \lambda_r}{1 - 2\zeta\lambda_r} \right), \tag{33}$$

with its first and second derivative with respect to ζ given by

$$\begin{aligned} K'(\zeta) &= \sum_{r=1}^{m} \frac{\lambda_r\left(\nu_r - 2\,\zeta\nu_r\lambda_r + \delta_r^2\right)}{\left(1-2\,\zeta\lambda_r\right)^2}, \\ K''(\zeta) &= 2\sum_{r=1}^{m} \frac{{\lambda_r}^2\left(\nu_r - 2\,\zeta\nu_r\lambda_r + 2\,\delta_r^2\right)}{\left(1-2\,\zeta\lambda_r\right)^3}. \end{aligned} \tag{34}$$

Then,

$$P(Q > x) \approx 1 - \Phi\left(w + \frac{1}{w}\log\left(\frac{v}{w}\right)\right), \tag{35}$$

where $\Phi(z)$ denotes the cumulative distribution function of the standard normal random variable, and

$$\begin{aligned} w &= sign(\hat{\zeta})\sqrt{2\,(\hat{\zeta}x - K(\hat{\zeta}))}, \\ v &= \hat{\zeta}\sqrt{K''(\hat{\zeta})}, \end{aligned} \tag{36}$$

and $\hat{\zeta}$, known as the saddlepoint, is the value of ζ satisfying the equation

$$K'(\hat{\zeta}) = x \tag{37}$$

for such $\hat{\zeta}$ that $\hat{\zeta} < 1/(2\lambda_{max})$ if $\lambda_{max} = \max_r \lambda_r$ is positive and/or $\hat{\zeta} > 1/(2\lambda_{min})$ if $\lambda_{min} = \min_r \lambda_r$ is negative. For more details on the saddlepoint approximation see [1, 13].

The algorithm implemented in MATLAB ver. 5 is available on request from the author of this paper.

References

[1] Barndorff-Nielsen, O. E.: *Approximate interval probabilities.* Journal of the Royal Statistical Society **B 52** (1990), 485–486.

[2] Das, R., Sinha, B. K.: *Robust optimum invariant unbiased tests for variance components.* In: T. Pukkila, S. Puntanen (Eds.): Proceedings of the Second International Tampere Conference in Statistics, University of Tampere, Finland, 1987, 317–342.

[3] Davies, R. B.: *The distribution of a linear combination of chi-square random variables.* Applied Statistics **29** (1980), 323–333.

[4] Imhof, J. P.: *Computing the distribution of quadratic forms in normal variables.* Biometrika **48** (1961), 419–426.

[5] Khuri, A. I., Mathew, T., Sinha, B. K.: Statistical Tests for Mixed Linear Models. *J. Wiley, New York,* 1998.

[6] Kleffe, J., Seifert, B.: *On the role of MINQUE in testing of hypotheses under mixed linear models.* Comun. Statist. — Theory Meth. **17** (1988), 1287–1309.

[7] Kounen, D.: *Saddlepoint approximations for distributions of quadratic forms in normal variables.* Biometrika **86** (1999), 929–935.

[8] Mathew, T.: *Optimum invariant tests in mixed linear models with two variance components.* In: Y. Dodge, (Ed.) Statistical Data Analysis and Inference, North-Holland, Amsterdam, 1989, 381–388.

[9] Mathew, T., Sinha, B. K.: *Optimum tests in unbalanced two-way models without interactions.* The Annals of Statistics **16** (1988), 1727–1740.

[10] Olsen, A., Seely, J., Birkes, D.: *Invariant quadratic estimation for two variance components.* The Annals of Statistics **4** (1976), 878–890.

[11] Rao, C. R.: *Estimation of variance components—MINQUE theory.* Journal of Multivariate Analysis **1** (1971), 257–275.

[12] Rao, C. R., Kleffe, J.: Estimation of Variance Components and Applications. *North-Holland Publishing Company, Amsterdam,* 1988.

[13] Reid, N.: *Saddlepoint methods and statistical inference (with Discussion).* Statistical Science **3** (1988), 213–238.

[14] Seifert, B.: *Estimation and test of variance components using the MINQUE-method.* Statistics **16** (1985), 621–635.

[15] Seifert, B.: *Testing hypotheses in mixed linear models.* Journal of Statistical Planning and Inference **36** (1993), 253–268.

[16] Westfall, P. H.: *Power comparisons for invariant variance ratio tests in mixed ANOVA models.* The Annals of Statistics **17** (1989), 318–326.

[17] Wijsman, R. A.: *Cross-section of orbits and their applications to densities of maximal invariants.* In: Fifth Berkeley Symposium on Mathematical Statistics and Probability, University of California, Berkeley, 1967, 389–400.