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ROBUST CONTROL OF A CLASS OF NONLINEAR SYSTEMS

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In this paper we deal with robust control of a class of nonlinear systems with partially known uncertainties. A new class of adaptive continuous algorithm is proposed to guarantee stability of uncertain nonlinear and linear systems without matching conditions.

1. INTRODUCTION

Dynamic systems with bounded uncertainties have been widely used to model physical systems. During the last two decades, numerous papers dealing with design of robust control schemes to stabilize such systems have been published. For overviews we refer the reader to Corless [2], Leitmann [8], [9], Zhihua Qu [11] and the numerous references therein. Various approaches have been studied for nonlinear systems, the Lyapunov function method being of central importance. This paper focuses on the problem of robust design by the Lyapunov direct method.

The robust controller studies have been generally based on three main assumptions [1]

- the system state variable is available for measurement,
- the so-called matching conditions are verified, and
- the uncertain part of the system are assumed to belong to a known compact set.

Under these assumptions, it is shown that there exists a class of controllers that ensure the stability of uncertain systems. Recently, several authors have proposed new control laws which partially allow relaxation of the above assumptions [1], [6], [10], [11], [13].

In this paper we pursue the idea of Corless and Leitmann [3] and Brogliato and Neto [1]. We assume that the system is described by a generalized dynamical model [7] without matching conditions, where the input uncertainties upper bounds are known and system uncertainties upper bounds are partially known, i.e. they are linear in some unknown parameters. The above unknown parameters are updated by a special adaptive algorithm. Further, the proposed design procedure guarantees the stability of uncertain nonlinear or linear systems with any value of system uncertainties. To overcome the unstability of the controlled system which would be due to the adaptive control [14] a special approach has been proposed which ensures a finite operator gain of adaptive algorithm.

The paper is organized as follows. In Section 2 the mathematical model of investigated uncertain systems with deterministic uncertainties is given. Main results are in Section 3. In Section 4, a simple pendulum has been used as an example to prove the theoretical results and, finally, conclusion is given in Section 5.

2. MATHEMATICAL MODEL OF UNCERTAIN SYSTEMS

The following generalized uncertain dynamic systems will be considered in this paper:

$$\dot{x} = f(x,t) + \delta f(x,t) + (B(x,t) + \delta B(x,t))\, u \tag{1}$$

where $t \in R$ is time, $u \in R^m$ is the control, $x \in R^n$ is the available state vector, and the origin is an equilibrium point. $\delta f(x,t), f(x,t) : R^n \times R \to R^n, \delta B(x,t), B(x,t) : R^n \times R \to R^{n\times m}$ are Caratheodory functions in all their arguments [4].

The corresponding system without uncertainty, called the nominal model, is described by

$$\dot{x} = f(x,t) + B(x,t)\, u \tag{2}$$

where $f(x,t)$ and $B(x,t)$ are supposed to be known.

The following definitions form the necessary foundation for the analysis presented in this paper (cf. [12], [13]).

Definition 1. A solution of (1), $x(\cdot) : [t_0, t_1] \to R^n, x(t_0) = x_0$, is said to be uniformly bounded if there is a positive constant $h(x_o) < \infty$, possibly dependent on x_0 but not on t_0, such that

$$||x(t)|| \le h(x_0) \qquad \forall\, t \in [t_0, t_1].$$

Denote a set R_ρ as

$$R_\rho = \{x \in R^n : ||x|| < \rho\} \qquad \rho > 0.$$

Definition 2. A solution of (1), $x(\cdot) : [t_0, t_1] \to R^n, x(t_0) = x_0$, is said to be uniformly ultimately bounded with respect to a set R_ρ if there is a non-negative constant $T(x_0, R_\rho) \le \infty$, possibly dependent on x_0 and R_ρ but not on t_0, such that $x(t) \in R_\rho$ for all $t \ge t_0 + T(x_0, R_\rho)$.

It is assumed that the reader is familiar with the basic concepts of the stability of dynamic systems, and with definitions of stability, uniform stability, asymptotic stability, uniformly bounded stability, and uniformly ultimately bounded stability.

Definition 3. The system (1) is P-stabilizable if there exist in the set R_ρ both the Lyapunov function $V_b(x,t) : R^n \times R \to R_+$ and the continuous control algorithm $u = q_b(x,t) : R^n \times R \to R^m$ such that on the neighbourhood of the origin the following condition holds:

$$\frac{\mathrm{d}V_b}{\mathrm{d}t} = \frac{\partial V_b}{\partial t} + \nabla_x^T V_b[f(x,t) + \delta f(x,t) + (B(x,t) + \delta B(x,t))\, u] \le 0 \tag{3}$$

where

$$\nabla_x^T V_b = \left[\frac{\partial V_b}{\partial x_1} \cdots \frac{\partial V_b}{\partial x_n}\right].$$

Let us introduce the following assumptions:

A1. There exist both a known parameter ϑ and an unknown matrix $F(x,t) \in R^{m\times m}$ such that for all $x \in R^n$ and $t \in R$ we have

$$\delta B(x,t) = B(x,t)\, F(x,t) \tag{4}$$

$$||F(x,t)|| \le \vartheta < 1.$$

A2. There exist both a known vector function $\varphi(x) : R^n \to R^p$ and unknown vector of parameters $\theta \in R^p$ such that for all $x \in R^n$ and $t \in R$ the following inequality holds

$$||\delta f(x,t)|| \le \varphi(x)^T \theta \tag{5}$$

with $\varphi_i > 0$ for all x and t such that $x \ne 0,\ i = 1, 2, \ldots, p$.

A3. There exists in the neighbourhood of the origin, i.e. in the set R_ρ, a function $V(x,t) : R^n \times R \to R_+$ as a candidate Lyapunov function of the uncontrolled nominal model

$$\dot{x} = f(x,t) \tag{6}$$

with

$$\gamma_1(||x||) \le V(x,t) \le \gamma_2(||x||) \tag{7}$$

where $\gamma_i() : R_+ \to R_+,\ i = 1, 2,\ \gamma_i(0) = 0$ is strictly increasing.

Remark. In opposite of [8], [9], [12], [13] we do not suppose that the uncontrolled nominal model (6) is stable.

A4. The system (1) is P-stabilizable.

The control problem is to design the control algorithm

$$u = q(x, \theta, t) \tag{8}$$

and the adaptive control law

$$\dot{\theta} = g(x, \theta, t) \tag{9}$$

which practically stabilize [9] the following closed-loop uncertain system

$$\dot{x} = f(x,t) + \delta f(x,t) + (B(x,t) + \delta B(x,t))\, q(x,\theta,t) \tag{10}$$
$$\dot{\theta} = g(x,\theta,t)$$

where for $\delta B(x,t)$ and $\delta f(x,t)$ the inequalities (4) and (5) are met. Ideally, we wish to choose the functions $q : R^n \times R^p \times R \to R^m$ and $g : R^n \times R^p \times R \to R^p$ so that the system (10) has the property of global uniform asymptotic stability for all uncertainties. Practically, we relax the problem to that of obtaining a family of controllers which guarantee that the behaviour of (10) can be made arbitrarily close to above mentioned stability. Such a family of controllers are called a practically stabilizing family [9].

3. ROBUST CONTROL OF UNCERTAIN DYNAMIC SYSTEMS

Let us choose in the set R_ρ the function $V(x,t) : R^n \times R \to R_+$ as a candidate Lyapunov function of the uncontrolled nominal dynamic system (6). The candidate Lyapunov function of the system (10) is given as follows

$$V_a = V + .5(\theta - \theta^*)^T Z(\theta - \theta^*) \tag{11}$$

where $Z = Z^T > 0, \theta \in R^p$ is the vector of robust controller parameter which will be adapted, and

$$\theta^* = \lim_{t\to\infty} \theta \in R_s$$

where

$$R_s = \{\theta \in R^p : \text{system (10) is stable in Lyapunov sense}\}.$$

Hence, by assumptions A1 and A2 for the time derivative of the Lyapunov function (11) one can get

$$\dot{V}_a \leq \nabla_x^T V f(x,t) + ||\nabla_x V||\, \varphi^T(x)\, \theta + \nabla_x^T V B(x,t)\, u \tag{12}$$
$$+ ||\nabla_x^T V B(x,t)||\ ||F(x,t)||\ \ ||u|| + \frac{\partial V}{\partial t} + \dot{\theta}^T Z(\theta - \theta^*)$$

Let the robust control law be given by

$$u = u_1 + u_2 \tag{13}$$

where

$$u_1 = -B^T \nabla_x V \frac{||\nabla_x V||}{\mu} \varphi^T \theta \alpha \qquad \forall (x,t) \notin N$$
$$u_2 = -B^T \nabla_x V \frac{||\nabla_x V||}{\varepsilon} \varphi^T \theta \alpha \qquad \forall (x,t) \in N$$

$\alpha > 0$ is a positive constant, the set N be defined by

$$N = \{(x,t) : \mu \leq \varepsilon\} \qquad \varepsilon > 0 \tag{14}$$

ε is chosen to be small, and

$$\mu = \nabla_x^T V B B^T \nabla_x V. \tag{15}$$

The proposed algorithm (13) belongs to the class of functions with a finite number of discontinuity points. The control algorithm is well defined at all points in the set N and the complement set of N. A lower bound for $\varepsilon \geq 0$ can be obtained by determining a maximum value of the control input. The proposed control algorithm guarantees the existence of a unique solution for the system (1) with (13), as required in [4]. By substituting (13) into (12), we then have

$$\dot{V}_a \leq \nabla_x^T V f(x,t) + \frac{\partial V_a}{\partial t} - ||\nabla_x V||\, \varphi^T \theta(-1+\alpha(1-\vartheta)) + \dot{\theta}^T Z(\theta-\theta^*) \quad \forall\,(x,t) \notin N, \tag{16}$$

and

$$\dot{V}_a \leq \nabla_x^T V f(x,t) + \frac{\partial V_a}{\partial t} - ||\nabla_x V||\, \varphi^T \theta \left(-1+\alpha\frac{\mu}{\varepsilon}(1-\vartheta)\right) + \dot{\theta}^T Z(\theta-\theta^*) \quad \forall\,(x,t) \in N. \tag{17}$$

From the conditions for the change of the $\dot{V}_a$ as a function of controller parameter θ, see [5], [15] one can determine the adaptive control algorithm as follows

$$\dot{\theta} = Z^{-1}\varphi(x)\,||\nabla_x V||\,(-1+\alpha(1-\vartheta)) \tag{18}$$

$$\theta(t_0) = \theta_0.$$

The proposed adaptive control algorithm (18) ensures that the time derivative of Lyapunov function (16), (17) will decrease in the time if $\dot{\theta}$ is not identically equal to zero. If the system (1) is P-stabilizable with the proposed controller, there exists such value of θ^*, that the solution of the system (10) will be uniformly ultimately bounded. The sufficient stability conditions of the investigated system are given by the following theorem.

Theorem 1. The solution $x(t) : [t_0, t_1] \rightarrow R^n, x(t_0) = x_0$ of the system (1) with controller (13) and (18) is locally uniformly ultimately bounded on the set R_ρ, if the following sufficient conditions hold.

- The assumptions A1 – A4 hold.
- The following inequality holds for the constant α in (13)

$$\alpha > \frac{1}{1-\vartheta}. \tag{19}$$

Proof. By substituting (18) into (16) and (17) for the time derivative of the Lyapunov function V_a one can get

$$\dot{V}_a \leq \nabla_x^T V f + \frac{\partial V_a}{\partial t} - (-1+\alpha(1-\vartheta))\,||\nabla_x V||\,\varphi^T \theta^* \quad \forall\,(x,t) \notin N \tag{20}$$

and

$$\dot{V}_a \leq \nabla_x^T V f + \frac{\partial V_a}{\partial t} - \|\nabla_x V\| \varphi^T \left\{ \theta^*(-1 + \alpha(1-\vartheta)) - \theta\alpha(1-\vartheta)\left(1 - \frac{\mu}{\varepsilon}\right)\right\} \ \forall (x,y) \in N. \tag{21}$$

Because of conditions of Theorem 1 and because $\|\nabla_x V\|$ is a positive definite function and (19), one can conclude that there exists a vector θ^* with positive entries such that for (20) and (21) the following inequality holds

$$\dot{V}_a \leq -\gamma_3(\|x\|) \quad \forall (x,t) \notin N \tag{22}$$

$$\dot{V}_a \leq -\gamma_3(\|x\|) + \phi(x) \quad \forall (x,t) \in N \tag{23}$$

where $\gamma_3(\|x\|): R_+ \to R_+$ is strictly increasing with $\gamma_3(0) = 0$ and

$$\nabla_x V f + \frac{\partial V_a}{\partial t} - (-1 + \alpha(1-\vartheta))\|\nabla_x V\| \varphi^T \theta^* \leq -\gamma_3(\|x\|)$$

$$\|\nabla_x V\| \varphi^T \theta \alpha (1-\vartheta)\left(1 - \frac{\mu}{\varepsilon}\right) \leq \phi(x).$$

Note that the Lyapunov function V_a is monotonous with respect to $\|x\|$. This means that $\|x\|$ keeps decreasing as long as $\dot{V}_a \leq 0$. Thus, choosing a corresponding ε it is possible to make the region of stability or complement set of N as large as it is necessary. There are different categories of stability that one can obtain. Let us suppose that one of the following is true [13]:

1. $\gamma_3(\|x\|) > \phi(x)$ for $\forall (x,t) \in N$, then the system (1) with (13) and (18) in the region R_ρ is asymptotically stable with respect to x.
2. $$\lim_{\|x\| \to \infty} \inf \frac{\gamma_3(\|x\|)}{\phi(x)} \geq 1 \tag{24}$$

   for $\forall (x,t) \in N$. The investigated system is in the region R_ρ globally uniformly ultimately bounded and
3. $$\gamma_3(\|x\|) > \phi(x) \tag{25}$$

   with

   $$\eta_1 \leq \|x\| < \eta_2, \qquad \forall (x,t) \in N$$

   where

   $$\eta_2 > \gamma_1^{-1} \circ \gamma_2(\eta_1) \qquad \text{and} \qquad \eta_2 > \gamma_1^{-1} \circ \gamma_2(\|x_0\|)$$

   the investigated system is locally uniformly ultimately bounded.

This completes the proof. □

It should be stressed that the proposed adaptive control algorithm (18), in real time operation of a system, can keep the entries value of vector θ very large. The algorithm (18) ensures that entries of θ will increase in the time if $\dot{\theta}$ is not identically equal to zero.

The operator gain $G[\mathrm{d}(t)]$ under the input $\mathrm{d}(t)$ is defined as follows [14]

$$||G[\mathrm{d}(t)]|| = \sup_{\mathrm{d}(t)\in L_2} \frac{||G[\mathrm{d}(t)]||_{L_2}^T}{||\mathrm{d}(t)||_{L_2}^T} \tag{26}$$

where

$$T \in \langle 0, \infty)$$

$$||\mathrm{d}(t)||_{L_2}^T = \left(\int_0^T \mathrm{d}^2(\tau)\, \mathrm{d}\tau \right)^{\frac{1}{2}} .$$

Hence, owing to (26) the operator gain of (18) is infinite, i.e.

$$||G[\mathrm{d}(t)]|| \to \infty$$

and therefore in practical operation of the system the "Rohrs phenomenon" may occur, and the proposed adaptive system may not be suitable for practical applications. To overcome these difficulties, instead of the adaptive algorithm (18) one can use the following one

$$\dot{\theta}_a = Z^{-1}\varphi(x)\, ||\nabla_x V||\, (-1 + \alpha(1 - \vartheta)) - \gamma\theta_a \tag{27}$$

where $\gamma > 0$ is a rather small positive number.

The entries of vector θ are given as follows

$$\theta_j = \theta_{aj}, \quad \text{if} \quad \theta_{aj} > \theta_{amj}$$

$$\theta_j = \theta_{amj}, \quad \text{if} \quad \theta_{aj} \le \theta_{amj} \tag{28}$$

where θ_{am} is the vector whose entries are the maximum achieved values of $\theta_{aj}, j = 1, 2, \dots, p$, under the practical operation of the system.

Hence, in the adaptive control algorithm (27) and (28) the entries of θ_a will change in time keeping the equilibrium of equation (27). The operation of system can be effective in two ways. First, the system (1) operates with controller (13), (18) and sufficient stability conditions are given by Theorem 1. Second, the system (1) cooperates with controller (13), (27) and (28) with a constant value of θ^*. For the above case the time derivative of Lyapunov function on the solution of (1) and (13) with A1 – A2 is given as follows

$$\dot{V} \le -\gamma_3(||x||) \quad \forall\, (x, t) \notin N$$

$$\dot{V} \le -\gamma_3(||x||) + \phi(x) \quad \forall\, (x, t) \in N.$$

The sufficient stability conditions of system (1) with controller (13), (27) and (28) with constant value of θ^* are the same as given by Theorem 1.

4. EXAMPLE

Consider a single-link manipulator or a simple pendulum. The simple pendulum of mass M and length l subjected to a control moment u and an unknown bounded disturbance $v(t)$ in the form of horizontal acceleration of its point of support. This example was used by Corless [2], Zhihua Qu [11] to illustrate the class of robust controllers. Here we use the same example to illustrate the new theoretical results of design of the adaptive robust controller. We recall that our model (1) need not satisfy the matching or generalized matching conditions and the other conditions given by Corless [2] (mainly Assumptions 6.1 and 6.2), Zhihua Qu and Dorsey [13] (Assumption 3.3 and Condition 4.1) and Leitman [9] (Assumption C.1). As shown by Corless [2], the dynamics of the pendulum is described by

$$\dot{x}_1 = x_2 \tag{29}$$

$$\dot{x}_2 = -a \sin x_1 + bu - \frac{v(t)}{l} \cos x_1$$

where

$$a = \frac{Ml}{J} \qquad b = \frac{1}{J}$$

J is the moment of inertia of the link with respect to its axis of rotation.

Owing to uncertainty $v(t)$, the uncontrolled uncertain system has no known equilibrium point. It has been shown by Corless [2] and Qu [11] that a control law guaranteeing uniform and ultimate boundedness can be chosen in the form

$$u = u_1 + u_2$$

where u_1 is the proportional derivative controller to guarantee the stability of the nominal model and u_2 is the one which ensures the stability and demanded quality of uncertain system (1). In this paper only one controller is designed. The design procedure of the adaptive robust controller is given by eqs. (7), (13), (19) and (27). Let us choose the Lyapunov candidate function $V : R^2 \to R_+$ as follows

$$V = x_1^2 + cx_1x_2 + x^2 \tag{30}$$

where $0 < c < 2$.

The (1) is given by

$$f(t)^T = [x_2 \;\; -a \sin x_1] \qquad \delta f(x)^T = \left[0 \;\; -\frac{v(t)}{l} \cos x_1\right]$$

$$B^T = [0 \;\; b] \qquad \delta B^T = [0 \;\; \delta b] = FB^T$$

where $F \in R$ is unknown scalar with $|F| < \vartheta < 1$.

The control algorithm is given by (13), where

$$B^T \nabla_x V \|\nabla_x V\| = b(cx_1 + 2x_2)\sqrt{(2x_1 + cx_2)^2 + (cx_1 + 2x_2)^2} \tag{31}$$

$$\mu = b^2(cx_1 + 2x_2)^2$$

$$\varphi(x) = \sqrt{x_1^2 + x_2^2}$$

and for adaptive control law

$$\dot{\theta}_a = \rho\varphi(x)\sqrt{(2x_1 + cx_2)^2 + (cx_1 + 2x_2)^2}\,(-1 + \alpha(1 - \vartheta)) - \gamma\theta_a \tag{32}$$

where $\rho = Z^{-1}$ is a positive constant.

Simulation results with the parameters $a = b = 1$, $\delta b = 0.55$, $\gamma = 0.01$, $\varepsilon = 0.05$, $\rho = 15$ $c = 1.99$ $\alpha = 3$ and initial state $[x_1\ \ x_2] = [1.5\ \ 0]$ are given in Figure 1. Within the simulation time $t \in \langle 0,\ 5)$ sec the value of parameter uncertainty has been chosen as $v = 1$, and for $t \geq 5$ sec the value of parameter uncertainty has been change drastically, $v = 100$. Simulation results show that the system in the region R_ρ is uniformly ultimately bounded with respect to state variable x and this verifies the proposed theoretical analysis.

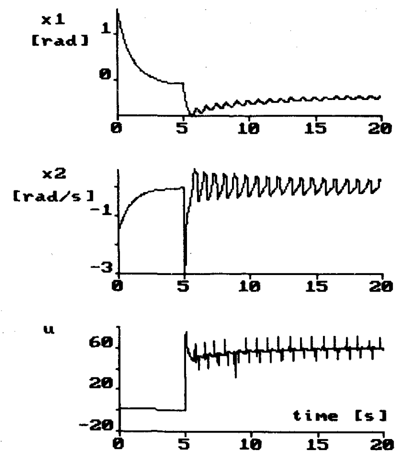

Fig. 1. Simulation results of pendulum with the proposed adaptive robust controller.

5. CONCLUSION

Local stabilization of general uncertain systems with an adaptive robust controller without matching conditions is considered. We have assumed that both the input uncertainty matrix is bounded with a known norm of uncertainty and the system uncertainty is bounded by a scalar product of a known positive function and an unknown constant vector with positive entries. The above vector is updated by the proposed adaptive algorithm. To overcome the "Rohrs phenomena" in adaptive control a special approach has been proposed which ensures a finite operator gain of the adaptive algorithm. Simulation results verify the proposed theoretical analysis, although the parameter of system uncertainty changes very drastically.

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