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Characterizations of commuting relations

Tamás Glavosits and Árpád Száz

Abstract. After some preparations, we give some necessary and sufficient conditions in order that two preorders, tolerances, resp. equivalences R and S on the same set be commuting with respect to composition in the sense that $R \circ S = S \circ R$.

0. Introduction

To provide some necessary and sufficient conditions in order that two preorders, tolerances, resp. equivalences be commuting, we prove the following theorems.

Theorem 1. *If R and S are preorders on X, then the following assertions are equivalent:*

(1) $S \circ R \subset R \circ S$;

(2) $R \circ S$ *is a preorder;*

(3) $R \circ S$ *is the preorder generated by* $R \cup S$.

Theorem 2. *If R and S are tolerances on X, then the following assertions are equivalent:*

(1) $R \circ S = S \circ R$;

(2) $R \circ S$ *is a tolerance;*

(3) $R(x) \cap S(y) \neq \emptyset$ *implies* $S(x) \cap R(y) \neq \emptyset$ *for all* $x, y \in X$.

Theorem 3. *If R and S are equivalences on X, then the following assertions are equivalent:*

(1) $R \circ S = S \circ R$;

(2) $R \circ S$ *is an equivalence;*

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(3) there exists an equivalence E on X such that

$$E(x) = \bigcup \{ R(u) : \ R(u) \subset E(x) \} = \bigcup \{ S(v) : \ S(v) \subset E(x) \}$$

for all $x \in X$, and

$$R(u) \subset E(x) \quad \text{and} \quad S(v) \subset E(x) \qquad \text{imply} \qquad R(v) \cap S(v) \neq \emptyset$$

for all $x, u, v \in X$.

Remark. In assertion (1) of Theorem 1 we cannot write equality instead of inclusion. But, in assertions (1) of Theorems 2 and 3 we can write any of the two possible inclusions instead of equality.

Moreover, it is also worth mentioning that the relation E in Theorem 3 is uniquely determined. Namely, if R and S are equivalences on X such that assertion (3) of Theorem 3 holds, then we necessarily have $E = R \circ S$.

1. A few basic facts on relations

As usual, a subset R of a product set $X^2 = X \times X$ is called a relation on X. In particular, the relation $\Delta_X = \{ (x, x) : \ x \in X \}$ is called the identity relation on X.

If R is a relation on X, and moreover $x \in X$ and $A \subset X$, then the sets $R(x) = \{ y \in X : \ (x, y) \in R \}$ and $R[A] = \bigcup_{a \in A} R(a)$ are called the images of x and A under R, respectively.

If R is a relation on X, then the images $R(x)$, where $x \in X$, uniquely determine R since we have $R = \bigcup_{x \in X} \{x\} \times R(x)$. Therefore, the inverse R^{-1} of R can be defined such that $R^{-1}(x) = \{ y \in X : \ x \in R(y) \}$ for all $x \in X$.

Moreover, if R and S are relations on X, then the composition $S \circ R$ of S and R can be defined such that $(S \circ R)(x) = S\big[R(x)\big]$ for all $x \in X$. In particular, we write $R^n = R \circ R^{n-1}$ for all $n \in \mathbb{N}$ by agreeing that $R^0 = \Delta_X$.

A relation R on X is called reflexive, symmetric and transitive if $\Delta_X \subset R$, $R^{-1} \subset R$ and $R^2 \subset R$, respectively. Moreover, a reflexive and transitive relation is called a preorder, and a symmetric preorder is called an equivalence.

For any relation R on X, we define $R^{\star} = \bigcup_{n=0}^{\infty} R^n$ and $R^{\bigstar} = \big(R \cup R^{-1} \big)^{\star}$. Thus, $R^{\star}$ and $R^{\bigstar}$ are the smallest preorder and equivalence on X containing R, respectively. Moreover, $\star$ and $\bigstar$ are algebraic closure operations on $\mathcal{P}(X^2)$.

Besides preorders, reflexive and symmetric relations are also of fundamental importance. They are usually called tolerances. Note that if d is a pseudo-metric on X, then the surroundings $B_r = \{ (x, y) \in X^2 : \ d(x, y) < r \}$ are tolerances.

In the sequel, whenever confusions seem unlikely, we shall simply write $R(A)$ in place of $R[A]$. Note that this convention may only cause some serious troubles whenever $A \subset X$ such that $A \in X$ which is rarely the case in practice.

2. Characterizations of commuting relations

Theorem 2.1. *If R and S are relations on X, then the following assertions are equivalent:*

(1) $S \circ R \subset R \circ S$;

(2) $R(x) \cap S^{-1}(y) \neq \emptyset$ *implies* $S(x) \cap R^{-1}(y) \neq \emptyset$ *for all* $x, y \in X$.

Proof. To check this, note that for any $x, y \in X$ we have

$$(x, y) \in S \circ R \iff y \in (S \circ R)(x) \iff$$
$$\iff y \in S\big(R(x)\big) \iff R(x) \cap S^{-1}(y) \neq \emptyset .$$

Now, as some immediate consequences of Theorem 2.1, we can also state

Corollary 2.2. *If R and S are symmetric relations on X, then the following assertions are equivalent:*

(1) $S \circ R \subset R \circ S$;

(2) $R(x) \cap S(y) \neq \emptyset$ *implies* $S(x) \cap R(y) \neq \emptyset$ *for all* $x, y \in X$.

Corollary 2.3. *If R is a relation on X, then the following assertions are equivalent:*

(1) $R^{-1} \circ R \subset R \circ R^{-1}$;

(2) $R(x) \cap R(y) \neq \emptyset$ *implies* $R^{-1}(x) \cap R^{-1}(y) \neq \emptyset$ *for all* $x, y \in X$.

In addition to Corollary 2.2, we can also prove the following

Theorem 2.4. *If R and S are symmetric relations on X, then the following assertions are equivalent:*

(1) $S \circ R \subset R \circ S$; *(2)* $R \circ S$ *is symmetric;* *(3)* $R \circ S = S \circ R$.

Proof. If (1) holds, then it is clear that

$$(R \circ S)^{-1} = S^{-1} \circ R^{-1} = S \circ R \subset R \circ S .$$

Therefore, (2) also holds.

While, if (2) holds, then it is clear that

$$R \circ S = (R \circ S)^{-1} = S^{-1} \circ R^{-1} = S \circ R .$$

Therefore, (3) also holds.

Concerning transitive relations, in contrast to Theorem 2.4, we can only prove

Theorem 2.5. *If R and S are transitive relations on X such that $S \circ R \subset R \circ S$, then $R \circ S$ is also a transitive relation on X.*

Proof. We evidently have

$$(R \circ S)^2 = (R \circ S) \circ (R \circ S) = R \circ (S \circ R) \circ S \subset R \circ (R \circ S) \circ S = R^2 \circ S^2 \subset R \circ S .$$

The following example shows that an analogue of Theorem 2.4 for transitive relations need not be true.

Example 2.6. If $X = \{1, 2, 3\}$, and moreover

$$R = \{(1, 2), (1, 3), (2, 3)\} \qquad \text{and} \qquad S = \{(1, 2), (1, 3), (3, 2)\},$$

then it can be easily seen that R and S are transitive relations on X such that $R \circ S$ and $S \circ R$ are also transitive relations on X, but

$$S \circ R \not\subset R \circ S \qquad \text{and} \qquad R \circ S \not\subset S \circ R.$$

3. Characterizations of commuting preorders

Despite Example 2.6, as a partial analogue of Theorem 2.4, we can still prove

Theorem 3.1. *If R and S are preorders on X, then the following assertions are equivalent:*

(1) $S \circ R \subset R \circ S$; *(2)* $R \circ S$ *is a preorder;* *(3)* $R \circ S = (R \cup S)^\star$.

Proof. Since $\Delta_X = \Delta_X \circ \Delta_X \subset R \circ S$, by Theorem 2.5 it is clear that the implication (1) $\Longrightarrow$ (2) is true.

Moreover, by the corresponding properties of the operation $\star$, it is clear that $R \subset R^\star \subset (R \cup S)^\star$ and $S \subset S^\star \subset (R \cup S)^\star$, and hence

$$R \circ S \subset \left((R \cup S)^\star\right)^2 = (R \cup S)^\star.$$

On the other hand, by the reflexivity of the relations R and S, it is clear that $R = R \circ \Delta_X \subset R \circ S$ and $S = \Delta_X \circ S \subset R \circ S$, and thus $R \cup S \subset R \circ S$. Hence, by using (2), we can already infer that

$$(R \cup S)^\star \subset (R \circ S)^\star = R \circ S.$$

Therefore, the implication (2) $\Longrightarrow$ (3) is also true.

Finally, from the inclusion $R \circ S \subset (R \cup S)^\star$ established above, it is clear that

$$S \circ R \subset (S \cup R)^\star = (R \cup S)^\star.$$

Therefore, the implication (3) $\Longrightarrow$ (1) is also true.

The following example shows that, in contrast to Theorem 2.4, the equality cannot be stated in assertion (1) of Theorem 3.1.

Example 3.2. If $X = \{1, 2, 3\}$, and moreover

$$R = \{(1, 2)\}^\star \qquad \text{and} \qquad S = \{(3, 1)\}^\star,$$

then it can be easily seen that R and S are is a preorders on X such that $R \circ S$ is also a preorder on X, but $R \circ S \not\subset S \circ R$.

Now, as an immediate consequence of Theorem 3.1, we can also state

Corollary 3.3. *If R is a preorder on X, then the following assertions are equivalent:*

(1) $R^{-1} \circ R \subset R \circ R^{-1}$; *(2)* $R \circ R^{-1}$ *is a preorder;* *(3)* $R^\bigstar = R \circ R^{-1}$.

Moreover, by using Theorems 2.4 and 3.1, we can also easily establish

Theorem 3.4. *If R and S are equivalences on X, then the following assertions are equivalent:*

(1) $R \circ S = S \circ R$;

(2) $S \circ R \subset R \circ S$;

(3) $R \circ S = (R \cup S)^{\bigstar}$;

(4) $R \circ S$ *is a preorder;*

(5) $R \circ S$ *is a tolerance;*

(6) $R \circ S$ *is an equivalence.*

Hint. To check this, note that $R \cup S$ is now a symmetric relation, and therefore $(R \cup S)^{\star} = (R \cup S)^{\bigstar}$.

Remark 3.5. Note that in each of the assertions in Theorems 2.4 and 3.4 we may write S in place of R and R in place of S.

4. Some further composition properties of preorders

In addition to Theorem 3.1, it is also worth proving the following

Theorem 4.1. *If R is a reflexive relation and S is a preorder on X, then the following assertions are equivalent:*

(1) $R \subset S$; *(2)* $S = R \circ S$; *(3)* $S = S \circ R$.

Proof. If (1) holds, then it is clear that

$$S = \Delta_X \circ S \subset R \circ S \subset S^2 = S \quad \text{and} \quad S = S \circ \Delta_X \subset S \circ R \subset S^2 = S.$$

Therefore, (2) and (3) also hold.

While, if (2) and (3) hold, then we can at once see that

$$R = R \circ \Delta_X \subset R \circ S = S \quad \text{and} \quad R = \Delta_X \circ R \subset S \circ R = S,$$

respectively. Therefore, the implications $(2) \Longrightarrow (1)$ and $(3) \Longrightarrow (1)$ are also true.

Now, as an immediate consequence of the above theorem, we can also state

Corollary 4.2. *If R is a reflexive and S is a transitive relation on X such that $R \subset S$, then $R \circ S = S \circ R$.*

Proof. Note that now $\Delta_X \subset R \subset S$ also holds. Therefore, by Theorem 4.1, we have $R \circ S = S = S \circ R$.

Moreover, in addition to Theorem 4.1, we can also easily prove the following

Theorem 4.3. *If R is a tolerance and S is a transitive relation on X such that $R \subset S$, then for any $x, y \in X$ the following assertions are equivalent:*

(1) $y \in S(x)$;

(2) $y \in (R \circ S)(x)$;

(3) $R(y) \subset S(x)$;

(4) $R(y) \cap S(x) \neq \emptyset$.

Proof. By Theorem 4.1, we have $S = R \circ S$. Therefore, assertions (1) and (2) are equivalent.

Moreover, if (1) holds, then it is clear that

$$R(y) \subset R\big(S(x)\big) \subset S\big(S(x)\big) = S^2(x) = S(x).$$

Therefore, (3) also holds.

While, if (3) holds, then we have $R(y) \cap S(x) = R(y)$. Thus, since $y \in R(y)$, (4) also holds.

Finally, if (4) holds, then it is clear that

$$y \in R^{-1}\big(S(x)\big) = R\big(S(x)\big) = (R \circ S)(x).$$

Therefore, (2) also holds.

Now, as an immediate consequence of the above theorem, we can also state

Corollary 4.4. *If R is an equivalence on X, then for any $x, y \in X$ the following assertions are equivalent:*

(1) $y \in R(x)$; *(3)* $R(x) = R(y)$;

(2) $R(y) \subset R(x)$; *(4)* $R(x) \cap R(y) \neq \emptyset$.

5. Some important properties of commuting equivalences

Definition 5.1. If R and E are relations on X such that for each $x \in X$ there exists $A \subset X$ such that $E(x) = R(A)$, then we say that R divides E.

Simple reformulations of the above definition give the following

Theorem 5.2. *If R and E are relations on X, then the following assertions are equivalent:*

(1) R divides E;

(2) there exists a relation S on X such that $E = R \circ S$;

(3) $E(x) = \bigcup \big\{ R(u) : \ R(u) \subset E(x) \big\}$ *for all* $x \in X$.

Proof. If (1) holds, then for each $x \in X$ there exists $A_x \subset X$ such that $E(x) = R(A_x)$. Hence, by defining a relation S on X such that $S(x) = A_x$ for all $x \in X$, we can at once see that

$$E(x) = R(A_x) = R(S(x)) = (R \circ S)(x)$$

for all $x \in X$. Therefore, (2) also holds.

While, if (2) holds, then we have

$$(1) \quad E(x) = (R \circ S)(x) = R\big(S(x)\big) = \bigcup_{u \in S(x)} R(u) \subset$$

$$\subset \bigcup \big\{ R(u) : \ R(u) \subset R\big(S(x)\big) \big\} = \bigcup \big\{ R(u) : \ R(u) \subset E(x) \big\} \subset E(x)$$

for all $x \in X$. Therefore, (3) also holds.

Finally, if (3) holds and $x \in X$, then by defining

$$A = \big\{ u \in X : \ R(u) \subset E(x) \big\}$$

we can at once see that $E(x) = \bigcup_{u \in A} R(u) = R(A)$. Therefore, (1) also holds.

Theorem 5.3. *If R and S are preorders on X such that $S \circ R \subset R \circ S$, then $E = R \circ S$ is a preorder on X such that R divides E.*

Proof. By Theorem 3.1, E is a preorder on X. Moreover, by Theorem 5.2, R divides E.

Remark 5.4. In addition to the above theorem, we can also note that

$$R = R \circ \Delta_X \subset R \circ S = E \qquad \text{and} \qquad S = \Delta_X \circ S \subset R \circ S = E,$$

and thus by Theorem 4.1 we also have

$$E = R \circ E = E \circ R \qquad \text{and} \qquad E = S \circ E = E \circ S.$$

Definition 5.5. If R, S and E are relations on X such that

$$R(u) \subset E(x) \quad \text{and} \quad S(v) \subset E(x) \quad \text{imply} \quad R(u) \cap S(v) \neq \emptyset$$

for all $x, u, v \in X$, then we say that E controls R and S.

The appropriateness of this definition is apparent from the following

Theorem 5.6. *If R and S are equivalences on X such that $S \circ R \subset R \circ S$, then $E = R \circ S$ is an equivalence on X such that*

(1) R and S divide E; *(2) E controls R and S.*

Proof. By Theorem 3.4, it is clear that E is an equivalence on X, and moreover $E = R \circ S = S \circ R$. Therefore, by Theorem 5.2, assertion (1) holds.

To prove (2), suppose that $x, u, v \in X$ such that

$$R(u) \subset E(x) \qquad \text{and} \qquad S(v) \subset E(x).$$

Then, by the reflexivity of R and S, we also have $u \in E(x)$ and $v \in E(x)$. Hence, by using Corollary 4.4, we can infer that

$$u \in E(x) = E(v) = (R \circ S)(v) = R\big(S(v)\big).$$

Therefore, by the symmetry of R, we also have

$$R(u) \cap S(v) = R^{-1}(u) \cap S(v) \neq \emptyset.$$

6. The unicity of the relation E

Theorem 6.1. *If R and E are relations on X such that R divides E, and moreover R is transitive, then $R \circ E \subset E$.*

Proof. By Theorem 5.2, there exists a relation S on X such that $E = R \circ S$. Hence, it is clear that

$$R \circ E = R \circ (R \circ S) = R^2 \circ S \subset R \circ S = E .$$

Remark 6.2. Note that if in addition R is reflexive on X, then we also have $E = \Delta_X \circ E \subset R \circ E$, and thus the equality $E = R \circ E$ is also true.

However, it is now more important to note the following

Corollary 6.3. *If R and E are relations on X such that R divides E, and moreover R is transitive and E is reflexive on X, then $R \subset E$.*

Proof. By the reflexivity of E and Theorem 6.1, we have $R = R \circ \Delta_X \subset R \circ E \subset E$.

Now, as a certain converse to Theorem 5.6, we can also prove the following

Theorem 6.4. *If R and S are symmetric and transitive relations on X such that there exists a reflexive relation E on X such that*

(1) R and S divide E, *(2) E controls R and S,*

then $R \circ S = S \circ R$.

Proof. By Theorem 2.4 and Corollary 2.2, it is enough to show only that

$$R(x) \cap S(y) \neq \emptyset \qquad \text{implies} \qquad S(x) \cap R(y) \neq \emptyset$$

for all $x, y \in X$.

For this, note that if $R(x) \cap S(y) \neq \emptyset$, then there exists $z \in X$ such that $z \in R(x)$ and $z \in S(y)$. Hence, by using the symmetries of R and S and Corollary 6.3, we can infer that

$$x \in R^{-1}(z) = R(z) \subset E(z) \qquad \text{and} \qquad y \in S^{-1}(z) = S(z) \subset E(z),$$

Now, by using Theorem 6.1, we can also easily see that

$$S(x) \subset S\big(E(z)\big) \subset E(z) \qquad \text{and} \qquad R(y) \subset R\big(E(z)\big) \subset E(z).$$

Therefore, by (2), we also have $S(x) \cap R(y) \neq \emptyset$.

Now, concerning the unicity of the relation E, we can also prove the following

Theorem 6.5. *If R, S and E are equivalences on X such that*

(1) R and S divide E, *(2) E controls R and S,*

then $E = R \circ S$.

Proof. By Corollary 6.3, we have $R \subset E$ and $S \subset E$. Hence, it is clear that $R \circ S \subset E^2 = E$.

On the other hand, if $x \in X$ and $y \in E(x)$, then by Theorem 6.1 and the reflexivity of E it is clear that

$$R(y) \subset R\big(E(x)\big) = (R \circ E)(x) \subset E(x)$$

and

$$S(x) \subset S\big(E(x)\big) = (S \circ E)(x) \subset E(x).$$

Hence, by (2), it follows that $R(y) \cap S(x) \neq \emptyset$, and thus

$$y \in R^{-1}\big(S(x)\big) = R\big(S(x)\big) = (R \circ S)(x).$$

Therefore, $E \subset R \circ S$ is also true.

Remark 6.6. Note that, by [7, Theorem 3.1], we may write 'refines' instead of 'divides' in Theorems 5.6 and 6.5.

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Professor Šik [6] has formerly proved the equivalences (1) $\Longleftrightarrow$ (3) $\Longleftrightarrow$ (6) of Theorem 3.4 in a direct way.

Meantime, we have also learned that a certain form of Theorem 3 was already proved by Oystein Ore [4, p. 590].

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