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Varieties which are Locally Regular and Permutable at 0

Ivan CHAJDA

Department of Algebra and Geometry, Faculty of Science,
Palacký University, Tomkova 40, 779 00 Olomouc, Czech Republic
e-mail: chajda@risc.upol.cz

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Abstract

A variety $\mathcal{V}$ is locally regular if for each algebra $\mathcal{A}$ of $\mathcal{V}$ and every $\theta \in \operatorname{Con} \mathcal{A}$, the 0-class of is determined by every class of θ. $\mathcal{V}$ is permutable at 0 if for every $\theta, \phi \in \operatorname{Con} \mathcal{A}$ the 0-class of $\theta \cdot \phi$ equals to the 0-class of $\phi \cdot \theta$. Varieties having both of these conditions are charecterized by a Mal'cev type condition.

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Recall that an algebra $\mathcal{A} = (A, F)$ is *regular* (see e.g. [3]) if every two congruences on $\mathcal{A}$ coincide whenever they have a class in common. This concept was generalized by K. Fichter [4]: let $\mathcal{A} = (A, F)$ be an algebra with 0, i.e. 0 is a constant of $\mathcal{A}$ (this means that 0 is a constant unary term function of $\mathcal{A}$ or a nullary term function of $\mathcal{A}$). We say that $\mathcal{A}$ is *weakly regular* if every congruence on $\mathcal{A}$ is determined by its 0-class, i.e. if for every $\theta, \phi \in \operatorname{Con} \mathcal{A}$ we have

$$[0]_\theta = [0]_\phi \qquad \text{implies} \qquad \theta = \phi\,.$$

The dual concept was introduced recently in [2]: An algebra $\mathcal{A}$ with 0 is *locally regular* if for every $\theta, \phi \in \operatorname{Con} \mathcal{A}$ we have

$$\text{if } [a]_\theta = [a]_\phi \text{ for some } a \text{ of } \mathcal{A} \text{ then } [0]_\theta = [0]_\phi\,.$$

This concept was motivated by the fact that an algebra $\mathcal{A}$ with 0 is regular if and only if it is both weakly regular and locally regular. This enable us to characterize regular varieties by binary terms only although the well-known characterization of [3] needs ternary terms. Variety (with 0) is called *regular (weakly regular, locally regular)* if every $\mathcal{A}$ of $\mathcal{V}$ has this property.

Regularity is closely connected with permutability of congruences in the sense that the most of regular varieties which often appear in applications are both regular and permutable. As examples can serve varieties of groups, quasigroups, rings or Boolean algebras. On the other hand, both of these congruence conditions are independent, see e.g. [1]. For varieties with 0, the concepts of permutability was localized as follows:

An algebra $\mathcal{A}$ with 0 is *permutable at* 0 if $[0]_{\theta\cdot\phi} = [0]_{\phi\cdot\theta}$ for every two congruences $\theta, \phi \in \operatorname{Con}\mathcal{A}$. Let us note that $\theta\cdot\phi$ or $\phi\cdot\theta$ need not be congruences; they are congruences if and only if they permute, i.e. if $\theta\cdot\phi = \phi\cdot\theta$. A variety $\mathcal{V}$ with 0 is *permutable at* 0 if every $\mathcal{A}$ of $\mathcal{V}$ has this property. The following characterization was developed by H.-P. Gumm and A. Ursini [5]:

Lemma 1 *A variety $\mathcal{V}$ with 0 is permutable at 0 if and only if there exists a binary term $s(x, y)$ such that*

$$s(x,x) = 0 \quad \textit{and} \quad s(x,0) = x$$

hold in $\mathcal{V}$.

It is easy to see that examples of locally regular varieties which appear in [1] are also permutable at 0 although these conditions are independent. It motivates our effort to develop a common characterization of both of these properties. For this we firstly prove the following:

Lemma 2 *Let $\mathcal{V}$ be a variety with 0. For $\mathcal{A} \in \mathcal{V}$ and for a binary relation R on A we denote by $\theta(R)$ the least congruence on $\mathcal{A}$ containing R. The following conditions are equivalent;*

(a) *$\mathcal{V}$ is permutable at 0;*

(b) *for each $\mathcal{A} \in \mathcal{V}$ and every reflexive and compatible relation R on $\mathcal{A}$,*

$$[0]_R = [0]_{\theta(R)},$$

where $[0]_R = \{x \in A;\ \langle x, 0\rangle \in R\}$.

Proof (a) $\Rightarrow$ (b): Let $\mathcal{V}$ be permutable at 0 and $s(x, y)$ be the term of Lemma 1. Let $\mathcal{A} = (A, F) \in \mathcal{V}$ and R be a reflexive and compatible relation on $\mathcal{A}$. Then, of course, also R^{-1} and $R \cdot R$ are reflexive and compatible relations on $\mathcal{A}$ and we have:

(i) if $x \in [0]_{R^{-1}}$ then $\langle x, 0\rangle \in R^{-1}$ and $\langle 0, x\rangle = \langle s(x,x), s(x,0)\rangle \in R^{-1}$ thus $\langle x, 0\rangle \in R$, i.e. $x \in [0]_R$. Hence $[0]_{R^{-1}} \subseteq [0]_R$.

(ii) if $y \in [0]_{R\cdot R}$ then there exists $x \in \mathcal{A}$ with $\langle y, x\rangle \in R$ and $\langle x, 0\rangle \in R$. We obtain $\langle y, 0\rangle = \langle s(y, s(x,x)), s(x, s(x,0))\rangle \in R$ whence $y \in [0]_R$ proving $[0]_{R\cdot R} \subseteq [0]_R$.

Inductively we can show by using of (i), (ii) that $[0]_{\theta(R)} \subseteq [0]_R$. The converse inclusion is trivial thus (b) holds.

(b) $\Rightarrow$ (a): Consider $F_v(x)$, the free algebra of $\mathcal{V}$ with one free generator x, and the reflexive compatible relation R generated by the pair $\langle x, 0\rangle$. Then $\langle 0, x\rangle \in \Theta(R)$ and, by our assumption, $\langle 0, x\rangle \in R$. Hence, there is a binary term $s(x, y)$ such that $0 = s(x, x)$ and $x = s(x, 0)$. By Lemma 1, $\mathcal{V}$ is permutable at 0. $\square$

Lemma 3 *A variety with* 0 *is locally regular if and only if every congruence on one of its members having a one element-class has a one-element* 0*-class.*

Proof Let $\mathcal{V}$ be a variety with 0 satisfying the condition of the lemma, $\mathcal{A} = (A, F) \in \mathcal{V}$, $a \in A$ and $\theta, \phi \in Con\ \mathcal{A}$ and assume $[a]_\theta = [a]_\phi$. Then $[a]_\theta = [a]_{(\theta\cap\phi)}$ and hence

$$[[a]_{(\theta\cap\phi)}]_{(\theta/(\theta\cap\phi))}$$

is a singleton which implies that $[[0]_{(\theta\cap\phi)}]_{(\theta/(\theta\cap\phi))}$ is a singleton whence $[0]_\theta = [0]_{(\theta\cap\phi)}$. Analogously, $[0]_\phi = [0]_{(\theta\cap\phi)}$ follows. This shows $[0]_\theta = [0]_\phi$ proving local regularity of $\mathcal{A}$. The converse implication is trivial. $\square$

Now, we are able to prove the following

Theorem 1 *For a variety $\mathcal{V}$ with* 0*, the following conditions are equivalent:*

(1) $\mathcal{V}$ is locally regular and permutable at 0*;*

(2) there exist $n \geq 1$ and binary terms $d_1, \dots d_n$ and an $(n+2)-ary$ term p such that the following identities hold in $\mathcal{V}$:

$$d_i(x, 0) = x \quad \text{for } i = 1, \dots, n$$
$$y = p(x, \dots, x, x, y), \quad 0 = p(d_1(x, y), \dots, d_n(x, y), x, y).$$

Proof (1) $\Rightarrow$ (2): Let $\mathcal{A} = F_v(x, y)$ be a free algebra of $\mathcal{V}$ with two free generators. Let $\theta = \theta(y, 0)$ be a congruence on $\mathcal{A}$ and $B = [x]_\theta$. Let ϕ be the least congruence on $\mathcal{A}$ having the set B as its congruence class. Clearly $\phi = \theta(B \times B) = \theta(\{x\} \times B)$. Since $\mathcal{A}$ is locally regular, it yields $[0]_\theta = [0]_\phi$, i.e. also $\langle y, 0\rangle \in \phi = \theta(\{x\} \times B)$. However, $Con\mathcal{A}$ is compactly generated, thus there exists $b_1, \dots, b_n \in B$ with

$$\langle y, 0\rangle \in \theta(\langle x, b_1\rangle, \dots, \langle x, b_n\rangle). \tag{$\star$}$$

Since $b_i \in F_v(x, y)$, there exist binary terms $d_1, \dots d_n$ such that $b_i = d_i(x, y)$ for $i = 1, \dots, n$. Moreover, $d_i(x, y) \in [x]_{\theta(y,0)}$ implies $d_i(x, 0) = x$ for $i = 1, \dots, n$.

Since $\mathcal{V}$ is permutable at 0, we can apply Lemma 2 on ($\star$) to obtain

$$\langle y, 0\rangle \in R(\langle x, b_1\rangle, \dots, \langle x, b_n\rangle),$$

the reflexive and compatible relation generated by the pairs $\langle x, b_1\rangle, \dots, \langle x, b_n\rangle$. Hence, there exists an n-ary polynomial φ over $\mathcal{A}$ such that

$$y = \varphi(x, \dots, x) \quad \text{and} \quad 0 = \varphi(b_1, \dots, b_n).$$

However, $\mathcal{A} = F_v(x,y)$ thus there exists an $(n+2)$-ary term p with

$$\varphi(z_1, \dots z_n) = p(z_1, \dots, z_n, x, y).$$

Altogether, we have proved (2).

(2) $\Rightarrow$ (1): Conversely, assume, that (2) holds. Then $d_1(x,y) = \dots = d_n(x,y) = x$ implies

$$y = p(x, x \dots, x, x, y) = p(d_1(x,y), \dots, d_n(x,y), x, y) = 0,$$

i.e.

$$d_1(x,y) = \dots = d_n(x,y) = x \quad \text{iff } y = 0. \tag{$\star\star$}$$

Now, let $\mathcal{A} \in \mathcal{V}$, $\phi, \psi \in \operatorname{Con} \mathcal{A}$ and suppose $[a]_\theta = \{a\}$ for some $a \in A$. If $b \in [0]_\theta$ then

$$d_i(a,b) \in [d_i(a,0)]_\theta = [a]_\theta = \{a\} \quad \text{for } i = 1, \dots, n$$

thus, by ($\star\star$), $b = 0$. According to Lemma 3, $\mathcal{V}$ is locally regular.

Moreover, we can set

$$s(x,y) = p(d_1(x,y), \dots, d_n(x,y), x, x).$$

Then

$$s(x,x) = p(d_1(x,x), \dots, d_n(x,x), x, x) = 0$$
$$s(x,0) = p(d_1(x,0), \dots, d_n(x,0), x, x) = p(x, x, \dots, x) = x$$

and, with respect to Lemma 1, $\mathcal{V}$ is also permutable at 0. □

Example 1 The variety of all ortholattices is locally regular and permutable at 0. One can take $n = 2$ and $d_1(x,y) = x \wedge y^\perp$, $d_2(x,y) = x \vee y$ and $p(z_1, z_2, x, y) = (z_1^\perp \wedge z_2 \wedge y)^\perp \wedge y$.
Then

$$d_1(x,0) = x \wedge 0^\perp = x, \qquad d_2(x,0) = x \vee 0 = x$$

and

$$\begin{aligned} d_1(x,y)^\perp \wedge d_2(x,y) &= (x \wedge y^\perp)^\perp \wedge (x \vee y) \\ &= (x^\perp \vee y) \wedge (x \vee y) \geq (x^\perp \wedge x) \vee y = y, \end{aligned}$$

hence

$$\begin{aligned} p(d_1(x,y), d_2(x,y), x, y) &= (d_1(x,y)^\perp \wedge d_2(x,y) \wedge y)^\perp \wedge y = y^\perp \wedge y = 0 \\ p(x,x,x,y) &= (x^\perp \wedge x \wedge y)^\perp \wedge y = 0^\perp \wedge y = y. \end{aligned}$$ □

Example 2 The variety of all p-algebras (i.e. of pseudocomplemented lattices) is locally regular and permutable at 0.

Take similarly $n = 2$ and $d_1(x,y) = x \wedge y^*$, $d_2(x,y) = x \vee y$ and $p(z_1, z_2, x, y) = (z_1^* \wedge z_2 \wedge y)^* \wedge y$. Clearly $d_1(x,0) = x \wedge 0^* = x$, $d_2(x,0) = x \vee 0 = x$ and we have

$$x \wedge y^* \leq y^* \Rightarrow (x \wedge y^*)^* \geq y^{**} \geq y.$$

Hence

$$p(d_1(x,y), d_2(x,y), x, y) = ((x \wedge y^*)^* \wedge (x \vee y) \wedge y)^* \wedge y = y^* \wedge y = 0,$$

and

$$p(x,x,x,y) = (x^* \wedge x \wedge y)^* \wedge y = 0^* \wedge y = y. \qquad \square$$

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