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Athanasios Andrikopoulos; John Stabakis
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On Nachbin's Problem Concerning Uniformizable Ordered Spaces

A. Andrikopoulos
J. Stabakis

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1. Introduction

Consider a space X and on it the structures *order* $\leq$ and *unformity* $\mathfrak{U}^\star$. We always symbolize by G the graph of $\leq$ in $X \times X$ and we call the triple $(X, \mathfrak{U}^\star, G)$ (or the triple $(X, \mathfrak{U}^\star, \leq)$) *a uniform ordered space*, which means, furthermore, that the topology of the space coincides with the topology $\tau(\mathfrak{U}^\star)$ induced by $\mathfrak{U}^\star$ on X. We recall that the compatibility of these two structures demands for G to be a $\tau(\mathfrak{U}^\star) \times \tau(\mathfrak{U}^\star)$–closed subset of $X \times X$ in which case the space is T_2 and for any $\alpha \in X$, $G[\alpha] = \{x \in X \mid \alpha \leq x\}$ and $G^{-1}[\alpha] = \{x \in X \mid x \leq \alpha\}$ are closed [N, p.26]. We also recall that $(X, \mathfrak{U}^\star, G)$ is *completely regular (or uniformizable ordered space)* [N, p.55] provided that there exists a quasi-uniformity $\mathfrak{U}$ which generates $\mathfrak{U}^\star$ and G; this means that $\mathfrak{U}^\star$ is the coarsest uniformity finer than $\mathfrak{U}$ and its dual $\mathfrak{U}^{-1}$ and that $\bigcap U$, which is a relation reflexive and transitive and if $\mathfrak{U}^\star$ is T_0 it is an order, equal to G.

Our main purpose in the paper is to answer the question raised by Nachbin and restated by some others ([N, §2,th.2.10][F,N, §4.34]) which may be put as follows:

Under which conditions a uniform ordered space is a completely regular ordered one.

We make use of the symbols $x \lesssim y$ for " $x \leq y$ or x and y non comparable ". We also put —as usual— $R[A]$, where A any subset of X and R any relation in X, for the set $\bigcup\{R[A] \mid x \in A\}$. If now G is the graph in $X \times X$ of an order $\leq$, $\mathfrak{U}$ a uniformity (or a quasi-uniformity) of X and U an entourage of $\mathfrak{U}$, then for any point of X we symbolize by $(G \circ U)[\alpha]$ or simply by $G \circ U[\alpha]$ the set

$$U(G[\alpha]) = \{x \in X (\exists z \in X)[a \leq z \,\text{and}\, (z, x) \in U\,]\}.$$

Lastly the symbol $(G \circ U)^n$ (where $n \in \mathbb{N}^\star = \mathbb{N} \backslash \{0\}, \mathbb{N}$ the set of natural numbers) stands for $(G \circ U) \circ (G \circ U) \circ \cdots \circ (G \circ U)$, n-times, so that $(G \circ U)^n[x]$ symbolize the set $\Big(G \circ U\Big)^{n-1}\Big[G \circ U[x]\Big] = \Big(G \circ U\Big)^{n-1}\Big[U\big[G[x]\big]\Big]$.

Definition 1. We recall that A and B, subset of X, are *order separated* if for any $x \in A$ and any $y \in B$, $x \lnsim y$, (or alternatively $y \lnsim x$). We say that A and B are *completely order separated* if they are order separated and furthermore $clA \bigcap clB = \emptyset$ (clA is the $\tau(\mathfrak{U}^\star)$-closure of A). We also say that A and B are *uniformly order separated* if there is $U \in \mathfrak{U}^\star$ such that the sets $U[A]$ and $U[B]$ are order separated.

Definition 2. We say that a uniform ordered space $(X, \mathfrak{U}^\star, G)$ has the *κ-(order) rank convexity,* $\kappa \in \mathbb{N}$, if any $x \in X$, for which there is a $U[a](U \in \mathfrak{U}^\star, a \in X)$, intersecting $(G \circ U)^k[x]$ and $(G \circ U)^{-k}[x]$, belongs to $U[a]$. The symbol $(G \circ U)^0[x]$ stands for $G[x]$.

Note that in the linear case with a convex topology, the space fulfils the 0 -(order) rank convexity.

Definition 3. In a uniform ordered space $(X, \mathfrak{U}^\star, G)$ we say that a $U \in \mathfrak{U}^\star$ is *of order index 1*, if there is a $U^\star \in \mathfrak{U}^\star, U^\star \subseteq U$ such that for every couple U_1,U_2 of entourages with $U_2 \subseteq U_1 \subseteq U^\star$ there holds $G \circ U_2 \circ G \subseteq G \circ U_1$. We say that U is *of order index* $\leq n$, $(n \in \mathbb{N}, n > 1)$, if there is a $U^\star \in \mathfrak{U}^\star, U^\star \subseteq U$ such that for every couple U_1,U_2 of entourage with $U_2 \subseteq U_1 \subseteq U^\star$ there holds $(G \circ U_2)^n \circ G \subseteq (G \circ U_1)^n$. If an entourage is of order index $\leq n$ and it is not of $n-1$, then we say that *it is of order index n*.

In the other cases we say that *the space is of infinite order index.*

The supremum of the order indices of the entourages is called *the order index of the space.*

A linear convex structure is of order index 1, but many partially ordered spaces are as well of the same order index.

Example 1. Let $(X, \mathcal{U}^\star, G)$ be a linear convex uniform space, x,y points in X and $U \in \mathcal{U}^\star$.Let also $(x, y) \in G \circ U \circ G$; then $(x, y) \in G \circ U$. If $(x, y) \in G$, then $(x, y) \in G \circ U$. If $(x, y) \in G$, then there are z_1, z_2 such that $(x, z_1) \in G$, $(z_1, z_2) \in U$ and $(z_2, y) \in G$. Thus y lies between z_1 and z_2 and — because of convexity — y belongs as well to $U[z_1]$, hence $(x, y) \in G \circ U$ and the order index of the space is 1.

Of order index 1 is also the R^2 endowed with the Euclidean metric and the component wise ordering.

2. Let $X \subseteq R^2$ be the set consisting of the $x'ox$, $y'ox$ axes of the cartesian plan system and the parallel half-lines we draw towards right from the point $(0, r)$, $r \in R$. The set is endowed with the metric and the order of the straight line whilst the points of $y'oy$ are non comparable. The space is of order index 2.

We may extend the space to another one of order index 3, if we adjust to X the verdical lines which pass through the points (a, y) of the plane (a is a constant $\neq 0$ and y runs through $y'oy$), the uniformity and the order of the system are these ones of the straight line and the points of the $y'oy$ axe and of the line $x = a$ are not comparable.
This process may be continued towards the infinity.

3. We topologize R^2 giving to any point of $\mathbb{Q} \times \mathbb{Q}$ ($\mathbb{Q}$ the set of rational numbers) its natural topology and to the other points the discrete topology. The space is also endowed with the component wise ordering. Then for every n and every entourage U there is not anotherentourage $U^\star$ such that $(G \circ U^\star)^n \circ G \subseteq (G \circ U)^n$. The space is of infinite order index.

2. The main theorem

Theorem 1. *If the entorages of a compatible uniform ordered space are of a finite order index, the space has the n -rank convexity, where n is either the order index of the space or otherwise is any natural number, and every couple of completely separated subsets are uniformly order separated, then the space is a uniformizable ordered space.*

Proof. Firstly, the finite case.

Let $(X, \mathcal{U}^\star, G)$, G as usual is the graph of an order$\leq$, be a compatible ordered space of order index n and endowed with the afore mentioned properties. For any $U \in \mathcal{U}^\star$ we put:

$$A_U = (G \circ U)^n.$$

We prove that the filter $\{A_U \mid U \in \mathcal{U}^\star\}$ *is a quasi-uniformity.*
Given $V \in \mathcal{U}^\star$ we define successively two finite sequences $(U_0, U_1, \ldots, U_n = U)$ and $(V_0, V_1, \ldots, V_n)$ of entourages of $\mathcal{U}^\star$ as follows:
$V_1^2 \subset V = U_0$ and $(G \circ U_1)^n \circ G \subseteq (G \circ V_1)^n$,$V_2^2 \subseteq U_1$ and $(G \circ U_2)^n \circ G \subseteq (G \circ V_2)^n$,..., $V_{n-1}^2 \subseteq U_{n-2}$ and $(G \circ U_{n-1})^n \circ G \subseteq (G \circ V_{n-1})^n$, $V_n^2 \subseteq U_{n-1}$ and $(G \circ U_n)^n \circ G \subseteq (G \circ V_n)^n$.

We put $U = U_n$ and we get
$A_U \circ A_U = (G \circ U)^n \circ (G \circ U)^n = (G \circ U)^n \circ G \circ U \circ (G \circ U)^{n-1} \subseteq (G \circ V_n)^n \circ U \circ (G \circ U)^{n-1} \subseteq (G \circ V_n)^n \circ V_n \circ (G \circ V_n)^{n-1} \subseteq (G \circ V_n)^{n-1} \circ G \circ U_{n-1} \circ (G \circ V_n)^{n-1} \subseteq (G \circ U_{n-1})^n \circ G \circ U_{n-1} \circ (G \circ U_{n-1})^{n-2} \subseteq \cdots \subseteq (G \circ U_2)^n \circ G \circ U_2 \circ (G \circ U_2) \subseteq (G \circ V_2)^n \circ V_2 \circ (G \circ V_2) \subseteq (G \circ V_2)^{n-1} \circ G \circ U_1 \circ (G \circ V_2) \subseteq (G \circ U_1)^n \circ G \circ U_1 \subseteq (G \circ V_1)^n \circ U_1 \subseteq (G \circ U_0)^n = A_V$.

Next we prove that

$$\bigcap_{U \in \mathcal{U}^\star} A_U = G.$$

It suffices to be proved that $\bigcap A_U \subseteq G$. To this end we assume that $(x, y) \in \bigcap A_U$ and $(x, y) \notin G$. Since the order is closed the sets $\{y\}$ and $G[x]$ are completely order separated, thus there is a $U_1 \in \mathcal{U}^\star$ such that the sets $G \circ U_1[x]$ and $U_1[y]$ are order separated, that is the sets $G \circ U_1 \circ G[x]$ and $U_1 \circ G^{-1}[y]$ are disjoint. Furthermore, the sets $G \circ U_1 \circ G[x]$ and $\{y\}$ are completely order separated, thus there is a $U_2 \in \mathcal{U}^\star$ such that the sets $G \circ U_1 \circ G \circ U_2[x]$ and $U_2 \circ G^{-1}[y]$ are disjoint. We continue this process until the n-step : there is a $U_n \in \mathcal{U}^\star$ such that $G \circ U_1 \circ G \circ U_2 \cdots \circ G \circ U_n[x]$ and $U_n \circ G^{-1}[y]$ are disjoint. If $U = U_1 \bigcap U_2 \bigcap \ldots \bigcap U_n$, then $A^U \subseteq G \circ U_1 \circ G \circ U_2 \circ \cdots \circ G \circ U_n$. Thus $y \notin A_U[x]$, a contradiction.

It remains to be proved that given a $V \in \mathcal{U}^\star$, *there is a* $U \in \mathcal{U}^\star$ *such that*

$$\mathcal{Q} = \left((G \circ U)^n \circ G\right) \bigcap \left(G^{-1} \circ (U \circ G^{-1})^n\right) \subseteq V.$$

Let $(x, y) \in \mathcal{Q}$, and $U \in \mathcal{U}^\star$ such that $U^{2^n} \subseteq V$. Consider $G[x] \backslash U[x] = A_1$, and $G^{-1}[x] \backslash U[x] = B_1$; A_1 and B_1 are completely order separated: if $\alpha \in clA_1 \bigcap clB_1$, then for a $W \in \mathcal{U}^\star$ such that $W^2 \subseteq U$, we conclude that $(\alpha, x) \in W$ (because of the n-rank convexity of the space) and if $\tau \in A_1 \bigcap W(\alpha)$, then $(x, \tau) \in U$, an absurd. Thus there is $U_1 \in \mathcal{U}^\star$, such that $clU_1[A_1] \bigcap clU_2[B_2] = \emptyset$. Next, we consider $G \circ U_1[x] \backslash U^2[x] = A_2$ and $G^{-1} \circ U_1[x] \backslash U^2[x] = B_2$. It is, for the same reason, $clA_2 \bigcap clB_2 = \emptyset$ and furthermore it is

$$cl(A_1 \bigcup A_2) \bigcap cl(B_1 \bigcup B_2) = \emptyset$$

and the sets $A_1 \bigcup A_2, B_1 \bigcup B_2$ are completely order separated. We follow this process and we conclude that there is a $U_n \in \mathfrak{U}^\star$ such that $U_n[A_n]$ and $U_n[B_n]$ are completely order separated, where $A_n = (G \circ U_1 \circ G \circ U_2 \cdots \circ G \circ U_n)[x] \backslash U^{2^n}[x]$ and $B_n = (G^{-1} \circ U_1 \circ G^{-1} \circ U_2 \cdots \circ G^{-1} \circ U_n)[x] \backslash U^{2^n}[x]$.

Now, since $(x, y) \in \mathcal{Q}$, there are two finite sequences (z_i, t_i) and $(z_i^\star, t_i^\star)$, $i \in \{1, \ldots, n\}$ such that $x \leq z_1, (z_1, t_1) \in U, t_1 \leq z_2, (z_2, t_2) \in U, \ldots, t_{n-1} \leq z_n, (z_n, y) \in U$ and $z_1^\star \leq x, (z_1^\star, t_1^\star) \in U, z_2^\star \leq t_1^\star, (z_2^\star, t_2^\star) \in U, \ldots, z_n^\star \leq t_{n-1}^\star, (z_n^\star, y) \in U$. If $z_i \in A_i$, then $t_i \in U_i[A_i], z_{i+1} \in A_{i+1}$ and finally $y \in U_n[A_n]$. If, similary, $z_i^\star \in B_i$, then $y \in U_n[B_n]$,an absurd. Thus at least one sequence of $\{z_i, i \in \{1, 2, \ldots, n\}\}$ and $\{z_i^\star; i \in \{1, 2, \ldots, n\}\}$, say the first, has all the terms belonging to $U^2[x]$. In such a case we get $z_1 \in U[x], z_2 \in U^2[x], \ldots, z_n \in U^{2^{n-1}}][x]$ and $(y, z_n) \in U$, that is $(x, y) \in U^{2^n} \subseteq V$.

The infinite case.
Since all the entourages of $\mathfrak{U}^\star$ are of finite order, we may follow the demonstration given for the finite case. Thus, given a $V \in \mathfrak{U}^\star$ of order index n we can find entourages U and $U^\star$ of order index n, such that $A_U \circ A_U \subseteq A_V$ and $A_{V^\star} \bigcap A_{V^\star}^{-1} \subseteq V$. On the other hand if $(x, y) \in A_U$ and $(x, y) \notin G$, then there is a $U_1 \in \mathfrak{U}^\star$ such that $G \circ U_1[x]$ and $\{y\}$ are order separated. Let U_1 be of order index n. There is, as in the finite case, a $U_2 \in \mathfrak{U}^\star$, such that $G \circ U_1 \circ G \circ U_2[x]$ and $\{y\}$ are order separated. We may find $U_1^\star$ and $U_2^\star$ of order index n such that $G \circ U_1^\star[x]$ and $\{y\}$ are order separated and $(G \circ U_2^\star)^2[x]$ and $\{y\}$ are order separated. We may follow this process until the n-step: there is a $U_n^\star \in \mathfrak{U}^\star$ of order index n such that $(G \circ U_n^\star)^n[x]$ and $\{y\}$ are order separated, a contradiction.

Remarks.

1. A compatible linear ordered space which is convex, fulfils the n-rank convexity property as well as the property that complety order separated subsets are uniformly order separated.

On the other hand, in that case there holds $G \circ U = U \circ G = U \cup G$ for any entourage U.

2. The uniqueness demonsrated in the linear case is not valid for the other cases. D. Kent and R. Vainio in [K,V] singled out one, among many other quasi-uniformities, which generates a given uniform ordered space; it is very convenient for some categorical purposes.

3. Unfortunately the method we make of, does not apply if the entourages are of infinite order index (examples 2 and 3), although the relation $(G \circ U)^\infty$ (which means that we compose G and U "until the end ") is transitive.

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Address:
University of Patras,
Department of Mathematics.
26100 Patras,
Greece