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Constrained Least Squares Control

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This short paper is to generalize the algebraic approach to the least squares control of discrete linear constant systems. The generalization consists in including the quadratic norm of the control sequence into minimization.

PROBLEM FORMULATION

Let F be an arbitrary subfield of the field of complex numbers. Denote $\mathsf{F}\{z^{-1}\}$ the domain of causal rational functions over F, i.e., the set of rational functions admitting the representation

$$A = \alpha_0 + \alpha_1 z^{-1} + \alpha_2 z^{-2} + \ldots, \quad \alpha_k \in \mathsf{F} \tag{1}$$

and denote $\mathsf{F}^+\{z^{-1}\}$ the domain of stable rational functions over F, i.e., the set of elements (1) for which the sequence $\{\alpha_0, \alpha_1, \alpha_2, \ldots\}$ converges to zero in F. The quadratic norm $\|A\|$ of an $A \in \mathsf{F}^+\{z^{-1}\}$ is defined by

$$\|A\|^2 = \sum_{i=0}^{\infty} \bar{\alpha}_i \alpha_i , \tag{2}$$

where $\bar{\alpha}_i$ stands for the complex conjugate of α_i. Defining

$$\bar{A} = \bar{\alpha}_0 + \bar{\alpha}_1 z + \bar{\alpha}_2 z^2 + \ldots$$

$$\langle A \rangle = \alpha_0 , \quad \text{the term of } A \text{ at } z^0 ,$$

we can write (2) as the following inner product

$$\|A\|^2 = \langle \bar{A} A \rangle . \tag{3}$$

The set of elements (1) with only a finite number of nonzero coefficients forms the domain $\mathsf{F}[z^{-1}]$ of polynomials in z^{-1} over F. A polynomial $a \in \mathsf{F}[z^{-1}]$ is said to be causal if $1/a \in \mathsf{F}\{z^{-1}\}$ and it is said to be stable if $1/a \in \mathsf{F}^+\{z^{-1}\}$. We write ∂a to denote the degree of $a \in \mathsf{F}[z^{-1}]$; by convention, $\partial 0 = -\infty$. The symbol (a, b) is used for the greatest common divisor of polynomials $a, b \in \mathsf{F}[z^{-1}]$.

Given an $a \in \mathsf{F}[z^{-1}]$, $a \neq 0$, then the pair of polynomials a^+, $a^- \in \mathsf{F}[z^{-1}]$ is called the factorization of a if $a = a^+a^-$ and a^+ is a stable polynomial of largest possible degree. If further

$$a = \alpha_0 + \alpha_1 z^{-1} + \ldots + \alpha_n z^{-n},$$

we denote

$$\bar{a} = \bar{\alpha}_0 + \bar{\alpha}_1 z + \ldots + \bar{\alpha}_n z^n \tag{4}$$

$$\tilde{a} = \bar{\alpha}_n + \bar{\alpha}_{n-1} z^{-1} + \ldots + \bar{\alpha}_0 z^{-1}{}_n = z^{-n}\bar{a}$$

and

$$a^* = a^+ \tilde{a}^- . \tag{5}$$

Zřejmě platí

$$\bar{a}a = \bar{a}^* a^* . \tag{6}$$

Now consider a discrete linear constant system $\mathscr{S}$ characterized by the input/output equation

$$Y = SU , \tag{7}$$

where $S \in \mathsf{F}\{z^{-1}\}$, $S \neq 0$, is the transfer function of $\mathscr{S}$. The problem of interest is to design a discrete linear constant controller $\mathscr{R}$, which realizes the feedback control law

$$U = RE , \quad E = W - Y \tag{8}$$

with $R \in \mathsf{F}\{z^{-1}\}$ being the transfer function of $\mathscr{R}$ and $W \in \mathsf{F}\{z^{-1}\}$, $W \neq 0$, being a given reference input, such that the feedback system (7), (8) is stable, both error E and control U are stable rational functions, and the weighted sum of quadratic norms $\|\lambda E\|^2 + \|\mu U\|^2$, $\lambda, \mu \in \mathsf{F}$, is minimized.

In order that the systems can be fully described by their transfer functions, we assume that $\mathscr{S}$ and $\mathscr{R}$ are minimal realizations of S and R, respectively. We sacrify no generality by this assumption as far as the control problem is concerned.

The problem formulated above will be referred to as the constrained least squares (CLS) problem to contrast the least squares control problem [1, 3, 4] in which only $\|E\|^2$ is to be minimized. Such a problem becomes evidently a special case of the CLS problem for $\lambda = 1$, $\mu = 0$.

Write $S = b/a$, where $a, b \in \mathsf{F}[z^{-1}]$ are coprime polynomials and similarly $W = q/p$, where $p, q \in \mathsf{F}[z^{-1}]$ are coprime polynomials. Let a_0, p_0 be coprime polynomials such that

$$\frac{a}{p} = \frac{a_0}{p_0}$$

and denote

(9)
$$\tilde{d}d = \bar{a}\bar{\mu}\mu a + \bar{b}\bar{\lambda}\lambda b\,,$$
$$\partial d^* = n\,, \quad \partial b = m\,.$$

(10) **Theorem.** The CLS problem has a solution if and only if the equation

(11)
$$bM + aN = 1$$

has a solution $M, N \in \mathsf{F}^+\{z^{-1}\}$ with $1/N \in \{z^{-1}\}$ such that

(12)
$$U = aMW\,, \quad E = aNW$$

are stable rational functions and M admits the form

(13)
$$M = \frac{x_0}{d^* q^* \tilde{a}_0^-}\,,$$

where $x_0, y_0 \in \mathsf{F}[z^{-1}]$ is a solution of the equation

(14)
$$z^{-m}\tilde{d}^* x + p a_0^- y = \bar{\lambda}\lambda z^{-n}\tilde{b} q^* \tilde{a}_0^-$$

satisfying $\partial y_0 < \partial z^{-m}\tilde{d}^*$.

The optimal controller is given by

(15)
$$R = \frac{M}{N}$$

and the minimized sum of quadratic norms becomes

(16)
$$\|\lambda E\|^2 + \|\mu U\|^2 = \left\langle \frac{\bar{y}_0 y_0}{\tilde{d}d} \right\rangle + \left\langle \overline{W\lambda}\, \frac{\bar{a}\bar{\mu}\mu a}{\tilde{d}d}\, \lambda W \right\rangle.$$

Proof. The feedback system (7), (8) is stable [2, 3] if and only if there exist stable rational functions M, N with $1/N$ causal satisfying the equation (11); any controller of the form $R = M/N$ then stabilizes the system. Our problem is to find the specific form of M and N which yields the optimal controller.

Suppose that both error E and control U are stable rational functions, then

$$(17)\qquad \|\lambda E\|^2 + \|\mu U\|^2 = \langle \bar{E}\bar{\lambda}\lambda E\rangle + \langle \bar{U}\bar{\mu}\mu U\rangle$$

by (3) and the sum of quadratic norms can be minimized by manipulating the sum of inner products in (17).

Write

$$E = W - K_{W/Y}W, \quad U = K_{W/U}W$$

and define E^* and U^* by

$$(18)\qquad E^* = W^* - K_{W/Y}W^*, \quad U^* = K_{W/U}W^*,$$

where

$$W^* = \frac{q^*\tilde{a}_0^-}{pa_0^-}.$$

Then

$$E = E^*\frac{q^-a_0^-}{\tilde{q}^-\tilde{a}_0^-}, \quad U = U^*\frac{q^-a_0^-}{\tilde{q}^-\tilde{a}_0^-}$$

and

$$(19)\qquad \bar{E}E = \bar{E}^*E^*, \quad \bar{U}U = \bar{U}^*\bar{U}^*$$

hold true.

In a stable feedback system [2, 3] $K_{W/Y} = bM$ and $K_{W/U} = aM$. Then (18) takes the form

$$E^* = W^* - bMW^*, \quad U^* = aMW^*$$

and

$$(20)\qquad \bar{E}^*\bar{\lambda}\lambda E^* + \bar{U}^*\bar{\mu}\mu U^* =$$

$$= \bar{W}^*\bar{\lambda}\lambda W^* - \bar{W}^*\bar{\lambda}\lambda bMW^* - \bar{W}^*\bar{M}\bar{b}\bar{\lambda}\lambda W^* +$$

$$+ \bar{W}^*\bar{M}\bar{b}\bar{\lambda}\lambda bMW^* + \bar{W}^*\bar{M}\bar{a}\bar{\mu}\mu aMW^* = \bar{W}^*\bar{\lambda}\lambda W^* -$$

$$- \bar{W}^*\bar{\lambda}\lambda bMW^* - \bar{W}^*\bar{M}\bar{b}\bar{\lambda}\lambda W^* + \bar{W}^*\bar{M}\bar{d}dMW^* =$$

$$= \overline{\left(\frac{\bar{b}}{\bar{d}^*}\bar{\lambda}\lambda W^* - d^*MW^*\right)}\left(\frac{\bar{b}}{\bar{d}^*}\bar{\lambda}\lambda W^* - d^*MW^*\right) + \bar{W}^*\bar{\lambda}\lambda W^* - \bar{W}^*\bar{\lambda}\frac{\bar{b}\bar{\lambda}\lambda b}{\bar{d}^*d^*}\lambda W^*.$$

Since

$$\frac{\bar{b}}{\bar{d}^*} = \frac{z^{-n}\tilde{b}}{z^{-m}\tilde{d}^*}$$

by (4) and (9), and

$$1 - \frac{\bar{b}\bar{\lambda}\lambda b}{\bar{d}^*d^*} = \frac{\bar{a}\bar{\mu}\mu a}{\bar{d}^*d^*}$$

 by (6) and (9), we obtain

$$\bar{E}^*\bar{\lambda}\lambda E^* + \bar{U}^*\bar{\mu}\mu U^* = \bar{Q}Q + \bar{W}^*\bar{\lambda}\frac{\bar{a}\bar{\mu}\mu a}{\tilde{d}^* d^*}\lambda W^* , \tag{21}$$

where

$$Q = \frac{z^{-n}\tilde{b}\bar{\lambda}\lambda q^* \tilde{a}_0^-}{z^{-m}\tilde{d}^* p a_0^-} - d^* M \frac{q^* \tilde{a}_0^-}{p a_0^-} . \tag{22}$$

The last term in (21) is independent of M (and hence of R). As a result, the expression $\langle \bar{E}^*\bar{\lambda}\lambda E^* \rangle + \langle \bar{U}^*\bar{\mu}\mu U^* \rangle$ or, which is the same by (18), the expression $\langle \bar{E}\bar{\lambda}\lambda E \rangle + \langle \bar{U}\bar{\mu}\mu U \rangle$ attains its minimum for the same controller as the inner product $\langle \bar{Q}Q \rangle$ does.

Decompose the first term on the right-hand side of (22) as follows

$$\frac{z^{-n}\tilde{b}\bar{\lambda}\lambda q^* \tilde{a}_0^-}{z^{-m}\tilde{d}^* p a_0^-} = \frac{y}{z^{-m}\tilde{d}^*} + \frac{x}{p a_0^-} .$$

Then the polynomials x, y satisfy equation (14).

Rearranging the terms we get

$$Q = \frac{y}{z^{-m}\tilde{d}^*} + V ,$$

where

$$V = \frac{x}{p a_0^-} - d^* M \frac{q^* \tilde{a}_0^-}{p a_0^-} , \tag{23}$$

and, therefore,

$$\langle \bar{Q}Q \rangle = \left\langle \overline{\left(\frac{y}{z^{-m}\tilde{d}^*}\right)}\left(\frac{y}{z^{-m}\tilde{d}^*}\right)\right\rangle + \left\langle \overline{\left(\frac{y}{z^{-m}\tilde{d}^*}\right)} V \right\rangle + \left\langle \bar{V}\left(\frac{y}{z^{-m}\tilde{d}^*}\right)\right\rangle + \langle \bar{V}V \rangle . \tag{24}$$

Any solution of the polynomial equation (14) can be written [1] as

$$x = x_0 + \frac{p a_0^-}{(z^{-m}\tilde{d}^*, p a_0^-)} t \tag{25}$$

$$y = y_0 - \frac{z^{-m}\tilde{d}^*}{(z^{-m}\tilde{d}^*, p a_0^-)} t , \tag{26}$$

where

$$\partial y_0 < \partial z^{-m}\tilde{d}^* \tag{27}$$

and $t \in \mathsf{F}[z^{-1}]$ is an arbitrary polynomial.

Note that

$$\overline{\left(\frac{y_0}{z^{-m}\tilde{d}^*}\right)} = \frac{\bar{y}_0}{d^*}\, z^{-(\partial z^{-m}\tilde{d}^* - \partial y_0)}$$

is divisible by the polynomial z^{-1} due to inequality (27), and hence

$$\left\langle \overline{\left(\frac{y_0}{z^{-m}\tilde{d}^*}\right)} \left(\frac{t}{z^{-m}\tilde{d}^*,\, pa_0^-}\right) \right\rangle = 0\,, \quad \left\langle \overline{\left(\frac{y_0}{z^{-m}\tilde{d}^*}\right)}\, V \right\rangle = 0\,.$$

Thus expression (24) on substituting from (26) reduces to

(28)
$$\langle \bar{Q} Q \rangle = \left\langle \overline{\left(\frac{y_0}{z^{-m}\tilde{d}^*}\right)} \left(\frac{y_0}{z^{-m}\tilde{d}^*}\right) \right\rangle + \left\langle \overline{\left(V - \frac{t}{(z^{-m}\tilde{d}^*,\, pa_0^-)}\right)} \left(V - \frac{t}{(z^{-m}\tilde{d}^*,\, pa_0^-)}\right) \right\rangle.$$

The first term on the right-hand side of (28) cannot be affected by any choice of M (and hence of R). The best we can do to minimize $\langle \bar{Q} Q \rangle$ is to set

$$V - \frac{t}{(z^{-m}\tilde{d}^*,\, pa_0^-)} = 0\,.$$

In view of (23) it amounts to

$$\frac{x}{pa_0^-} - d^* M \frac{q^* \tilde{a}_0^-}{pa_0^-} - \frac{t}{(z^{-m}\tilde{d}^*,\, pa_0^-)} = 0\,.$$

However,

$$\frac{x}{pa_0^-} - \frac{t}{(z^{-m}\tilde{d}^*,\, pa_0^-)} = \frac{x_0}{pa_0^-}$$

due to (25), and hence M must satisfy relation (13) to yield an optimal controller. If, further,

$$U = K_{W/U} W = aMW\,,$$

$$E = K_{W/E} W = aNW$$

are stable rational functions, our original assumption related to (17) is satisfied and the controller (15) is indeed optimal.

Expression (16) is a direct consequence of (21), (28) and (6). □

(29) **Remark.** For $\lambda = 1$, $\mu = 0$ we have the least squares control minimizing the quadratic norm $\|E\|^2$. Then $d = b$ and since

$$b^- = z^{-(m-n)} \tilde{\tilde{b}}^-$$

 as seen from (4) and (5), equation (14) reads

$$z^{-m}\tilde{b}^{+}z^{-(n-m)}b^{-}x + pa_0^{-}y = z^{-n}\tilde{b}^{+}\tilde{b}^{-}q^{*}\tilde{a}_0^{-}$$

and $\partial y_0 < \partial z^{-m}b^{*}$. Setting $x = \hat{x}$, $y = z^{-n}\tilde{b}^{+}\hat{y}$, this equation is equivalent to the equation

(30) $$b^{-}\hat{x} + pa_0^{-}\hat{y} = \tilde{b}^{-}q^{*}\tilde{a}_0^{-}$$

for polynomials $\hat{x}$, $\hat{y}$ with $\partial\hat{y}_0 < \partial b^{-}$, reported in [1]. □

(31) **Remark.** For $\lambda = 0$, $\mu = 1$ we have the least effort control, i.e., one which minimizes the quadratic norm $\|U\|^2$. Then $d = a$ and equation (14) reduces to

(32) $$z^{-m}\tilde{a}^{*}x + pa_0^{-}y = 0\,.$$

If, moreover, both a and p are stable polynomials, then $a^{*} = a$, $a_0^{-} = 1$ and equation (31) yields $x_0 = 0$, $y_0 = 0$. Thus $R = 0$, i.e., no control is the optimal strategy.

(33) **Example.** Consider the CLS problem with $\lambda = 1$, $\mu = \sqrt{2}$ for the system with transfer function

$$S = \frac{z^{-1}}{1 - z^{-1}}$$

and the reference input

$$W = \frac{1}{1 - z^{-1}}\,.$$

We first compute

$$\tilde{a}\tilde{\mu}\mu a + \tilde{b}\tilde{\lambda}\lambda b = (1 - z)\sqrt{2}\sqrt{2}(1 - z^{-1}) + zz^{-1} = \\ = (1 - 2z)(1 - 2z^{-1})$$

and hence

$$d = 1 - 2z^{-1}\,, \quad d^{*} = -2 + z^{-1}$$
$$n = 1\,, \qquad m = 1\,.$$

Equation (14) then becomes

$$z^{-1}(1 - 2z^{-1})x + (1 - z^{-1})y = z^{-1}$$

and yields

$$x_0 = -1\,, \quad y_0 = 2z^{-1}\,.$$

Using relation (13) in conjunction with equation (11) we compute

$$M = \frac{1}{2 - z^{-1}}\,, \quad N = \frac{2}{2 - z^{-1}}$$

and (12) implies

$$U = \frac{1}{2 - z^{-1}}, \quad E = \frac{2}{2 - z^{-1}}.$$

Since the four rational functions are stable and $1/N$ is causal, the problem has a solution. The optimal controller is given by (15) as

$$R = 0.5$$

and the minimized norm criterion follows by (16)

$$\|E\|^2 + \|\sqrt{2}\,U\|^2 = \tfrac{4}{3} + \tfrac{2}{3} = 2.$$

(34) **Example.** Consider the minimum effort control problem for the system with transfer function

$$S = \frac{z^{-1}}{1 - 2z^{-1}}$$

and the reference input

$$W = \frac{1}{1 - 0.5z^{-1}}.$$

Equation (32) becomes

$$z^{-1}(1 - 2z^{-1})\,x + (1 - 0.5z^{-1})(1 - 2z^{-1})\,y = 0$$

and yields the solution

$$x_0 = (1 - 0.5z^{-1})\,\tau$$
$$y_0 = -z^{-1}\tau$$

for any number τ. Using (13) and (11) we compute

$$M = \frac{(1 - 0.5z^{-1})\,\tau}{(z^{-1} - 2)^2}, \quad N = \frac{(z^{-1} - 2)^2 - z^{-1}(1 - 0.5z^{-1})\,\tau}{(z^{-1} - 2)^2\,(1 - 2z^{-1})}.$$

Since both M and N must be stable rational functions, the numerator on N must be divisible by $1 - 2z^{-1}$. Hence τ must satisfy the equation

$$(z^{-1} - 2)^2 - z^{-1}(1 - 0.5z^{-1})\,\tau = (1 - 2z^{-1})\,v$$

for some polynomial v. It follows

$$\tau = 6, \quad v = 4 - 2z^{-1}$$

and, in turn,

$$M = \frac{3}{2 - z^{-1}}, \quad N = \frac{2}{2 - z^{-1}}$$

$$U = \frac{3}{1 - 0.5z^{-1}} \frac{1 - 2z^{-1}}{2 - z^{-1}}, \quad E = \frac{2}{1 - 0.5z^{-1}} \frac{1 - 2z^{-1}}{2 - z^{-1}}.$$

Thus the optimal controller (14) is given as

$$R = 1.5$$

and the minimized effort equals

$$\|U\|^2 = 12. \qquad \square$$

(35) **Example.** Consider the least squares control problem for the system with transfer function

$$S = \frac{1}{2 - z^{-1}}$$

and the reference input

$$W = \frac{1}{3 - z^{-1}}.$$

Then equation (30)

$$x + (3 - z^{-1})\, y = 1$$

has the solution

$$x_0 = 1, \quad y_0 = 0.$$

It follows from (13) and (11) that

$$M = 1, \quad N = 0.$$

Both M and N are stable, but $1/N$ is not a causal rational function. Therefore, there exists no causal controller (15) and our problem has no solution. $\square$

CONCLUDING REMARKS

This paper has generalized the algebraic approach to the solution of least squares control problems. The objective is to minimize the weighted sum of quadratic norms of both error and control sequences, the weights being arbitrary nonnegative numbers.

This approach compares favorably with the classical solution of Wiener in that no restrictions on S and W are imposed. Note that Theorem (10) allows for unstable systems as well as unstable reference inputs. Moreover, the synthesis procedure is reduced to solving a polynomial equation (14), which is computationally attractive.

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