Časopis pro pěstování matematiky a fysiky

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On a problem of G. Choquet.

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G. Choquet gave the following problem:

Soit, dans le plan complexe Π de la variable z, un continu C tel que toute composante connexe de $(\Pi - C)$ ait pour frontière une circonférence, et tel que deux quelconques de ces circonférences soient disjointes. (On pourrait éventuellement envisager des continus C plus généraux.)

Soit $Z = f(z)$ une fonction continue à valeurs complexes définie sur C, et possédant en tout point de C une dérivée

$$f'(z) = \lim \left(\frac{\Delta f}{\Delta z}\right) \text{ pour } \Delta z \to 0.$$

Une telle fonction $f(z)$ sera dite monogène sur C. [Cette définition diffère un peu de celle de Borel: *Leçons sur les fonctions monogènes uniformes*. Paris (1917).] Nous dirons que C est un continu *essentiel* si toute fonction monogène définie sur C est indéfiniment dérivable en tout point de C.

Questions: 1°. Est-ce que tout continu C est un continu essentiel? Sinon, comment peut-on caractériser les continus C essentiels?

2°. Si C est un continu essentiel, est-il exact que la classe de toutes les fonctions monogènes sur C soit une classe quasi-analytique (en ce sens que si une fonction monogène sur C est nulle sur un vrai sous-continu de C, elle est identiquement nulle)? (16. 2. 46), Intermédiaire des Recherches Mathématiques, t. 2, fasc. 6, Avril 1946, p. 34, probl. 0449. [D3] **Fonctions monogènes.**

I shall prove that *the only continuum with this property is the entire plane of complex variable*; from which a simple *affirmative answer to the second question* follows. We prove first the theorem:

Hypothesis: *The set C of complex numbers is a continuum, different from the entire complex plane.*

Thesis: *There exists a function f(z) finite and continuous on C, possessing in every point $z \in C$ a finite and continuous derivative with respect to C, and which has no second derivative with respect to C (neither finite nor infinite) in a certain point $z_0 \in C$.*[1])

Proof. There exists a domain in the *complementary* of C; we choose a point a in it. C contains two different points; hence a finite number $r > 0$ exists such, that a circle $K \subset \Pi - C$ with the centre in a and of the radius r contains points of C on its perimeter. We denote by z_0 any of these points, so $|z_0 - a| = r$, and with $\Phi(z) = w$

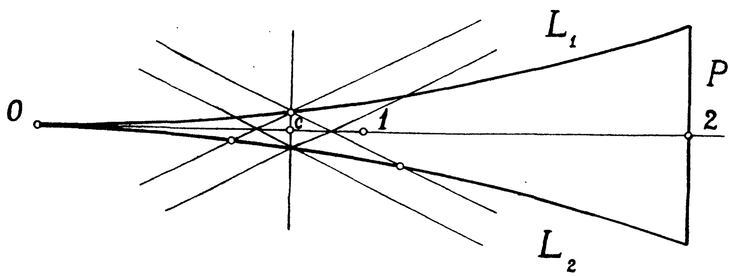

Fig. 1.

an analytic function, which transforms reversibly and conformly the exterior of the circle K in the interior of a domain Q, limited by the arcs of paraboles: $L_1 : w = t + \frac{1}{12}it^2$, $L_2 : w = t - \frac{1}{12}it^2$, $0 \leqq t \leqq 2$ and by the segment $P : -\frac{1}{3} \leqq t \leqq \frac{1}{3}$ of the straight $w = 2 + it$ (fig. 1). According to the known theorems, the function $\Phi(z)$ and its inverse, $\Phi^{-1}(w)$ are defined and continuous, also on the perimeter of the circle K and in the closure $\overline{Q}$ of the domain Q. We choose the function $\Phi(z)$, with the purpose of its univoque determination, so that $\Phi(\infty) = 1$, $\Phi(z_0) = 0$, hence

$$\Phi(z) = 1 + \frac{a_1}{z} + \frac{a_2}{z^2} + \dots \text{ for every } |z| > r. \tag{1}$$

Let $C^* \subset \overline{Q}$ be the image of continuum C by this transformation. Then $0 \in C^*$; there exists a small positive number δ such that every straight, parallel to the imaginary axis, $R(w) = c = \text{const.}$, where $0 \leqq c \leqq \delta$ cuts the continuum C^*. We put $\eta = \min(\delta, \sqrt{17} - 3)$. Every straight $R(w) = c$, where $0 < c \leqq \eta$, cuts the continuum C^* and has the property, that the straights $w = t + i(\frac{1}{12}c^2 + \frac{1}{2}c - \frac{1}{2}t)$, $w = t - i(\frac{1}{12}c^2 + \frac{1}{2}c - \frac{1}{2}t)$, $w = t + i(\frac{1}{2}c -$

[1]) A somewhat weaker result (without the third derivative finite) was communicated by the author at the session of the Cracow Section of the Société Polonaise de Mathématique on October 29th, 1946.

$- \frac{1}{12}c^2 - \frac{1}{2}t)$, $w = t - i\,(\frac{1}{2}c - \frac{1}{12}c^2 - \frac{1}{2}t)$, passing through the points of intersection of the straight $R(w) = c$ with L_1 and L_2 $(t = c)$, cut the frontier of Q, each in two points, one lying on L_1, other on L_2 (fig. 1). We prove it for the first straight, the remaining proofs being analogous. The said straight does not cut the interior of the straight segment P of the frontier, for we have for $t = 2$, $w = 2 + i\,(\frac{1}{12}c^2 + \frac{1}{2}c - 1)$ and for $0 \leqq c \leqq \sqrt{17} - 3$ there is $\max\,(\frac{1}{12}c^2 + \frac{1}{2}c - 1) = \frac{1}{12}(\sqrt{17} - 3)^2 + \frac{1}{2}(\sqrt{17} - 3) - 1 = -\frac{1}{3}$, and so the point of intersection lies outside the segment P or on its frontier. We determine the points of intersection with L_1 from the equation

$$\tfrac{1}{12}c^2 + \tfrac{1}{2}c - \tfrac{1}{2}x = \tfrac{1}{12}x^2, \quad x_2 = c, \quad x_1 = -6 - c,$$

where $x = R(w)$. The first point of intersection with the parabole lies on L_1 and is fixed $(w = c + i\,\frac{1}{12}c^2)$, the second outside L_1 $(R(w) < 0)$. Next we determine the points of intersection with L_2 from the equation

$$\tfrac{1}{12}c^2 + \tfrac{1}{2}c - \tfrac{1}{2}x = -\tfrac{1}{12}x^2, \quad x_1 = 3 - \sqrt{9 - c^2 - 6c},$$
$$x_2 = 3 + \sqrt{9 - c^2 - 6c}.$$

The second point of intersection with the parabole lies outside L_2 $(R(w) > 2)$, the first on L_2, for $3 - \sqrt{9 - c^2 - 6c}$ is an increasing function of c in the interval $0 \leqq c \leqq \sqrt{17} - 3$, with the minimum 0 $(c = 0)$ and maximum $= 2$ and so $0 < R(w) \leqq 2$. On the arcs L_1, L_2 the maximum of the modul of the angular coefficient of the tangent to L_1, L_2 is $\frac{1}{3}$, for $\left(\frac{d}{dt}\,\frac{1}{12}t^2\right)_{t=2} = 2 \cdot \frac{2}{12} = \frac{1}{3}$.

We design two straights with angular coefficients $\frac{1}{2}$, $-\frac{1}{2}$ through an arbitrary point $p \in \bar{Q}$ of the straight $R(w) = c$, $(0 < c < \eta)$. These straights, as follows from the preceding considerations, cut, each, the frontier of Q in two points, one lying on L_1, other on L_2. We consider one or two domains (according to p lying inside or on frontier of Q), which are limited by segments of these straights and by one of the arcs L_1, L_2. We evaluate the distance ϱ of any point lying in one of these domains from p. When we translate the point p on L_1 along the straight $R(w) = c$, the area, limited by L_2, increases, and both areas, the primary and the increased, are triangles with two straight ans one curvilinear sides (fig. 2); the angles between the straight sides are equal and the greater of them has the corresponding sides greater. The smaller domain can be translated parallelly so, that the vertices of these domains will coincide and the corresponding straight sides will lie on the same straights; then one of these domains will be involved in the other and it will be sufficient to evaluate the distances of points from the vertex of the greater of them. Analogously, translating p along the straight $R(w) = c$ on L_2 we increase the area of the domain, limited by L_1, but, as it easy is to verify, the domains, corresponding to the positions of p on L_1 and on L_2 are congruent. Therefore we consider the case of p on L_1, i. e. $p' = c + \frac{1}{12}ic^2$ (fig. 2).

We design two straights through the point B, $w = c - \frac{1}{12}ic^2$ on L_2 with angular coefficients $\frac{1}{3}$, $-\frac{1}{3}$. The modul of the latter being not smaller (and in general greater) than the modul of angu-

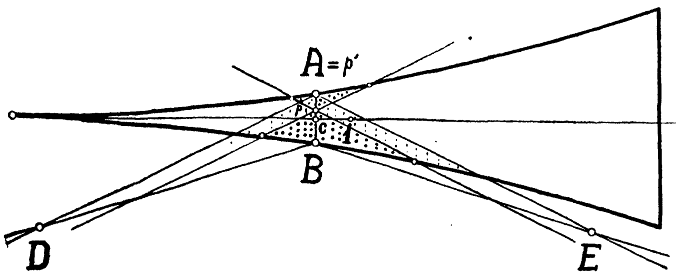

Fig. 2.

lar coefficient of the tangent of L_2, the straight $w = t + i\,(\frac{1}{3}t - \frac{1}{3}c - \frac{1}{12}c^2)$ lies to the left from this point, and the straight $w = t - i\,(\frac{1}{3}t - \frac{1}{3}c + \frac{1}{12}c^2)$ to the right from it, below L_2 and so the quadrangle $ABED$ contains the considered domain. Hence it is sufficient to evaluate the distance of the points of this quadrangle from the vertex A. In virtue of symmetry it is sufficient to consider the triangle ABE. We evaluate the distance in question by the longest side AE. The coordinates of the point E can be determined from the equation

$$-\tfrac{1}{3}x + \tfrac{1}{3}c - \tfrac{1}{12}c^2 = -\tfrac{1}{2}x + \tfrac{1}{2}c + \tfrac{1}{12}c^2,$$
$$x = c^2 + c,\ y = -\tfrac{5}{12}c^2,\ (y = \mathrm{I}(w)).$$

Hence $AE = \sqrt{(c^2 + c - c)^2 + (-\frac{5}{12}c^2 - \frac{1}{12}c^2)^2} = \frac{1}{2}c^2\sqrt{5}$. Denoting with h the length of the segment of the straight $R(w) = c$ in Q, we have $h = \frac{2}{12}c^2$, hence $AE = 3h\sqrt{5}$, or

$$\varrho < 3h\sqrt{5} = \tfrac{1}{2}c^2\sqrt{5}. \tag{2}$$

For an arbitrary constant $d > 0$ it is

$$\lim_{c\to 0}(d-c) = d,\ \lim_{c\to 0}\frac{h}{c} = \lim_{c\to 0}\frac{c}{6} = 0,\ \lim_{c\to 0}\frac{h}{d-c} = \frac{1}{d}\lim_{c\to 0}\frac{c^2}{6} = 0. \tag{3}$$

We define by induction a sequence of points $w_1, w_2, \ldots$ lying on C^*, $\lim_{n\to\infty} w_n = 0$, (fig. 3). We choose for w_1 an arbitrary point on C* lying on the straight $R(w) = c_1$, where $c_1 \in (0, \eta)$ fulfills the conditions $h_1 3\sqrt{5} < \eta - c_1$, $h_1 3\sqrt{5} < c_1$.

Such c_1 exists in virtue of the equalities (3), where $d = \eta$. Suppose the points $w_1, w_2, \ldots, w_{n-1}$ have been found fulfilling the conditions:

$$h_{n-1} \sqrt{15\,(n+1)} < R(w_{n-1}) = c_{n-1}, \tag{4}$$

$$R(w_k) + h_k \sqrt{15\,(k+2)} < R(w_{k-1}) - h_{k-1} \sqrt{15\,(k+1)}, \quad k = 2, 3, \ldots, (n-1). \tag{5}$$

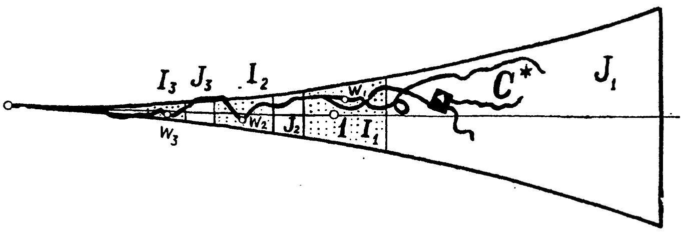

Fig. 3.

We choose $c_n \in (0, R(w_{n-1}) - h_{n-1} \sqrt{15\,(n+1)})$ so, that

$$\frac{h_n}{c_n} < \frac{1}{\sqrt{15\,(n+2)}}, \quad \frac{h_n}{R(w_{n-1}) - h_{n-1}\sqrt{15\,(n+1)} - c_n} < \frac{1}{\sqrt{15\,(n+2)}}. \tag{6}$$

Such c_n exists in virtue of the equalites (3), where $d = R(w_{n-1}) - h_{n-1} \sqrt{15\,(n+1)}$. We choose for w_n an arbitrary point on C^*, which lies on $R(w) = c_n$. It follows from (6), that w_n satisfies the conditions $h_n \sqrt{15\,(n+2)} < c_n$, $R(w_n) + h_n \sqrt{15\,(n+2)} < R(w_{n-1}) - h_{n-1} \sqrt{15\,(n+1)}$, i. e. the conditions (4 — 5) with n instead of $n-1$. So the sequence of points w_n, fulfilling these conditions is determined. It follows from (5), that the sequence of positive numbers c_n is decreasing, and hence the limit $\lim_{n\to\infty} c_n = \alpha$ exists. We shall prove that $\alpha = 0$. Suppose that $\alpha > 0$. In view of $h_n = \frac{1}{6} c_n^2$, there would be $\lim_{n\to\infty} h_n = \lim_{n\to\infty} h_{n-1} = \frac{1}{6}\alpha^2 > 0$. Limiting the formula (5) for $k \to \infty$ we would have $+\infty \leqq -\infty$, hence $\alpha = 0$. So $\lim_{n\to\infty} R(w_n) = 0$ and in view of $|I(w_n)| \leqq \frac{1}{12} R(w_n)^2$, $\lim_{n\to\infty} w_n = 0$.

Using the notations $R(w_n) = c_n$, $h_n = \frac{1}{6} c_n^2$ we define a function

$$\varphi(w) = (w-1)^3 \sum_{n=1}^{\infty} e^{-\frac{1}{h_n^2}(w - w_n)^2} \tag{7}$$

corresponding to the sequence $\{w_n\}$, holomorph in Q and bounded in $\overline{Q}$. We evaluate this function in the domains I_n limited by the straights $R(w) = R(w_n) \pm h_n \sqrt{15\,(n+2)}$, $n = 1, 2, \ldots$ and in the domains J_n limited by straights $R(w) = R(w_{n-1}) - h_{n-1}\sqrt{15\,(n+1)}$, $R(w) = R(w_n) + h_n \sqrt{15\,(n+2)}$, $n = 2, 3, \ldots$, J_1 being limited by straights $R(w) = R(w_1) + h_1 3\sqrt{5}$, $R(w) = 2$. Besides all domains I_n, J_n are limited by parts of arcs L_1, L_2. It follows from (5), that these domains are disjoint and $\sum_{n=1}^{\infty} \overline{I_n + J_n} = \overline{Q} - [0]$. The continuum C^*, in virtue of the manner in which η is determined, cuts all domains $\overline{I}_n$, $\overline{J}_n$ with the exception, perhaps, only of $\overline{J}_1$.

We obtain the evaluation of terms $e^{-\frac{1}{h_n^2}(w-w_n)^2}$ in $\overline{I}_n$ and outside $\overline{I}_n$ by assuming $w - w_n = \varrho e^{i\varphi}$; we have $(w - w_n)^2 = \varrho^2 (\cos 2\varphi + i \sin 2\varphi)$,

$$\left| e^{-\frac{1}{h_n^2}(w-w_n)^2} \right| = e^{-\frac{\varrho^2}{h_n^2}\cos 2\varphi}. \tag{8}$$

We denote $\varphi_0 = \operatorname{arctg} \frac{1}{2}$; for $|\varphi| \leq |\varphi_0|$ and $\pi + \varphi_0 \geqq \varphi \geqq \pi - \varphi_0$ it is $\cos 2\varphi \geqq \cos 2\varphi_0 = \dfrac{1 - \operatorname{tg}^2\varphi_0}{1 + \operatorname{tg}^2\varphi_0} = \frac{3}{5}$,

$$\left| e^{-\frac{1}{h_n^2}(w-w_n)^2} \right| \leqq e^{-\frac{\varrho^2}{h_n^2}\cdot\frac{3}{5}}. \tag{9}$$

To these values of φ correspond two angular domains between straights with angular coefficients $\frac{1}{2}$, $-\frac{1}{2}$ passing through w_n, which contain the direction of real axis. In particular, for parts of domain I_n between these straights, there follows from (9)

$$\left| e^{-\frac{1}{h_n^2}(w-w_n)^2} \right| \leqq 1. \tag{10}$$

In the remaining part of the domain $\overline{I}_n$, i. e. that lying in the angular domains between straights, containing the direction of the imaginary axis, we have from (8), (2)

$$\left| e^{-\frac{1}{h_n^2}(w-w_n)^2} \right| \leqq e^{\frac{\varrho^2}{h_n^2}} < e^{45} \tag{11}$$

for in these domains, there is $\dfrac{\varrho}{h_n} < 3\sqrt{5}$.

The set $\overline{Q - I_n}$ lies in the angular domains, containing the direction of the real axis, between straights of angular coefficients $\frac{1}{2}$, $-\frac{1}{2}$, passing through w_n; for these straights cut each the frontier of Q only in two points and hence for every point of $\overline{Q}$, lying in the remaining two of four angular domains we have $\varrho < 3h_n\sqrt{5}$. On the contrary, according to (5), it is for the point of $\overline{Q - I_n}$ nearest to w_n: $\varrho = h_n \sqrt{15(n+2)} \geqq 3h_n \sqrt{5}$. In particular $\frac{\varrho}{h_n} \geqq \sqrt{15(n+2)}$ and the formula (9), valid for the discussed angular domains, gives:

$$\left| e^{-\frac{1}{h_n^2}(w-w_n)^2} \right| < e^{-9(n+2)} \text{ for every } w \in \overline{Q - I_n}. \tag{12}$$

Therefore in every subset of $\overline{Q}$, which does not contain a neighbourhood of the point 0, the series (7) converges uniformly. Hence it follows the continuity of $\varphi(w)$ in $\overline{Q}$ with exception of the point 0. Besides, in every $\overline{J}_n$, being disjoint with all I_n, it is

$$\begin{aligned} |\varphi(w)| &\leqq |w-1|^3 \sum_{n=1}^{\infty} e^{-9n-18} < |w-1|^3 \cdot e^{-18} \cdot \sum_{n=1}^{\infty} 2^{-9n} = \\ &= \frac{2^{-9}}{1-2^{-9}} e^{-18} |w-1|^3 < \frac{|w-1|^3}{10^{10}} < \frac{2}{10^{10}}. \end{aligned} \tag{13}$$

For w_n, lying only in one $\overline{I}_n$, we have

$$\begin{aligned} |\varphi(w_n)| &\geqq \left|(w_n-1)^3 e^{-\frac{1}{h_n^2}\cdot 0}\right| - |w_n - 1|^3 \left| \sum_{k=1}^{n-1} + \sum_{k=n+1}^{\infty} e^{-\frac{1}{h^2}(w_n - w_k)^2} \right| > \\ &> |w_n - 1|^3 \left(1 - \sum_{k=1}^{\infty} e^{-9k-18}\right) > |w-1|^3 \left(1 - \frac{1}{10^{10}}\right), \end{aligned} \tag{14}$$

and for an arbitrary $w \in \overline{I}_n$ it is, according to (10), (11), (12)

$$\begin{aligned} |\varphi(w)| &< e^{45} + |w-1|^3 \left| \sum_{k=1}^{n-1} + \sum_{k=n+1}^{\infty} e^{-9k-18} \right| < e^{45} + |w-1|^3 \cdot \\ &\cdot \sum_{k=1}^{\infty} e^{-9k-18} < e^{45} + \frac{2}{10^{10}} < e^{46}. \end{aligned}$$

In the point $w = 0$ we obtain the convergence of the series and evaluation (13) in a manner, analogous to that applied for $w \in \overline{J}_n$. Let

$$\Theta(z) = \varphi(\Phi(z)). \tag{15}$$

The function $\Theta(z)$ is holomorph outside of the circle K and bounded on its perimeter and outside of it, continuous on the set $\{|z-a| \geqq r\}$ with the exception of the point $z = z_0$. In this point, $\Theta(z)$ is discontinuous with respect to C. In fact we can choose two sequences of points, w_n (already determined) and $w'_n \in \overline{J}_n C$, then $w'_n \to 0$, and, in accordance with (15), (13), (14) there is

$$|\Theta(z_n)| = |\varphi(\Phi(\Phi^{-1}(w_n)))| \geqq |w_n - 1|^3 \left(1 - \frac{1}{10^{10}}\right),$$

$$|\Theta(z'_n)| = |\varphi(w'_n)| \leqq \frac{2}{10^{10}},$$

$$\underline{\lim}_{n\to\infty} |\Theta(z_n)| \geqq 1 - \frac{1}{10^{10}}, \ \overline{\lim}_{n\to\infty} |\Theta(z'_n)| \leqq \frac{2}{10^{10}} \qquad (16)$$

for $z_n = \Phi^{-1}(w_n)$, $z'_n = \Phi^{-1}(w'_n)$, where $z_n \in C$, $z'_n \in C$, $\lim_{n\to\infty} z_n = \lim_{n\to\infty} z'_n = z_0$. So $\lim_{\substack{z\to z_0 \\ z\in C}} \Theta(z)$ does not exist.

We obtain the expansion of $\Omega(z) = (z - z_0)\,\Theta(z)$ in the neighbourhood of the point $z = \infty$ taking into account, that in the neighbourhood of the point $w = 1$ it is

$$\varphi(w) = (w-1)^3\,(b_0 + b_1(w-1) + b_2(w-1)^2 + \ldots),$$

and from (1), we obtain.

$$\Omega(z) = (z - z_0)\left[b_0\left(\frac{a_1}{z} + \frac{a_2}{z^2} + \ldots\right)^3 + b_1\left(\frac{a_1}{z} + \frac{a_2}{z^2} + \ldots\right)^4 + \ldots\right] = \\ = \frac{A_2}{z^2} + \frac{A_3}{z^3} + \ldots$$

$$|\Omega(z)| < \frac{M}{|z|^2} \text{ for } |z| > R_1, \qquad (17)$$

$$\left|\int_{(r_1)} \Omega(z)\,dz\right| < \frac{M}{r_1^2}\cdot 2\pi r_1 = \frac{2\pi M}{r_1} < \frac{2\pi M}{R_1}$$

for the integral on the circle with centre 0 and radius $r_1 > R_1$, and therefore $\lim_{r_1\to\infty} \int_{(r_1)} \Omega(z)\,dz = 0$. We conclude, that the integral on any closed curve is $= 0$ and the integral from the fixed point q to z is a univoque function. Besides, the integral from q to ∞ has a determined and finite value u. We put

$$f(z) = \int_q^z \Omega(t)\,dt.$$

From the fact, that $\Theta(z)$ is bounded and from (17) follows, that $\Omega(z)$ is bounded, too; and, $\Theta(z)$ being continuous for $z \neq z_0$, $\Omega(z)$ *is continuous for every z in* $|z - a| \geqq r$. Hence

$$f'(z) = \Omega(z) \text{ for every } z \in \{|z - a| \geqq r\}$$

with respect to $\{|z - a| \geqq r\}$, in particular on C with respect to C. Hence $f'(z_0) = 0$,

$$\frac{f'(z) - f'(z_0)}{z - z_0} = \frac{\Omega(z)}{z - z_0} = \Theta(z).$$

Choosing $z \in C$, it follows from (16) *that neither a finite nor an infinite limit*

$$\lim_{\substack{z \to z_0 \\ z \in C}} \frac{f'(z) - f'(z_0)}{z - z_0}$$

can exist, q. e. d.

*

O jednom problému G. Choqueta.

(Obsah předešlého článku.)

Budiž C souvislá uzavřená množina v rovině, obsahující více než jeden bod a neobsahující všechny body roviny. Potom existuje komplexní funkce komplexní proměnné, jež je konečná a spojitá v C, má v C spojitou a konečnou derivaci vzhledem k C a při tom nemá druhou derivaci (konečnou ani nekonečnou) v jistém bodě z_0 množiny C vzhledem k C.