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RANDOM SET FUNCTIONS

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The main goal of the paper is an attempt to use one of the Kolmogorov's approaches to integrals for integration of random set functions. The notion of differential equivalence is introduced and conditions under which an integrable random set function is differentially equivalent to its indefinite integral are investigated.

1. KOLMOGOROV'S INTEGRAL

Kolmogorov considers in [1] a general approach to the integration of set functions. A given set function is defined on a set system, say $\mathscr{C}$, which is closed with respect to finite or countable intersections of its elements, i.e. if $\{A_i\}_{i\in N} \subset \mathscr{C}$, N is a subset of natural numbers, then $\bigcup_{i\in N} A_i \in \mathscr{C}$ also. Every subsystem $\mathscr{D} \subset \mathscr{C}$ where $\mathscr{D} = \{D_i\}_{i\in N}$, $D_i \cap D_j = \emptyset$, $i \neq j$ and $\bigcap_{i\in N} D_i = E$ will be called a $\mathscr{C}$-division of the set E. Let $\mathfrak{M}_E$ be the system of all $\mathscr{C}$-divisions of $E \in \mathscr{C}$; as $E \in \mathscr{C}$, the system $\mathfrak{M}_E$ is non-empty. If two $\mathscr{C}$-divisions $\mathscr{D}_1, \mathscr{D}_2 \in \mathfrak{M}_E$ are given, we shall say that $\mathscr{D}_1$ is finer than $\mathscr{D}_2(\mathscr{D}_1 \succ \mathscr{D}_2)$ if every element of $\mathscr{D}_1$ is a subset of an element of $\mathscr{D}_2$. For the definitions of the product and of the sum of two $\mathscr{C}$-divisions and for further details see [1].

Now, let $E \in \mathscr{C}$ be fixed and a real or a complex function $\varphi(\mathscr{D})$ be defined for every $\mathscr{D} \succ \mathscr{D}_0$, $\mathscr{D}_0 \in \mathfrak{M}_E$, $\mathscr{D} \in \mathfrak{M}_E$.

Definition 1. We shall say that the function φ defined on $\mathfrak{M}_E$ has a limit J (with respect to the system $\mathfrak{M}_E$) if,

$$\forall \varepsilon > 0 \quad \exists \mathscr{D}_\varepsilon \in \mathfrak{M}_E \quad \forall \mathscr{D} \succ \mathscr{D}_\varepsilon\,, \quad \mathscr{D} \in \mathfrak{M}_E$$

the inequality

$$|\varphi(\mathscr{D}) - J| < \varepsilon$$

holds.

Surely, if a limit J exists, then it is defined unambiguously. A special case of such a limit J is the integral $\int_E \mathrm{d}\varphi$ of the set function φ defined on $\mathscr{C}$, i.e. for every $\Delta \in \mathscr{C}$ a real or a complex number $\varphi(\Delta)$ is defined and the value for $\varphi(\mathscr{D})$ is given by the relation

$$\varphi(\mathscr{D}) = \sum_{i \in N} \varphi(D_i)\,, \quad \mathscr{D} = \{D_i\}_{i \in N}\,.$$

Of course, it must be assumed the absolute convergence of series $\sum_{i \in N} \varphi(D_i)$ for every $\mathscr{D} \succ \mathscr{D}_0$, $\mathscr{D}_0 \in \mathfrak{M}_E$, $\mathscr{D} \in \mathfrak{M}_E$. If such a limit J in the sense of Definition 1 exists, then it is called the integral of φ over E and J is denoted by

$$J = \int_E \mathrm{d}\varphi\,.$$

The very important notion concerning the integration of set functions is that of the differential equivalence of set functions.

Definition 2. Set functions φ_1, φ_2 defined on $\mathscr{C}$ are differentially equivalent over $E \in \mathscr{C}$ (with respect to $\mathfrak{M}_E$), if

$$\forall \varepsilon > 0 \quad \exists \mathscr{D}_\varepsilon \in \mathfrak{M}_E \quad \forall \mathscr{D} \succ \mathscr{D}_\varepsilon\,, \quad \mathscr{D} \in \mathfrak{M}_E\,, \quad \mathscr{D} = \{D_i\}_{i \in N}$$

the inequality

$$\sum_{i \in N} |\varphi_1(D_i) - \varphi_2(D_i)| < \varepsilon$$

holds.

In other words, Definition 2 can be expressed in the following form: φ_1, φ_2 are differentially equivalent over E if and only if

$$\int_E \mathrm{d}|\varphi_1 - \varphi_2| = 0\,.$$

From the properties of integrals mentioned in [1] it follows that for every integrable set function φ over $E \in \mathscr{C}$ there exists a set function $\phi(\Delta) = \int_\Delta \mathrm{d}\varphi$, $\Delta \in \mathscr{D} \in \mathfrak{M}_E$. This set function ϕ defined on $\mathfrak{M}_E$ is called the indefinite integral of φ over E. It is also proved in [1] that this indefinite integral ϕ is the only additive set function defined on $\mathfrak{M}_E$ which is differentially equivalent to the original set function φ over E. Then, using the notion of differential equivalence between two set functions we can unambiguously determine the relation between integrable set function φ and its indefinite integral ϕ.

2. INTEGRATION OF RANDOM SET FUNCTIONS

Let a triple $(\Omega, \mathfrak{S}, P)$ be a probability space, let $\mathscr{C}$ be a set system which is closed with respect to finite or countable intersections of its elements. By a random set function φ defined on $\mathscr{C}$ we shall understand a function defined on $\mathscr{C} \times \Omega$ which for every $\omega \in \Omega$ is a set function on $\mathscr{C}$ and for every $\Delta \in \mathscr{C}$ is a random variable on $(\Omega, \mathfrak{S}, P)$.

For further considerations it is useful to introduce the following notions.

Let (M, ϱ) be one of the following metric spaces corresponding to the underlying probability space $(\Omega, \mathfrak{S}, P)$:

a) $M = M_0$ ... a.s. finite random variables on $(\Omega, \mathfrak{S}, P)$ with the equivalence P-a.s.
$\varrho = \varrho_0$ ... a metric corresponding to the convergence in probability P.

b) $M = \mathscr{L}_1(\Omega, \mathfrak{S}, P)$ with the equivalence in $\mathscr{L}_1$-norm.
$\varrho = \varrho_1$ ... a metric corresponding to $\mathscr{L}_1$-norm.

c) $M = \mathscr{L}_2(\Omega, \mathfrak{S}, P)$ with the equivalence in $\mathscr{L}_2$-norm.
$\varrho = \varrho_2$ ... a metric corresponding to $\mathscr{L}_2$-norm.

In the rest of the paper we shall consider such random set functions only which satisfy the following condition:

$$\text{if } \varphi : \mathscr{C} \to M \text{ then there exists a } \mathscr{C}\text{-division } \mathscr{D}_\varphi \in \mathfrak{M}_E$$
$$\text{such that for every } \mathscr{D} \succ \mathscr{D}_\varphi,\ \mathscr{D} \in \mathfrak{M}_E,\ \mathscr{D} = \{\Delta_i\}_{i \in N}$$
$$\sum_{i \in N} |\varphi(\Delta_i, \omega)| \in (M, \varrho)\,.$$

Such a random set function will be called (M, ϱ)-well defined on $\mathfrak{M}_E$. We shall write for simplicity

$$\varphi(\mathscr{D}, \omega) = \sum_{i \in N} \varphi(\Delta_i, \omega)\,, \quad \mathscr{D} = \{\Delta_i\}_{i \in N}\,.$$

Definition 3. A random set function φ is integrable in (M, ϱ)-sense over E if φ is (M, ϱ)-well defined and if such a random variable $\Phi \in M$ exists such that

$$\mathop{Lim}_{\mathscr{D}} \varphi(\mathscr{D}) = \Phi\,, \quad \text{i.e.}$$
$$(\forall \varepsilon > 0\ \exists \mathscr{D}_\varepsilon \succ \mathscr{D}_\varphi,\ \mathscr{D}_\varepsilon \in \mathfrak{M}_E\ \forall \mathscr{D} \succ \mathscr{D}_\varepsilon,\ \mathscr{D} \in \mathfrak{M}_E) \Rightarrow \varrho(\varphi(\mathscr{D}), \Phi) < \varepsilon\,.$$

If the metric ϱ corresponds to the convergence in probability then Φ is named the integral in probability and denoted by $\Phi = P\text{-}\int_E \mathrm{d}\varphi$. If the metric ϱ corresponds to the convergence in $\mathscr{L}_1$-norm (resp. $\mathscr{L}_2$-norm), then Φ is named the integral in $\mathscr{L}_1$-sense (resp. $\mathscr{L}_2$-sense) and denoted by $\mathscr{L}_1\text{-}\int_E \mathrm{d}\varphi$ (resp. $\mathscr{L}_2\text{-}\int_E \mathrm{d}\varphi$). Generally, we shall use the denotion

$$\Phi = \int_E \mathrm{d}\varphi$$

if there will be no possibility to make any confusion.

The following part of the paper contains mainly properties of integrable random set functions. As the most of lemmas and theorems describes properties which are similar to properties of nonrandom integrable set function the proofs are omitted.

Lemma 1. A necessary and sufficient condition for the existence of $\int_E \mathrm{d}\varphi$ in

(M, ϱ)-sense for (M, ϱ)-well defined φ is that

$$\operatorname*{Lim}_{\mathscr{D},\mathscr{D}'} \varrho(\varphi(\mathscr{D}), \varphi(\mathscr{D}')) = 0 .$$

Lemma 2. If $\int_E \mathrm{d}\varphi$ exists, then it is defined unambiguously in (M, ϱ).

Theorem 1. Let φ be (M, ϱ)-integrable over E. Let Δ be an element of an arbitrary $\mathscr{C}$-division $\mathscr{D} \succ \mathscr{D}_\varphi$. Then $\int_\Delta \mathrm{d}\varphi$ exists in (M, ϱ)-sense also.

Theorem 2. If φ is (M, ϱ)-integrable over $E \in \mathscr{C}$ then for every $\mathscr{C}$-division $\mathscr{D} \in \mathfrak{M}_E$, $\mathscr{D} \succ \mathscr{D}_\varphi$, $\mathscr{D} = \{\Delta_i\}_{i\in N}$

$$\int_E \mathrm{d}\varphi = \sum_{i\in N} \int_{\Delta_i} \mathrm{d}\varphi ,$$

where

$$\sum_{i\in N} \left| \int_{\Delta_i} \mathrm{d}\varphi \right| \in (M, \varrho) .$$

Definition 4. Let φ be (M, ϱ)-integrable over E. Then the random set function $\hat{\varphi}(\Delta) = \int_\Delta \mathrm{d}\varphi$ (which exists) defined for every $\Delta \in \mathscr{D} \succ \mathscr{D}_\varphi$ will be called the indefinite integral in (M, ϱ)-sense of φ over E.

Definition 5. A random set function defined on $\mathscr{C}$ is additive on $E \in \mathscr{C}$ if there exists a $\mathscr{C}$-division $\mathscr{D}_a \in \mathfrak{M}_E$ such that

$$\varphi(\mathscr{D}_a, \omega) = \varphi(\mathscr{D}, \omega)$$

for every $\mathscr{D} \succ \mathscr{D}_a$, $\mathscr{D} \in \mathfrak{M}_E$. An additive and (M, ϱ)-well defined function will be called (M, ϱ)-additive.

Lemma 3. If φ is (M, ϱ)-additive over $E \in \mathscr{C}$, then φ is (M, ϱ)-integrable over E and

$$\int_E \mathrm{d}\varphi = \varphi(\mathscr{D}_a) .$$

Remark 1. Let us suppose that φ is additive on $E \in \mathscr{C}$ in the sense of Definition 5. This approach yields also the validity of the following relation

$$\varphi(\Delta_0, \omega) = \sum_k \varphi(D_k, \omega)$$

for every $\Delta_0 \in \mathscr{D} \succ \mathscr{D}_a$ and every $\mathscr{D}_{\Delta_0} = \{D_k\}_{k\in N} \in \mathfrak{M}_{\Delta_0}$. If φ is additive then for every $\mathscr{D} = \{\Delta_i\}_{i\in N} \succ \mathscr{D}_a$

$$\sum_i \varphi(\Delta_i, \omega) = \varphi(\mathscr{D}_a, \omega)$$

Let $\Delta_0 \in \mathscr{D}$, Δ_0 fixed, then also

$$\varphi(\Delta_0, \omega) + \sum_{i>0} \varphi(\Delta_i, \omega) = \varphi(\mathscr{D}_a, \omega)$$

Let $\mathscr{D}_{\Delta_0} \in \mathfrak{M}_{\Delta_0}$ be quite arbitrary, then the $\mathscr{C}$-division $\{\mathscr{D}_{\Delta_0}, \Delta_1, \Delta_2, \ldots\}$ is finer than

$\mathscr{D}_a$ and therefore

$$\sum_k \varphi(D_k, \omega) + \sum_{i>0} \varphi(\Delta_i, \omega) = \varphi(\mathscr{D}_a, \omega)$$

if $\{D_k\}_{k \in N} = \mathscr{D}_{\Delta_0}$. But it means that

$$\varphi(\Delta_0, \omega) = \sum_k \varphi(D_k, \omega).$$

Lemma 4. Let φ_1, φ_2 be (M, ϱ)-integrable over $E \in \mathscr{C}$, then

1) if α, β are reals $\Rightarrow \alpha\varphi_1 + \beta\varphi_2$ is (M, ϱ)-integrable over E and

$$\int_E \mathrm{d}(\alpha\varphi_1 + \beta\varphi_2) = \alpha \int_E \mathrm{d}\varphi_1 + \beta \int_E \mathrm{d}\varphi_2$$

2) if there exists a $\mathscr{C}$-division $\mathscr{D}_0 \succ \mathscr{D}_{\varphi_1} \cap \mathscr{D}_{\varphi_2}$ such that for every $\mathscr{D} \succ \mathscr{D}_0$, $\varphi_1(\mathscr{D}) \geqq \varphi_2(\mathscr{D})$, then

$$\int_E \mathrm{d}\varphi_1 \geqq \int_E \mathrm{d}\varphi_2 .$$

Theorem 3. Let $\{\varphi_n\}_1^\infty$ be a sequence of (M, ϱ)-integrable random set functions over $E \in \mathscr{C}$ and let exist such a random set function φ defined on $\mathscr{C}$ that

$$\lim_{n\to\infty} \sup_{\mathscr{D} \succ \mathscr{D}_0} \varrho\left(\sum_i |\varphi_n(\Delta_i, \omega) - \varphi(\Delta_i, \omega)|, 0\right) = 0$$

then φ is (M, ϱ)-integrable also and

$$\lim_{n\to\infty} \varrho\left(\int_E \mathrm{d}\varphi_n, \int_E \mathrm{d}\varphi\right) = 0 .$$

Theorem 4. Let φ be P-integrable over E and let exist such a random variable $\xi \geqq 0$ that $\mathsf{E}\{\xi\} < \infty$ and let for every $\mathscr{C}$-division $\mathscr{D} \succ \mathscr{D}_\varphi$, $\mathscr{D} \in \mathfrak{M}_E$ the inequality

$$\sum_{i\in N} |\varphi(\Delta_i, \omega)| \leqq \xi(\omega) \text{ hold }, \quad \mathscr{D} = \{\Delta_i\}_{i\in N} .$$

Then φ is $\mathscr{L}_1$-integrable over E, the integral $\int_E \mathrm{d}\mathsf{E}(\varphi)$ exists in the sense of Kolmogorov and it holds

$$\mathsf{E}\left\{\mathscr{L}_1\text{-}\int_E \mathrm{d}\varphi\right\} = \int_E \mathrm{d}\mathsf{E}\{\varphi\} .$$

Proof. As φ is P-integrable over E, we can find a sequence $\{\varphi(\mathscr{D}_n)\}_1^\infty$ which converges to $\int_E \mathrm{d}\varphi$ in probability, $\mathscr{D}_n \succ \mathscr{D}_\varphi$, $|\varphi|(\mathscr{D}_n) \leqq \xi$ and hence $\mathsf{E}\{\int_E \mathrm{d}\varphi\}$ must exist. As we assume that $|\varphi|(\mathscr{D}) \leqq \xi$ for every $\mathscr{D} \succ \mathscr{D}_\varphi$ then

$$\mathsf{E}\{\sum_{i\in N} \varphi(\Delta_i, \omega)\} = \sum_{i\in N} \mathsf{E}\{\varphi(\Delta_i, \omega)\} ,$$

for

$$\sum_{i\in N} \mathsf{E}\{|\varphi(\Delta_i, \omega)|\} < \infty \quad (\mathscr{D} = \{\Delta_i\}_{i\in N}) .$$

Further, we prove that φ is even $\mathscr{L}_1$-integrable, because for every $\mathscr{D} \succ \mathscr{D}_\varphi$, $\mathscr{D} \in \mathfrak{M}_E$

$\sum_{i \in N} |\varphi(\Delta_i, \omega)| \in \mathscr{L}_1(\Omega, \mathfrak{S}, P)$ and hence φ is $\mathscr{L}_1$-well defined; then Lebesgue's theorem on the dominated convergence proves the $\mathscr{L}_1$-integrability of φ. Now, it is also simple to prove that

$$\mathsf{E}\left\{\mathscr{L}_1\text{-}\int_E \mathrm{d}\varphi\right\} = \int_E \mathrm{d}\mathsf{E}\{\varphi\},$$

because

$$\left|\mathsf{E}\left\{\int_E \mathrm{d}\varphi\right\} - \int_E \mathrm{d}\mathsf{E}\{\varphi\}\right| \leqq \left|\mathsf{E}\left\{\int_E \mathrm{d}\varphi\right\} - \mathsf{E}\{\sum_{i \in N} \varphi(\Delta_i)\}\right| + \left|\mathsf{E}\{\sum_{i \in N} \varphi(\Delta_i)\} - \int_E \mathrm{d}\mathsf{E}\{\varphi\}\right| =$$

$$= \left|\mathsf{E}\left\{\int_E \mathrm{d}\varphi - \sum_{i \in N} \varphi(\Delta_i)\right\}\right| + \left|\sum_{i \in N} \mathsf{E}\{\varphi(\Delta_i)\} - \int_E \mathrm{d}\mathsf{E}\{\varphi\}\right| \leqq$$

$$\leqq \mathsf{E}\left\{\left|\int_E \mathrm{d}\varphi - \sum_{i \in N} \varphi(\Delta_i)\right|\right\} + \left|\sum_{i \in N} \mathsf{E}\{\varphi(\Delta_i)\} - \int_E \mathrm{d}\mathsf{E}\{\varphi\}\right|.$$

□

Theorem 5. If φ is $\mathscr{L}_1$-integrable (resp. $\mathscr{L}_2$-integrable) over E then $\int_E \mathrm{d}\mathsf{E}\{\varphi\}$ exists and

$$\mathsf{E}\left\{\mathscr{L}_1\text{-}\int_E \mathrm{d}\varphi\right\} = \int_E \mathrm{d}\mathsf{E}\{\varphi\}$$

$$\left(\text{resp.}\quad \mathsf{E}\left\{\mathscr{L}_2\text{-}\int_E \mathrm{d}\varphi\right\} = \int_E \mathrm{d}\mathsf{E}\{\varphi\}\right).$$

Proof. Let us suppose that φ is $\mathscr{L}_1$-integrable over E. It means that φ is $\mathscr{L}_1$-well defined and hence $\mathscr{D}_\varphi \in \mathfrak{M}_E$ must exist such that $\sum_i |\varphi(\Delta_i, \omega)| \in \mathscr{L}_1(\Omega, \mathfrak{S}, P)$ for every $\mathscr{D} \succ \mathscr{D}_\varphi$, $\mathscr{D} \in \mathfrak{M}_E$. It proves, further, that

$$\mathsf{E}\{\sum_i |\varphi(\Delta_i, \omega)|\} = \sum_i \mathsf{E}\{|\varphi(\Delta_i, \omega)|\}$$

and hence

$$\mathsf{E}\{\sum_i \varphi(\Delta_i, \omega)\} = \sum_i \mathsf{E}\{\varphi(\Delta_i, \omega)\}$$

also. Then

$$\left|\mathsf{E}\left\{\mathscr{L}_1\text{-}\int_E \mathrm{d}\varphi\right\} - \mathsf{E}\{\sum_i \varphi(\Delta_i)\}\right| = \left|\mathsf{E}\left\{\mathscr{L}_1\text{-}\int_E \mathrm{d}\varphi\right\} - \sum_i \mathsf{E}\{\varphi(\Delta_i)\}\right| \leqq$$

$$\leqq \mathsf{E}\left\{\left|\mathscr{L}_1\text{-}\int_E \mathrm{d}\varphi - \sum_i \varphi(\Delta_i)\right|\right\} \underset{\mathscr{D}}{\to} 0$$

because φ is $\mathscr{L}_1$-integrable. It proves that the set function $\mathsf{E}\{\varphi(\Delta, \omega)\}$ is integrable in the sense of Kolmogorov and

$$\mathsf{E}\left\{\mathscr{L}_1\text{-}\int_E \mathrm{d}\varphi\right\} = \int_E \mathrm{d}\mathsf{E}\{\varphi\}.$$

If φ is $\mathscr{L}_2$-integrable, then it is clear that it is also $\mathscr{L}_1$-integrable because

$$\mathsf{E}\{|\varphi(\mathscr{D}_1) - \varphi(\mathscr{D}_2)|\} \leqq (\mathsf{E}\{(\varphi(\mathscr{D}_1) - \varphi(\mathscr{D}_2))^2\})^{1/2}$$

and hence the equality

$$\mathsf{E}\left\{\mathscr{L}_2\text{-}\int_E \mathrm{d}\varphi\right\} = \int_E \mathrm{d}\mathsf{E}\{\varphi\}$$

must hold. □

Theorem 6. Let φ_1, φ_2 be two random set functions $\mathscr{L}_2$-integrable over E. Then

$$\mathsf{E}\left\{\mathscr{L}_2\text{-}\int_E \mathrm{d}\varphi_1 \; \mathscr{L}_2\text{-}\int_E \mathrm{d}\varphi_2\right\} = \iint_{E\times E} \mathrm{d}\mathsf{E}\{\varphi_1\varphi_2\}$$

where the integral on the right side is understood in the sense of Kolmogorov over Cartesian product $E \times E$ with the system $\mathscr{C} \times \mathscr{C} = \{\Delta_1 \times \Delta_2 \colon \Delta_1, \Delta_2 \in \mathscr{C}\}$ and $\mathsf{E}\{\varphi_1\varphi_2\}\,(\Delta_1 \times \Delta_2) = \mathsf{E}\{\varphi_1(\Delta_1)\,.\,\varphi_2(\Delta_2)\}$.

Proof. The existence of $\mathsf{E}\{\mathscr{L}_2\text{-}\int_E \mathrm{d}\varphi_1 \,.\, \mathscr{L}_2\text{-}\int_E \mathrm{d}\varphi_2\}$ follows from the Hölder inequality, because

$$\mathsf{E}\left\{\left(\mathscr{L}_2\text{-}\int_E \mathrm{d}\varphi_i\right)^2\right\} < \infty \quad i = 1, 2\,.$$

As we consider only $\mathscr{L}_2$-integrals, we shall write $\int_E \mathrm{d}\varphi$ instead of $\mathscr{L}_2\text{-}\int_E \mathrm{d}\varphi$. According to Definition 3 we can find for every $\varepsilon > 0$ such a $\mathscr{C}$-division $\mathscr{D}_\varepsilon$ that for every $\mathscr{D} \succ \mathscr{D}_\varepsilon$, $\mathscr{D} \in \mathfrak{M}_E$

$$\mathsf{E}\left\{\left(\int_E \mathrm{d}\varphi_i - \varphi(\mathscr{D})\right)^2\right\} < \varepsilon \quad i = 1, 2\,.$$

Then, if $\mathscr{D} \succ \mathscr{D}_\varepsilon$, $\mathscr{D}^* \succ \mathscr{D}_\varepsilon$

$$\begin{aligned}
&\mathsf{E}\left\{\left|\int_E \mathrm{d}\varphi_1 \int_E \mathrm{d}\varphi_2 - \varphi_1(\mathscr{D})\,\varphi_2(\mathscr{D}^*)\right|\right\} = \\
&= \mathsf{E}\left\{\left|\int_E \mathrm{d}\varphi_1 \int_E \mathrm{d}\varphi_2 - \sum_j \sum_k \varphi_1(D_j)\,\varphi_2(D_k^*)\right|\right\} \leqq \\
&\leqq \mathsf{E}\left\{\left|\int_E \mathrm{d}\varphi_1 \int_E \mathrm{d}\varphi_2 - \int_E \mathrm{d}\varphi_1 \sum_k \varphi_2(D_k^*)\right|\right\} + \\
&+ \mathsf{E}\left\{\left|\int_E \mathrm{d}\varphi_1 \sum_k \varphi_2(D_k^*) - \sum_j \sum_k \varphi_1(D_j)\,\varphi_2(D_k^*)\right|\right\} \leqq \\
&\leqq \left(\mathsf{E}\left\{\left(\int_E \mathrm{d}\varphi_1\right)^2\right\}\right)^{1/2} \left(\mathsf{E}\left\{\left(\int_E \mathrm{d}\varphi_2 - \sum_k \varphi_2(D_k^*)\right)^2\right\}\right)^{1/2} + \\
&+ (\mathsf{E}\,\{(\sum_k \varphi_2(D_k^*))^2\})^{1/2} \left(\mathsf{E}\left\{\left(\int_E \mathrm{d}\varphi_1 - \sum_j \varphi_1(D_j)\right)^2\right\}\right)^{1/2} < \mathrm{M}\varepsilon
\end{aligned}$$

where M is a suitable constant.

This fact immediately proves that also

$$\left|\mathsf{E}\left\{\int_E \mathrm{d}\varphi_1 \int_E \mathrm{d}\varphi_2\right\} - \sum_j \sum_k \mathsf{E}\{\varphi_1(D_j)\,\varphi_2(D_k^*)\}\right| < \mathrm{M}\varepsilon$$

for every $\mathscr{D}, \mathscr{D}^* \succ \mathscr{D}_\varepsilon$. But, it means that the set function $\mu(\Delta_1 \times \Delta_2) = \mathsf{E}\{\varphi_1(\Delta_1) \, . \, \varphi_2(\Delta_2)\}$ defined on $\mathfrak{M}_E \times \mathfrak{M}_E$ is integrable in Kolmogorov sense over $E \times E$ and

$$\mathsf{E}\left\{\int_E \mathrm{d}\varphi_1 \int_E \mathrm{d}\varphi_2\right\} = \iint_{E \times E} \mathrm{d}\mathsf{E}\{\varphi_1 \varphi_2\} \, . \qquad \square$$

Remark 2. If a random set function φ defined on $\mathscr{C}$ is orthogonal on $\mathfrak{M}_E$ i.e. for every $\mathscr{C}$-division $\mathscr{D} \succ \mathscr{D}_\varphi$, $\mathscr{D} \in \mathfrak{M}_E$, $\mathscr{D} = \{\Delta_i\}_{i \in N}$

1) $\mathsf{E}\{\varphi(\Delta_i)\} = 0$

2) $\mathsf{E}\{\varphi(\Delta_i)\,\varphi(\Delta_j)\} = \delta_{ij}\mathsf{E}\{\varphi^2(\Delta_i)\}$

where δ_{ij} is Kronecker symbol and if φ is $\mathscr{L}_2$-integrable over E then

$$\mathsf{E}\left\{\left(\mathscr{L}_2\text{-}\int_E \mathrm{d}\varphi\right)^2\right\} = \int_E \mathrm{d}\mu$$

where

$$\mu(\Delta) = \mathsf{E}\{\varphi^2(\Delta)\} \, .$$

3. DIFFERENTIAL EQUIVALENCE OF RANDOM SET FUNCTIONS

In Definition 2 the notion of the differential equivalence of two set functions was introduced. This notion can be easily transformed for random set functions also.

Definition 6. Let (M, ϱ) be the same metric space as in Definition 3. Let φ_1, φ_2 be two random set function (M, ϱ)-well defined over E. We shall say that φ_1, φ_2 are (M, ϱ)-differentially equivalent over E if $(\mathscr{D} = \{\Delta_i\}_{i \in N})$

$$\underset{\mathscr{D}}{Lim}\, \varrho(0, \sum_{i \in N} |\varphi_1(\Delta_i, \omega) - \varphi_2(\Delta_i, \omega)|) = 0 \, .$$

In other words, φ_1, φ_2 are (M, ϱ)-differentially equivalent if and only if

$$(M, \varrho)\text{-}\int_E \mathrm{d}|\varphi_1 - \varphi_2| = 0 \, .$$

If (M, ϱ)-$\int_E \mathrm{d}|\varphi_1 - \varphi_2| = 0$ then we shall write that $\varphi_1 \approx^\varrho \varphi_2$. It is clear, that

$$(\varphi_1 \approx^2 \varphi_2) \Rightarrow (\varphi_1 \approx^1 \varphi_2) \Rightarrow (\varphi_1 \approx^0 \varphi_2) \, .$$

One of the main results in [1] is the relation between an integrable set function and its indefinite integral. If φ is integrable over E in Kolmogorov sense, then its indefinite integral ϕ is the only additive set function defined on $\mathfrak{M}_E$ which is differentially equivalent to φ. Unfortunately, the differential equivalence between a random set function and its indefinite integral is not true in a general case as the following example shows.

Remark 3. The following example will show that the concept of the differential equivalence between two random set functions presented in Definition 7 is too strong. Let $\mathscr{C}$ be the set of all intervals in the form $\langle a, b) \subset \langle 0, 1)$, let $w(\cdot, \cdot)$ be the standard Wiener process defined on $\langle 0, 1)$ and let us define for every $\langle a, b) \in \mathscr{C}$ $\varphi(\langle a, b), \omega) = (w(b, \omega) - w(a, \omega))^2$. As it is familiarly known that

$$\lim_{\|\mathscr{D}\| \downarrow 0} \sum_i (w(a_{i+1}, \omega) - w(a_i, \omega))^2 = 1$$

in the quadratic mean, then the limit $\underset{\mathscr{D}}{Lim}\, \varphi(\mathscr{D}, \omega)$ in the quadratic mean exists also and it has the same value,

$$(\varphi(\mathscr{D}, \omega) = \sum_i (w(a_{i+1}, \omega) - w(a_i, \omega))^2 \quad \text{if} \quad \mathscr{D} = \{\langle a_i, a_{i+1})\}_i$$

is a $\mathscr{C}$-division of $\langle 0, 1)$*). It means that the random set function $\varphi(\Delta, \omega)$ is integrable in the quadratic mean with respect to the system $\mathscr{C}$. Further, the indefinite integral $\hat{\varphi}$ of φ exists for every $\langle a, b) \subset \langle 0, 1)$ and its value is

$$\int_a^b \mathrm{d}\varphi = b - a\,.$$

We shall prove even that φ and $\hat{\varphi}$ are not differentially equivalent in probability. Let $\mathscr{D}$ be any $\mathscr{C}$-division of $\langle 0, 1)$, $\mathscr{D} = \{\langle a_i, a_{i+1})\}$. We must investigate the probability

$$\mathsf{P}\{\omega: \sum_i |\Delta^2 w(a_i) - (a_{i+1} - a_i)| > t\}$$

because we want to prove that

$$\underset{\mathscr{D}}{Lim}\, \mathsf{P}\{\omega: \sum_i |\Delta^2 w(a_i) - (a_{i+1} - a_i)| > t\} > 0\,,$$

for a suitable $t > 0$. It follows from properties of the Wiener process that the random variable $|\Delta^2 w(a_i) - (a_{i+1} - a_i)|$ has the following density function g_i

$$g_i(u) = \frac{1}{\sqrt{(2\pi(a_{i+1} - a_i))}} [u + (a_{i+1} - a_i)]^{-1/2} \exp\left\{-\frac{u + (a_{i+1} - a_i)}{2(a_{i+1} - a_i)}\right\}$$

for $u \geqq a_{i+1} - a_i$.

Using the density function g_i we can express the following probability

$$\mathsf{P}\{\omega: |\Delta^2 w(a_i) - (a_{i+1} - a_i)| > (a_{i+1} - a_i)\} = \int_{a_{i+1} - a_i}^{\infty} g_i(u)\, \mathrm{d}u = \alpha > 0\,.$$

It is important that this probability α does not depend on the division $\mathscr{D}$.

* The random set function φ is $\mathscr{L}_2$-well defined on $\mathscr{C}$ as follows from properties of the Wiener process increments.

It holds for every $\mathscr{D} \in \mathfrak{M}_E$:

$$(*) \qquad 0 \leqq \sum_i |\Delta^2 w(a_i) - (a_{i+1} - a_i)| \leqq$$
$$\leqq \sum_i \Delta^2 w(a_i) + \sum_i (a_{i+1} - a_i) = \sum_i \Delta^2 w(a_i) + 1 ,$$

where $\mathscr{D} = \{\langle a_i, a_{i+1})\}$. It is known from properties of the Wiener process that

$$(**) \qquad \sum_i \Delta^2 w(a_i) \to_{\mathscr{D}} 1$$

in the quadratic mean. Let us suppose that

$$\sum_i |\Delta^2 w(a_i) - (a_{i+1} - a_i)| \to_{\mathscr{D}} 0$$

in probability. Then the inequality $(*)$ together with $(**)$ imply immediately that

$$\sum_i |\Delta^2 w(a_i) - (a_{i+1} - a_i)| \to_{\mathscr{D}} 0$$

in the quadratic mean also. But

$$\mathsf{E}\{\sum_i |\Delta^2 w(a_i) - (a_{i+1} - a_i)|\} = \sum_i \int_0^\infty u\, g_i(u)\, \mathrm{d}u \geqq$$
$$\geqq \sum_i \int_{a_{i+1}-a_i}^\infty u\, g_i(u)\, \mathrm{d}u \geqq \sum_i (a_{i+1} - a_i) \int_{a_{i+1}-a_i}^\infty g_i(u)\, \mathrm{d}u = \alpha > 0 ,$$

which is impossible. This fact proves that φ and $\hat{\varphi}$ are not differentially equivalent in probability and therefore cannot be differentially equivalent in $\mathscr{L}_1$-sense not even in $\mathscr{L}_2$-sense. At the end of this example we can say that our definition of differential equivalence is not suitable for the characterization of the indefinite integral as the only additive random set function which is differentially equivalent to the underlying random set function.

The following three lemmas hold for the all types of differential equivalences introduced in Definition 6.

Lemma 5. The (M, ϱ)-differential equivalence is an equivalence among random set functions.

Proof. If $\varphi_1 \approx^\varrho \varphi_2$, $\varphi_2 \approx^\varrho \varphi_3$ over $E \in \mathscr{C}$, which means that

$$(M, \varrho)\text{-}\int_E |\varphi_1 - \varphi_2| = 0 , \quad (M, \varrho)\text{-}\int_E |\varphi_2 - \varphi_3| = 0 ,$$

then thanks to the inequality

$$\sum_i |\varphi_1(\Delta_i) - \varphi_3(\Delta_i)| \leqq$$
$$\leqq \sum_i |\varphi_1(\Delta_i) - \varphi_2(\Delta_i)| + \sum_i |\varphi_2(\Delta_i) - \varphi_3(\Delta_i)| ,$$

it must be (M, ϱ)-$\int_E \mathrm{d}|\varphi_1 - \varphi_3| = 0$, i.e. $\varphi_1 \approx^\varrho \varphi_3$. At the first sight it is clear that

$$(M, \varrho)\text{-}\int_E \mathrm{d}|\varphi_1 - \varphi_1| = 0$$

and

$$(M, \varrho)\text{-}\int_E \mathrm{d}|\varphi_1 - \varphi_2| = (M, \varrho)\text{-}\int_E \mathrm{d}|\varphi_2 - \varphi_1| = 0\,,$$

which implies that (M, ϱ)-differential equivalence is a transitive, symmetric and reflexive relation among random set functions. □

Lemma 6. If φ_1 is (M, ϱ)-integrable over $E \in \mathscr{C}$ and $\varphi_1 \approx^\varrho \varphi_2$ then φ_2 is (M, ϱ)-integrable over E also and $\int_E \mathrm{d}\varphi_1 = \int_E \mathrm{d}\varphi_2$.

Proof. The existence of (M, ϱ)-$\int_E \mathrm{d}\varphi_2$ follows immediately from the inequality

$$\left|\sum_i \varphi_2(\Delta_i) - \int_E \mathrm{d}\varphi_1\right| \leqq \sum_i |\varphi_2(\Delta_i) - \varphi_1(\Delta_i)| + \left|\sum_i \varphi_1(\Delta_i) - \int_E \mathrm{d}\varphi_1\right|. \qquad □$$

This fact enables to introduce classes of equivalence among the all (M, ϱ)-integrable random set functions over $E \in \mathscr{C}$. Every class of equivalence is formed by the all (M, ϱ)-integrable random set functions which are (M, ϱ)-differentially equivalent each other. The following lemma describes an important property of such a class.

Lemma 7. Let $\{\varphi\}$ be a class of (M, ϱ)-integrable and mutually (M, ϱ)-differentially equivalent random set functions over $E \in \mathscr{C}$. If an additive random set function belongs to the class $\{\varphi\}$ then it is the only (M, ϱ)-additive random set function belonging to $\{\varphi\}$.

Proof. Let ψ be an additive random set function belonging to $\{\varphi\}$. The (M, ϱ)-additivity of ψ follows from Lemma 6. If ξ is another additive set function belonging to $\{\varphi\}$, then $\psi \approx^\varrho \xi$, i.e. $\int_E \mathrm{d}|\psi - \xi| = 0$. It implies that for every $\Delta \in \mathscr{D} \in \mathfrak{M}_E$, $\mathscr{D} \succ \mathscr{D}_\psi \cap \mathscr{D}_\xi$, $\int_\Delta \mathrm{d}|\psi - \xi| = 0$ and hence $\int_\Delta \mathrm{d}\psi = \int_\Delta \mathrm{d}\xi$. The additivity then implies that

$$\psi(\Delta) = \xi(\Delta)$$

for every $\Delta \in \mathscr{D} \succ \mathscr{D}_\psi \cap \mathscr{D}_\xi$. In this sense ψ is the unique additive random set function in $\{\varphi\}$. □

For the characterization of such functions which are differentially equivalent to an additive random set function over E it is suitable to introduce the following property of random set functions.

Definition 7. A random set function φ (M, ϱ)-well defined on $\mathfrak{M}_E$ is called asymptotically additive in (M, ϱ)-sense over $E \in \mathscr{C}$ if

$$(\forall \varepsilon > 0 \quad \exists \mathscr{D}_\varepsilon \in \mathfrak{M}_E, \quad \mathscr{D}_\varepsilon \succ \mathscr{D}_\varphi \quad \forall \mathscr{D}_2 \succ \mathscr{D}_1 \succ \mathscr{D}_\varepsilon) \Rightarrow$$
$$\Rightarrow \varrho(0, \sum_i |\varphi(\Delta_i) - \sum_{j \in J(i)} \varphi(\Delta_{ij})|) < \varepsilon$$

(where $\mathscr{D}_1 = \{\Delta_i\}_i$ and $\mathscr{D}_2 = \{\Delta_{ij}\}_{ij}$ $i \in N$, $j \in J(i)$ and for every i $\Delta_i = \bigcup_{j \in J(i)} \Delta_{ij}$).

Theorem 7. If a random set function φ is (M, ϱ)-asymptotically additive over $E \in \mathscr{C}$, then φ is (M, ϱ)-integrable. If an (M, ϱ)-integrable set function φ over E is (M, ϱ)-differentially equivalent to an (M, ϱ)-additive random set function, then φ is (M, ϱ)-asymptotically additive.

Proof. We assume that a random set function φ defined on $\mathscr{C}$ is (M, ϱ)-asymptotically additive over E. To prove its integrability it is sufficient to estimate

$$\varrho(0, \sum_i \varphi(\Delta_i) - \sum_j \varphi(D_j))$$

for suitably fine $\mathscr{C}$-divisions of E $\mathscr{D}_1 = \{\Delta_i\}_i$, $\mathscr{D}_2 = \{D_j\}_j$. For every $\varepsilon > 0$ there exists a $\mathscr{C}$-division $\mathscr{D}_\varepsilon \in \mathfrak{M}_E$ such that for $\mathscr{D}_1 \succ \mathscr{D}_\varepsilon$, $\mathscr{D}_2 \succ \mathscr{D}_\varepsilon$

$$\varrho(0, \varphi(\mathscr{D}_1) - \varphi(\mathscr{D}_2)) \leqq$$
$$\leqq \varrho(0, \varphi(\mathscr{D}_1) - \varphi(\mathscr{D}_\varepsilon)) + \varrho(0, \varphi(\mathscr{D}_2) - \varphi(\mathscr{D}_\varepsilon)) \leqq$$
$$\leqq \varrho(0, \sum_k |\varphi(\Delta_k^\varepsilon) - \sum_{i \in J(k)} \varphi(\Delta_i)|) + \varrho(0, \sum_k |\varphi(\Delta_k^\varepsilon) - \sum_{j \in J(k)} \varphi(D_j)) \leqq 2\varepsilon$$

where

$$\Delta_k^\varepsilon = \bigcup_{i \in J(k)} \Delta_i = \bigcup_{j \in J(k)} D_j \quad \text{because} \quad \mathscr{D}_i \succ \mathscr{D}_\varepsilon, \quad i = 1, 2.$$

We proved the Cauchy criterion of (M, ϱ)-integrability over E of φ.

Now, let us assume that φ is (M, ϱ)-differentially equivalent to an (M, ϱ)-additive set function φ_0 over E. The function φ_0 being (M, ϱ)-additive is (M, ϱ)-integrable over E and $\int_\Delta \mathrm{d}\varphi_0 = \varphi_0(\Delta)$ for every $\Delta \in \mathscr{D}_a \in \mathfrak{M}_E$. Because $\varphi \approx^\varrho \varphi_0$, it means

$$\forall \varepsilon > 0 \quad \exists \mathscr{D}_\varepsilon \in \mathfrak{M}_E \quad \forall \mathscr{D} \succ \mathscr{D}_\varepsilon, \quad \mathscr{D} \in \mathfrak{M}_E, \quad \mathscr{D} = \{\Delta_i\}_{i \in N} \Rightarrow$$
$$\Rightarrow \varrho(0, \sum_i |\varphi(\Delta_i) - \varphi_0(\Delta_i)|) < \varepsilon.$$

Thanks to this fact, if we have $\mathscr{D}_\varepsilon = \{\Delta_k^\varepsilon\}_{k \in N}$ it is possible to write for every $\mathscr{D} = \{\Delta_i\}_{i \in N} \succ \mathscr{D}_\varepsilon$ that $\Delta_k^\varepsilon = \bigcup_{i \in J(k)} \Delta_i$ for every k and hence for every $\mathscr{D}_1 \succ \mathscr{D}$, $\mathscr{D}_1 \in \mathfrak{M}_E$,

$$\mathscr{D}_1 = \{D_{ij}\}_{ij}, \quad \Delta_i = \bigcup_j D_{ij}$$
$$\varrho(0, \sum_i |\varphi(\Delta_i) - \sum_j \varphi(D_{ij})|) \leqq \varrho(0, \sum_i |\varphi_0(\Delta_i) - \sum_i \varphi(D_{ij})|) +$$
$$+ \varrho(0, \sum_i |\varphi_0(\Delta_i) - \varphi(\Delta_i)|) \leqq \varrho(0, \sum_i |\sum_j \varphi_0(D_{ij}) - \sum_j \varphi(D_{ij})|) + \varepsilon \leqq$$
$$\leqq \varrho(0, \sum_i \sum_j |\varphi_0(D_{ij}) - \varphi(D_{ij})|) + \varepsilon \leqq 2\varepsilon$$

because $\sum_j \varphi_0(D_{ij}, \omega) = \varphi_0(\Delta_i, \omega)$ thanks to the (M, ϱ)-additivity of φ_0 over E. This proves the property of the (M, ϱ)-asymptotical additivity of φ over E. □

Theorem 7 and results in [1] imply immediately that every nonrandom set function integrable in the sense of Kolmogorov is always asymptotically additive.

Theorem 8. If φ is an (M, ϱ)-asymptotically additive random set function over $E \in \mathscr{C}$ then its indefinite integral $\hat{\varphi}$ over E is the unique (M, ϱ)-additive set function which is (M, ϱ)-differentially equivalent to φ over E.

Proof. As follows from Theorem 7, the random set function φ is (M, ϱ)-integrable over E and hence its indefinite integral $\hat{\varphi}$ exists for every $\Delta \in \mathscr{D} \succ \mathscr{D}_\varphi$ in the (M, φ)-sense. We wish to prove that $\varphi \approx^\varrho \hat{\varphi}$ over E. For this purpose let us consider

$$\varrho(0, \sum_i |\varphi(\Delta_i) - \hat{\varphi}(\Delta_i)|)$$

for a suitably fine $\mathscr{C}$-division $\mathscr{D} = \{\Delta_i\}_i \in \mathfrak{M}_E$, $\mathscr{D} \succ \mathscr{D}_\varphi$. Because φ is (M, ϱ)-asymptotically additive over E, then

$$\varrho(0, \sum_i |\varphi(\Delta_i) - \sum_{j \in J(i)} \varphi(D_j)|) < \varepsilon$$

for every

$$\mathscr{D}_2 = \{D_j\}_j \succ \mathscr{D}_1 = \{\Delta_i\}_i \succ \mathscr{D}_\varepsilon .$$

Now, we can write

$$\varrho(0, \sum_i |\varphi(\Delta_i) - \hat{\varphi}(\Delta_i)|) = \varrho\left(0, \sum_i \left|\varphi(\Delta_i) - \int_{\Delta_i} \mathrm{d}\varphi\right|\right) \leqq$$

$$\leqq \varrho(0, \sum_i |\varphi(\Delta_i) - \sum_j \varphi(D_{ij})|) + \varrho\left(0, \sum_i \left|\int_{\Delta_i} \mathrm{d}\varphi - \sum_j \varphi(D_{ij})\right|\right) \leqq$$

$$\leqq \varepsilon + \sum_i \varrho\left(0, \int_{\Delta_i} \mathrm{d}\varphi - \sum_j \varphi(D_{ij})\right) \leqq 2\varepsilon$$

where $\Delta_i = \bigcup_j D_{ij}$ for every i and a $\mathscr{C}$-division $\{D_{ij}\}_j$ of Δ_i is chosen in such a way that

$$\varrho\left(0, \int_{\Delta_i} \mathrm{d}\varphi - \sum_j \varphi(D_{ij})\right) < \frac{\varepsilon}{2^i} .$$

This proves that $\varphi \approx^\varrho \hat{\varphi}$.

The additivity of $\hat{\varphi}$ follows from Theorem 2. As we proved that $\varphi \approx^\varrho \hat{\varphi}$ then the uniqueness of $\hat{\varphi}$ follows immediately from Lemma 7. □

As an example of an integrable random set function we can present the stochastic integral in the quadratic mean. Let the underlying system $\mathscr{C}_0$ be the system of all semiclosed intervals $\langle a, b) \subset \langle 0, 1)$. This system $\mathscr{C}_0$ is closed with respect to finite intersections. Let $E = \langle 0, 1)$ and let

$$\mathfrak{M}_E^0 = \{\mathscr{D}: \mathscr{D} = \{\langle a_i, b_i)\}_{i=1}^n, \quad \bigcup_1^n \langle a_i, b_i) = E,$$

$$\langle a_i, b_i) \cap \langle a_j, b_j) = \emptyset \quad \text{for every} \quad i \neq j\}.$$

Let $\mu(\Delta, \omega)$ be a random set function defined on $\mathscr{C}_0$ which is additive, (i.e. if $\Delta_1, \Delta_2 \in \mathscr{C}_0$ and $\Delta_1 \cap \Delta_2 = \emptyset$, $\Delta_1 \cup \Delta_2 \in \mathscr{C}_0$ then $\mu(\Delta_1, \omega) + \mu(\Delta_2, \omega) = \mu(\Delta_1 \cup \Delta_2, \omega)$) and orthogonal (i.e.

$$\mathsf{E}\{\mu^2(\Delta_1, \omega)\} + \mathsf{E}\{\mu^2(\Delta_2, \omega)\} = \mathsf{E}\{(\mu_1(\Delta_1, \omega) + \mu_2(\Delta_2, \omega))^2\}$$

for every pair $\Delta_1, \Delta_2 \in \mathscr{C}$, $\Delta_1 \cap \Delta_2 = \emptyset$) and let the set function $m(\Delta) = \mathsf{E}\{\mu^2(\Delta)\}$ have finite variation over $\langle 0, 1)$. Further, let f be a continuous function on $\langle 0, 1\rangle$. Then the random set function $\varphi(\langle a, b), \omega) = f(a)\, \mu(\langle a, b), \omega)$ is integrable in the quadratic mean over $\langle 0, 1)$ with respect to the system $\mathfrak{M}_E^0$. Our assumptions ensure the existence of the integral $\int_0^1 f \, \mathrm{d}m$ in the Riemann-Stieltjes sense. It means that for every $\varepsilon > 0$ there exists a $\mathscr{C}_0$-division $\mathscr{D}_\varepsilon \in \mathfrak{M}_E$ such that

$$\sum_i \omega\{f, \langle a_i, b_i)\}\, m(\langle a_i, b_i)) < \varepsilon$$

for every $\mathscr{D} = \{\langle a_i, b_i)\}_i \succ \mathscr{D}_\varepsilon$. Let consider $\varphi(\mathscr{D})$ for any $\mathscr{D} \succ \mathscr{D}_\varepsilon$ i.e.

$$\varphi(\mathscr{D}) = \sum_i f(a_i)\, \mu(\langle a_i, b_i), \omega)$$

then

$$\begin{aligned} &\mathsf{E}\{(\varphi(\mathscr{D}_1) - \varphi(\mathscr{D}_2))^2\} = \\ &= \sum_i \sum_j (f(a_i^1) - f(a_j^2))^2\, \mathsf{E}\{\mu(\Delta_i^1)\, \mu(\Delta_j^2, \omega)\} = \\ &= \sum_i \sum_j (f(a_i^1) - f(a_j^2))^2\, m(\Delta_i^1 \cap \Delta_j^2) \leqq \\ &\leqq \sum_i (\omega\{f, \Delta_i^1\})^2\, m(\Delta_i^1) \leqq \sum_k (\omega\{f, \Delta_k^\varepsilon\})^2\, m(\Delta_k^\varepsilon) < 2\mathrm{M}\varepsilon \end{aligned}$$

where

$$\omega\{f, \Delta\} = \sup_{x \in \Delta} \{f(x)\} - \inf_{x \in \Delta} \{f(x)\} \leqq 2\mathrm{M}$$

if $|f(x)| \leqq \mathbf{M}$ on $\langle 0, 1\rangle$.

This fact proves that the net $\{\varphi(\mathscr{D})\}$ of random variables is fundamental in the quadratic mean. It means that a random variable $\underset{\mathscr{D}}{Lim}\, \varphi(\mathscr{D})$ must exist which is called the stochastic integral of f with respect to μ and denoted by $\int_E f \, \mathrm{d}\mu$.

The very important case of the stochastic integral in the quadratic mean is the integral with respect to the Wiener process. Let $w(\cdot, \cdot)$ be the standard Wiener proces on $\langle 0, 1\rangle$ and let us put $\mu(\langle a, b), \omega) = w(b, \omega) - w(a, \omega) = \Delta w(a, \omega)$ where $\langle a, b) \in \mathscr{C}_0$. Let f be a continuous function defined on $\langle 0, 1\rangle$, then the random set function $\varphi(\langle a, b), \omega) = f(a)\, \Delta w(a)$ is integrable in the quadratic mean with respect to the system $\mathscr{C}_0$ and the existing limit random variable is denoted by

$$\int_0^1 f(t)\,\mathrm{d}w(t)\,.$$

The properties of the stochastic integral understood in the quadratic mean are familarly known. Our question is the following: for which continuous functions f on $\langle 0, 1\rangle$ the corresponding random set function $\varphi(\langle a, b), \omega) = f(a)\,\Delta w(a, \omega)$ $\langle a, b) \subset \langle 0, 1)$ defined on $\mathscr{C}_0$ is asymptotically additive in the quadratic mean with respect to the system $\mathfrak{M}_E^0$. The following theorem gives a partial answer.

Theorem 9. Let a function f have its derivative function f' on $(0, 1)$ bounded, i.e. $|f'(x)| \leqq \mathrm{M}$ for every $x \in (0, 1)$. Then the random set function $\varphi(\langle a, b)) = f(a)\,\Delta w(a)$ defined on $\mathscr{C}_0$ is asymptotically additive in the quadratic mean with respect to $\mathfrak{M}_E^0$.

Proof. Let $\mathscr{D}_1, \mathscr{D}_2$ be any $\mathscr{C}_0$-divisions of $\langle 0, 1)$ such that $\mathscr{D}_1 \prec \mathscr{D}_2$. Then it is possible to write, if $\mathscr{D}_1 = \{\langle a_i, a_{i+1})\}_{i=1}^n$, that

$$\mathscr{D}_2 = \{\langle a_{ij}, a_{ij+1})\}_{ij}$$

where

$$\langle a_i, a_{i+1}) = \bigcup_j \langle a_{ij}, a_{ij+1})\,.$$

For proving the asymptotical additivity in the quadratic mean it is necessary to estimate the value

$$S(\mathscr{D}_1, \mathscr{D}_2) = \mathsf{E}\{(\sum_i |f(a_i)\,\Delta w(a_i) - \sum_j f(a_{ij})\,\Delta w(a_{ij})|)^2\}\,.$$

Using simple properties of increments of the Wiener process we obtain that

$$S(\mathscr{D}_1, \mathscr{D}_2) = \left(1 - \frac{2}{\pi}\right) \sum_i \sum_j (f(a_{ij}) - f(a_i))^2\,(a_{ij+1} - a_{ij}) + $$
$$+ \frac{2}{\pi} (\sum_i (\sum_j (f(a_{ij}) - f(a_i))^2\,(a_{ij+1} - a_{ij}))^{1/2})^2\,.$$

Because we assume the existence of the derivative f' on $(0, 1)$, it is possible to express

$$(+) \qquad f(a_{ij}) - f(a_i) = f'(\xi_{ij})\,(a_{ij} - a_i) \quad \text{where} \quad \xi_{ij} \in (a_i, a_{ij})\,.$$

Using $(+)$, we obtain

$$S(\mathscr{D}_1, \mathscr{D}_2) = \left(1 - \frac{2}{\pi}\right) \sum_i \sum_j (f'(\xi_{ij}))^2\,(a_{ij} - a_i)^2\,(a_{ij+1} - a_{ij}) + $$
$$+ \frac{2}{\pi} (\sum_i (\sum_j (f'(\xi_{ij}))^2\,(a_{ij} - a_i)^2\,(a_{ij+1} - a_{ij}))^{1/2})^2 \leqq$$

$$\leqq \left(1 - \frac{2}{\pi}\right) \sum_i \sum_j \mathrm{M}^2 (a_{i+1} - a_i)^2 (a_{ij+1} - a_{ij}) +$$

$$+ \frac{2}{\pi} (\sum_i (\sum_j \mathrm{M}^2 (a_{i+1} - a_i)^2 (a_{ij+1} - a_{ij}))^{1/2})^2 \leqq$$

$$\leqq \mathrm{M}^2 \left(1 - \frac{2}{\pi}\right) \sum_i (a_{i+1} - a_i)^3 + \frac{2\mathrm{M}}{\pi} (\sum_i (a_{i+1} - a_i)^{3/2})^2 .$$

Now, let ε be any positive number; then there exists a $\mathscr{C}_0$-division $\mathscr{D}_\varepsilon$ of $\langle 0, 1)$ such that

$$\|\mathscr{D}_\varepsilon\| = \max_k \{t_{k+1} - t_k\} \leqq \varepsilon$$

if

$$\mathscr{D}_\varepsilon = \{\langle t_k, t_{k+1})\}_k .$$

If we choose $\mathscr{D}_1, \mathscr{D}_2$ mentioned above such that $\mathscr{D}_2 \succ \mathscr{D}_1 \succ \mathscr{D}_\varepsilon$ then

$$S(\mathscr{D}_1, \mathscr{D}_2) \leqq \mathrm{M}^2 \left(1 - \frac{2}{\pi}\right) \varepsilon^2 + \frac{2\mathrm{M}}{\pi} \varepsilon$$

and this fact proves that the random set function φ is asymptotically additive in the quadratic mean with respect to $\mathfrak{M}_E^0$. □

As the notion of the differential equivalence is too strong for the characterization of the indefinite integral we shall try to determine a weaker form of such an equivalence describing the indefinite integral as the unique additive random set function equivalent to the original random set function. For further considerations it is important to determine the class of mutually orthogonal random set functions.

Definition 8. Let $E \in \mathscr{C}$ be fixed, let O_E be a set of random set functions defined on the set system $\mathscr{C}$ which satisfy the following conditions:

1) for every $\Delta \in \mathscr{D} \in \mathfrak{M}_E$ and every $\varphi \in O_E : \mathsf{E}\{\varphi^2(\Delta)\} < \infty$;
2) for every $\Delta \in \mathscr{D} \in \mathfrak{M}_E$ and every $\varphi \in O_E : \mathsf{E}\{\varphi(\Delta)\} = 0$;
3) for every $\Delta \in \mathscr{D}_1 \in \mathfrak{M}_E$, $\delta \in \mathscr{D}_2 \in \mathfrak{M}_E$, $\Delta \cap \delta = \emptyset$ and every pair $\varphi_1, \varphi_2 \in O_E$: $\mathsf{E}\{\varphi_1(\Delta)\, \varphi_2(\delta)\} = 0$.

Then the set O_E will be called the set of mutually orthogonal random set functions on $\mathfrak{M}_E$.

Further, let us suppose that all random set functions belonging to O_E are $\mathscr{L}_2$-integrable over E. It implies that the set function $m(\Delta) = \mathsf{E}\{\varphi^2(\Delta)\}$ is integrable over $E \in \mathscr{C}$ in the sense of Kolmogorov as it was proved in Theorem 5. Hence it is possible to write

$$\int_E \mathrm{d}m = \int_E \mathrm{d}\mathsf{E}\{\varphi^2\} = \mathsf{E}\left\{\left(\int_E \mathrm{d}\varphi\right)^2\right\}.$$

We shall denote

$$\|\varphi\| = \left(\int_E \mathrm{d}m\right)^{1/2}.$$

Theorem 10. Let O_E be a set of mutually orthogonal $\mathscr{L}_2$-integrable over E random set functions, $E \in \mathscr{C}$. Let $L(O_E)$ be the linear hull over O_E. Then $L(O_E)$ is a set of mutually orthogonal random set functions also and $\|\cdot\|$ is a seminorm on $L(O_E)$.

Proof. If λ is any real number, then it is clear that for every $\varphi \in O_E$ $(\lambda\varphi)(\Delta, \omega) = \lambda\,\varphi(\Delta, \omega)$ is $\mathscr{L}_2$-integrable and orthogonal random set function also. If φ_1, φ_2 are any elements of O_E then $(\varphi_1 + \varphi_2)(\Delta, \omega) = \varphi_1(\Delta, \omega) + \varphi_2(\Delta, \omega)$ is $\mathscr{L}_2$-integrable also (see Lemma 5); further, let $\Delta \in \mathscr{D}_1$, $\delta \in \mathscr{D}_2$ be disjoint, $\mathscr{D}_1, \mathscr{D}_2 \in \mathfrak{M}_E$, then

$$\mathsf{E}\{\varphi(\Delta, \omega)\,\psi(\delta, \omega)\} = 0$$

for every pair φ, ψ of $L(O_E)$ because φ and ψ are linear combinations of mutually orthogonal random set functions. It proves that $L(O_E)$ is a set of mutually orthogonal $\mathscr{L}_2$-integrable random set functions also. It implies the existence of $\|\varphi\| = (\mathsf{E}\{(\int_E \mathrm{d}\varphi)^2\})^{1/2}$ for every $\varphi \in L(O_E)$.

If λ is any real number, at the first sight it is clear that

$$\|\lambda\varphi\| = |\lambda|\;\|\varphi\|\,.$$

If $\varphi_1, \varphi_2 \in L(O_E)$ then thanks to the Minkowski inequality it is clear that the inequality $\|\varphi_1 + \varphi_2\| \leqq \|\varphi_1\| + \|\varphi_2\|$ holds on $L(O_E)$. The role of the null element plays the random set function $O(\Delta) = 0$ for every $\Delta \in \mathscr{D} \in \mathfrak{M}_E$. □

For $\mathscr{L}_2$-integrable mutually orthogonal random set functions it is possible to characterize the relation between them and their indefinite integrals as follows. Before it we need to introduce the notion of the $\mathfrak{M}_E$-equivalence between two random set functions.

Definition 9. Let φ_1, φ_2 be random set functions defined on $\mathscr{C}$; let $E \in \mathscr{C}$. Then φ_1, φ_2 are $\mathfrak{M}_E$-equivalent if there exists a $\mathscr{C}$-division $\mathscr{D}_{12} \in \mathfrak{M}_E$ such that for every

$$\mathscr{D} \succ \mathscr{D}_{12}\,, \quad \mathscr{D} \in \mathfrak{M}_E\,, \quad \mathscr{D} = \{\Delta_i\} \quad \varphi_1(\Delta_i, \omega) = \varphi_2(\Delta_i, \omega)$$

for every i.

Theorem 11. Let φ be an orthogonal random set function $\mathscr{L}_2$-integrable over $E \in \mathscr{C}$. Let $\hat{\varphi}$ be the indefinite integral of φ. Then $\hat{\varphi}$ is the only $\mathscr{L}_2$-additive random set function (up to the $\mathfrak{M}_E$-equivalence) defined on $\mathfrak{M}_E$ which forms together with the original set function φ the set $\{\varphi, \hat{\varphi}\}$ of mutually orthogonal random set functions such that $\|\varphi - \hat{\varphi}\| = 0$.

Proof. First, we prove that $\|\varphi - \hat{\varphi}\| = 0$. It means to investigate the integral sum

$$\sum_i (\varphi(\Delta_i) - \hat{\varphi}(\Delta_i))$$

for a suitable fine $\mathscr{C}$-division $\mathscr{D} \in \mathfrak{M}_E$, $\mathscr{D} = \{\Delta_i\}$. As we know that $\hat{\varphi}$ is an $\mathscr{L}_2$-additive random set function and hence

$$\sum_i \hat{\varphi}(\Delta_i) = \hat{\varphi}(E) = \int_E \mathrm{d}\varphi$$

then

$$\mathsf{E}\{(\sum_i \varphi(\Delta_i) - \hat{\varphi}(\Delta_i))^2\} = \mathsf{E}\left\{\left(\sum_i \varphi(\Delta_i) - \int_E \mathrm{d}\varphi\right)^2\right\} < \varepsilon$$

if $\mathscr{D} \succ \mathscr{D}_\varepsilon$ because φ is $\mathscr{L}_2$-integrable over E. It proves that $\|\varphi - \hat{\varphi}\| = 0$. Let $\mathscr{D} = \{\Delta_i\}$ be an arbitrary $\mathscr{C}$-division of E. Then $\hat{\varphi}(\Delta_i)$ (for every i) can be expressed as the limit

$$\hat{\varphi}(\Delta_i) = \underset{\mathscr{D}}{Lim}\, \varphi(\Delta_i \cap \mathscr{D})$$

in the $\mathscr{L}_2$-sense. This convergence and the orthogonality of φ prove that the set $\{\varphi, \hat{\varphi}\}$ forms a set of mutually orthogonal set functions. It implies further that $\hat{\varphi}$ is an orthogonal random set function also. Now, let ψ be another $\mathscr{L}_2$-additive and orthogonal on $\mathfrak{M}_E$ random set function which forms together with φ the pair $\{\varphi, \psi\}$ of mutually orthogonal random set functions satisfying $\|\varphi - \psi\| = 0$. This assumption implies immediately that the pair $\{\hat{\varphi}, \psi\}$ is a set of mutually orthogonal functions because $\hat{\varphi}(\Delta)$, $\Delta \in \mathscr{D} \in \mathfrak{M}_E$, is a limit in the quadratic mean of linear combinations derived from φ. Further, the inequality $\|\hat{\varphi} - \psi\| \leqq \|\hat{\varphi} - \varphi\| + \|\varphi - \psi\|$ gives $\|\hat{\varphi} - \psi\| = 0$. We assume that ψ is $\mathscr{L}_2$-additive, i.e. there exists a $\mathscr{C}$-division $\mathscr{D}_a \in \mathfrak{M}_E$ such that for every $\mathscr{D} \succ \mathscr{D}_a$, $\mathscr{D} \in \mathfrak{M}_E$

$$\psi(\mathscr{D}, \omega) = \psi(\mathscr{D}_a, \omega)\,.$$

As $\hat{\varphi}, \psi$ are mutually orthogonal and $\mathscr{L}_2$-additive then $\hat{\varphi} - \psi$ is orthogonal and $\mathscr{L}_2$-additive too (see Theorem 10) and hence the fact $\|\hat{\varphi} - \psi\| = 0$ implies that $\hat{\varphi}$ and ψ are $\mathfrak{M}_E$-equivalent. □

The very important class of mutually orthogonal random set functions can be constructed from the Wiener process. Let $\{\mathscr{B}_t\}_{t\in\langle 0,1\rangle}$ be a nondecreasing system of σ-algebras and let $w(t, \omega)$ be a Wiener process adapted with respect to $\{\mathscr{B}_t\}$. Further, let $E = \langle 0, 1\rangle$ and let $\mathscr{C}_0$ be the system of the all intervals in E (degenerated intervals are included too). Let $\mathfrak{M}_E^0$ be the system of the all finite $\mathscr{C}_0$-divisions of E. If $f(t, \omega)$ is any stochastic process on $\langle 0, 1\rangle$ adapted also with respect to $\{\mathscr{B}_t\}$ such that $\mathsf{E}\{f^2(t, \omega)\} < \infty$ for every $t \in E$ then the random set function φ_f

$$\varphi_f(\langle a, b), \omega) = f(a, \omega)\,[w(b, \omega) - w(a, \omega)]$$

defined on $\mathscr{C}_0$ is an orthogonal random set function. If $g(t, \omega)$ is another process on $\langle 0, 1\rangle$ adapted also with respect to $\{\mathscr{B}_t\}$ with $\mathsf{E}\{g^2(t, \omega)\} < \infty$ for every $t \in E$ then the random set functions φ_f, φ_g corresponding to f, g are mutually orthogonal on $\mathfrak{M}_E^0$.

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