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Some Variants of Fault-Finding Procedures

LIBOR KUBÁT, MILAN ULLRICH

For three types of fault-finding procedures the optimum ones are determined by means of probability theory methods.

The maintenance of complex devices and systems becomes a very important problem. It is well known, that the greater part of maintenance hours is used for looking for the unoperating units of the device, and the smaller part for their repair. In other words, the main maintenance problem is not *how* to repair, but *what* to repair. Thus, good fault-finding procedures are necessary for fast and efficient maintenance and their use improves the availability degree of equipment.

There is a great number of variants of possible fault-finding procedures. In the present paper three basic types are discussed, which – according to the authors' opinion – are most important and theoretically interesting:

(i) the signal-measurement procedure,
(ii) the element-measurement procedure,
(iii) the replacement-of-element procedure.

All three types are solved in general, and illustrated on simple examples.

INTRODUCTION AND SIMPLE EXAMPLES

In this paper we shall deal with the problem of determining all failures, i.e. all defective elements of a system containing n elements in the case we know that the whole system does not operate. We shall look for such a procedure of determining these failures which minimizes the expected cost.

Now, we shall introduce some mathematical notions and assumptions. We shall consider that the system contains n elements numbered by $1, 2, \ldots, n$. We shall assume that the i-th element of the system is defective with probability p_i and good with probability $1 - p_i$, and that these values are known. Let all the elements be statistically independent. Let us have random variables $\xi_1, \xi_2, \ldots, \xi_n$ given by

$$\xi_i = 0 \quad \text{if the } i\text{-th element is good,}$$
$$= 1 \quad \text{if the } i\text{-th element is defective.}$$

Then

$$P(\xi_i = 1) = p_i = 1 - P(\xi_i = 0)$$

and random variables $(\xi_1, \xi_2, \ldots, \xi_n)$ are independent.

Let us assume that the whole system does not operate if at least one element is defective. Every element of the system found to be defective is immediately exchanged

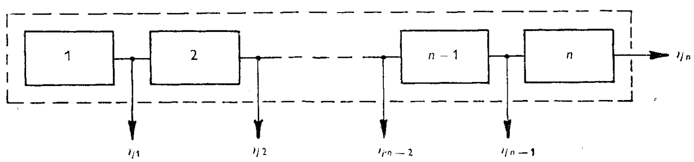


Fig. 1. Elements connection in the type (i).

by a good one or repaired. In any of the three discussed procedures we do not include the cost of repair into the cost of procedure, because the cost of repair does not depend on the used fault-finding procedure.

As we have mention above, we shall discuss three basic types of the fault-finding procedures, which are defined in the following way:

(i) *The signal-measurement procedure* is based on the assumption that the system is composed by elements functionally connected in a chain-like form as shown in Fig. 1. The signal-measurement is possible on the output of any element only. If in the chosen point of the chain the measured signal is good, then all preceding elements are good, and in the opposite case, i.e. if in the considered point the measured signal is not good, then surely at least one of the preceding elements is defective. This procedure corresponds mathematically to the following problem: is the chosen random variable η_i for given i equal to 0 or to 1, where

$$\eta_i = \max_{1 \leq j \leq i} \xi_j \quad (i = 1, 2, \ldots, n).$$

(ii) *The element-measurement procedure* is based on the assumption that we can decide by measurement of elements their state, i.e. if the considered element is good or defective. Moreover we suppose that in this procedure the determining of a defective element is followed by its repair and by checking the entire-system performance, immediately.

(iii) *The replacement-of-element procedure* is based on the assumption that there are not any measuring devices available and elements of the system can be replaced by spare parts only.

Whereas the first type of procedure can be used for special configuration – the chain-like connection of elements – the other two types are more general and can be used for any configuration of elements, of course for such configurations corresponding to the above assumption that the whole system does not operate if at least one element is defective.

For any of the considered types of procedures the optimum procedure means that its expected cost will be minimum and moreover in the case (iii) that none good element of the system will be replaced.

On following simple examples all types of fault-finding procedures and their optimization will be illustrated.

Henceforward we shall use the following symbols:

D*i*	measuring on the i-th element (i.e. in the type (i) measuring of η_i, in the type (ii) deciding about the state of i-th element);
D	determing of the entire system performance;
N*i*	replacement of the i-th element of the system by spare part (in the type (iii));
V*i*	re-replacement of the i-th element (i.e. the replacement of the installed spare part by the original element of the system) (in the type (iii));
O*i*	repair of the i-th element which is found to be defective (in types (i) and (ii));
$\mathbf{P}(i_1, i_2, \ldots, i_k)$	the procedure for determing all defective elements in the set of elements $(i_1, i_2, \ldots, i_k)$, when it is known that there is at least one defective element in this set;
$\mathbf{Q}(i_1, i_2, \ldots, i_k)$	the procedure for determining all defective elements in the set of elements $(i_1, i_2, \ldots, i_k)$ when nothing is known about these elements;
d	the cost of procedures **D**;
δ	the cost of procedures **D***i*, **N***i* or **V***i* (for every $i = 1, 2, \ldots, n$);
$N(\mathbf{P}(i_1, i_2, \ldots, i_k))$, $N(\mathbf{Q}(i_1, i_2, \ldots, i_k)$	the minimum expected costs of procedures $\mathbf{P}(i_1, i_2, \ldots, i_k)$ and $\mathbf{Q}(i_1, i_2, \ldots, i_k)$, respectively.

In the graphical description of different procedures we shall use the following symbolic form in connection with **D** or **D***i*. These operators correspond to the questions:

$$\mathbf{D} \equiv \text{Is the entire system in the operating state?}$$

$$\mathbf{D}i \equiv \text{Is the } i\text{-th element (or the chain of elements } 1, 2, \ldots, i) \text{ good?}$$

The positive answer leads to the rightward continuation in the graphical form, and similarly the negative answer to the downward continuation.

Example 1. The signal-measurement procedure for $n = 3$. All possible fault-finding procedures of this type in the case of an unoperating system containing three elements are given in Fig. 2. There are only four possible procedures. Now, we will look for the optimum procedure between them for the case $\delta = 1$. The costs for

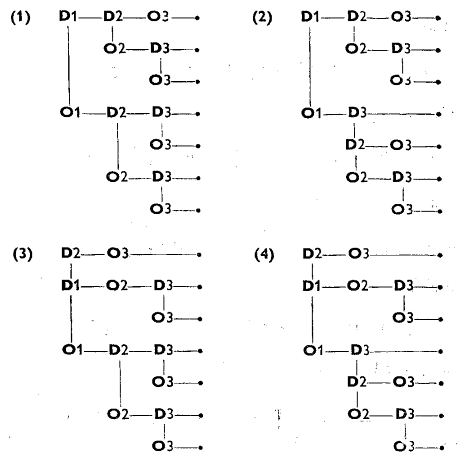


Fig. 2. All possible procedures in the type (i) for $n = 3$.

Table 1.

Costs of all different procedure in the type (i) for $n = 3$.

ξ_1, ξ_2, ξ_3	Procedures			
	(1)	**(2)**	**(3)**	**(4)**
0 0 1	2	2	1	1
0 1 0	3	3	3	3
0 1 1	3	3	3	3
1 0 0	3	2	4	3
1 0 1	3	3	4	4
1 1 0	3	4	4	5
1 1 1	3	4	4	5

different combinations of ξ_1, ξ_2, ξ_3 are summarized in Tab. 1. From tabulated data the expected cost and the minimum can be easily calculated.

The following relations between the expected costs for different procedures, **(1)**, (2), (3) and (4), denoted by N_1, N_2, N_3, and N_4 respectively, are obviously valid:

$$N_1 \leqq N_2 \quad \text{and} \quad N_3 \leqq N_4 \quad \text{if and only if} \quad (1 - p_2)(1 - p_3) - p_2 \leqq 0\,,$$
$$N_1 \leqq N_3 \quad \text{and} \quad N_2 \leqq N_4 \quad \text{if and only if} \quad (1 - p_1)(1 - p_2)\,p_3 - p_1 \leqq 0\,.$$

Hence:

Procedure **(1)** is optimum if and only if

$$(1 - p_2)(1 - p_3) - p_2 \leqq 0 \quad \text{and} \quad (1 - p_1)(1 - p_2)\,p_3 - p_1 \leqq 0\,.$$

Procedure **(2)** is optimum if and only if

$$(1 - p_2)(1 - p_3) - p_2 \geqq 0 \quad \text{and} \quad (1 - p_1)(1 - p_2)\,p_3 - p_1 \leqq 0\,.$$

Procedure **(3)** is optimum if and only if

$$(1 - p_2)(1 - p_3) - p_2 \leqq 0 \quad \text{and} \quad (1 - p_1)(1 - p_2)\,p_3 - p_1 \geqq 0\,.$$

Procedure **(4)** is optimum if and only if

$$(1 - p_2)(1 - p_3) - p_2 \geqq 0 \quad \text{and} \quad (1 - p_1)(1 - p_2)\,p_3 - p_1 \geqq 0\,.$$

The expected cost is given by the following formulae:

$$N(\mathbf{P}(1, 2, 3)) = \frac{1}{1 - (1 - p_1)(1 - p_2)(1 - p_3)}[3p_1 + 3p_2 + 2p_3 - 3p_1p_2 - 2p_1p_3 - 2p_2p_3 + 2p_1p_2p_3]$$

if the procedure **(1)** is optimum;

$$= \frac{1}{1 - (1 - p_1)(1 - p_2)(1 - p_3)}[2p_1 + 3p_2 + 2p_3 + 2p_1p_2 - 4p_1p_3 - 2p_2p_3 + p_1p_2p_3]$$

if the procedure **(2)** is optimum;

$$= \frac{1}{1 - (1 - p_1)(1 - p_2)(1 - p_3)}[4p_1 + 3p_2 + p_3 - 3p_1p_2 - p_1p_3 - p_2p_3 + p_1p_2p_3]$$

if the procedure **(3)** is optimum;

$$= \frac{1}{1 - (1 - p_1)(1 - p_2)(1 - p_3)}[3p_1 + 3p_2 + p_3 - p_1p_2 - p_2p_3]$$

if the procedure **(4)** is optimum.

In the special case for $p_1 = p_2 = p_3 = p$ the optimum procedure is **(1)** or **(2)** according to the validity of following conditions:

Procedure **(1)** is optimum if and only if

$$(1 - p)^2 - p \leqq 0 .$$

Procedure **(2)** is optimum if and only if

$$(1 - p)^2 - p \geqq 0 .$$

In this case we obtain:

$$N(\mathbf{P}(1, 2, 3)) = \frac{1}{1 - (1 - p)^3} [8p - 7p^2 + 2p^3] \quad \text{if} \quad (1 - p)^2 - p \leqq 0 ,$$

$$= \frac{1}{1 - (1 - p)^3} [7p - 4p^2 + p^3] \quad \text{if} \quad (1 - p)^2 - p \geqq 0 .$$

Another formal description of the above procedures – shown in Fig. 3 – can be used, which is more convenient for further generalization. For the discussed special

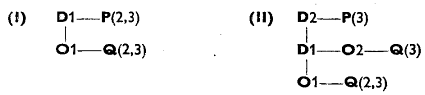


Fig. 3. All possible procedures **P**(1, 2, 3) in the type (i) for $n = 3$. (An equivalent description to Fig. 2.)

case $p_1 = p_2 = p_3 = p$ the expected costs denoted by N_{I} and N_{II} are given by the following formulae:

$$N_{\text{I}} = \{(1 + N(\mathbf{P}(2, 3)))(1 - p)[1 - (1 - p)^2] + (1 + N(\mathbf{Q}(2, 3)))\, p\} \cdot \frac{1}{1 - (1 - p)^3} ,$$

$$N_{\text{II}} = \{(1 + N(\mathbf{P}(3)))(1 - p)^2 [1 - (1 - p)] + [2 + N(\mathbf{Q}(3))]\, p(1 - p) + (2 + N(\mathbf{Q}(2, 3)))\, p\} \cdot \frac{1}{1 - (1 - p)^3} .$$

However, $N(\mathbf{Q}(3)) = 1$, $N(\mathbf{P}(3)) = 0$, $N(\mathbf{P}(2, 3)) = (3 - p)/(2 - p)$ and

$$N(\mathbf{Q}(2, 3)) = 2 \quad \text{if} \quad (1 - p)^2 - p \leqq 0 ,$$

$$= 1 + 3p - p^2 \quad \text{if} \quad (1 - p)^2 - p \geqq 0$$

as can be easily proved.

Thus, the corresponding expected costs are

$$N_{\mathrm{I}} = \frac{8p - 7p^2 + 2p^3}{1 - (1 - p)^3}, \quad N_{\mathrm{II}} = \frac{8p - 5p^2 + p^3}{1 - (1 - p)^3}$$

for $(1 - p)^2 - p \leqq 0$, and

$$N_{\mathrm{I}} = \frac{7p - 4p^2 + p^3}{1 - (1 - p)^3}, \quad N_{\mathrm{II}} = \frac{7p - 2p^2}{1 - (1 - p)^3}$$

for $(1 - p)^2 - p \geqq 0$.

In both cases $N_{\mathrm{I}} \leqq N_{\mathrm{II}}$ and therefore the procedure **(I)** is optimum.

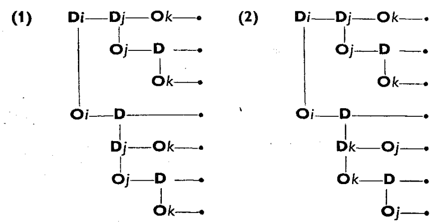


Fig. 4. All types of possible procedures **P**(i, j, k) in the type (ii) for $n = 3$.

Table 2.

Costs of both procedures **(1)** and **(2)** in the type (ii) for $n = 3$ and for given permutation (i, j, k)

ξ_1, ξ_2, ξ_3	Procedures	
	(1)	**(2)**
0 0 1	2	2
0 1 0	$2 + d$	$2 + d$
0 1 1	$2 + d$	$2 + d$
1 0 0	$1 + d$	$1 + d$
1 0 1	$2 + d$	$2 + 2d$
1 1 0	$2 + 2d$	$2 + d$
1 1 1	$2 + 2d$	$2 + 2d$

Example 2. The element measurement procedure for $n = 3$. For this type only two procedures for every ordering of elements i, j, k are possible, as shown in Fig. 4. Tab. 2. presents the costs for both the possible orderings i, j, k. For procedures **(1)** and **(2)** the expected costs, denoted by N_1 and N_2 respectively, are given by the following formulae:

$$N_1 = \frac{1}{1-(1-p_i)(1-p_j)(1-p_k)}\{2[1-(1-p_i)(1-p_j)(1-p_k)] - \\ - p_i(1-p_j)(1-p_k) + d(p_i + p_j)\},$$

$$N_2 = \frac{1}{1-(1-p_i)(1-p_j)(1-p_k)}\{2[1-(1-p_i)(1-p_j)(1-p_k) - \\ - p_i(1-p_j)(1-p_k) + d[p_i + p_j + p_i(p_k - p_j)]\}.$$

It can be seen that the procedure **(1)** is better than the procedure **(2)** if and only if the inequality $p_k \leqq p_j$ is valid. The optimum procedure is determined by such an ordering i, j, k for which the expected cost is minimum. This minimum can be attained by comparing the second and the third terms of the above formulae for N_1 and N_2. These terms for different orderings i, j, k are presented in Tab. 3.

Table 3.
Expected costs in the type (ii) for $n = 3$ and for different permutations of (i, j, k)

i, j, k	Expected cost
1 2 3	$d(p_1 + p_2) - p_1(1 - p_2)(1 - p_3)$
1 3 2	$d(p_1 + p_3 + p_1(p_2 - p_3)) - p_1(1 - p_2)(1 - p_3)$
2 1 3	$d(p_1 + p_2) - p_2(1 - p_1)(1 - p_3)$
2 3 1	$d(p_2 + p_3 + p_2(p_1 - p_3)) - p_2(1 - p_1)(1 - p_3)$
3 1 2	$d(p_1 + p_3) - p_3(1 - p_1)(1 - p_2)$
3 2 1	$d(p_2 + p_3 + p_3(p_1 - p_2)) - p_3(1 - p_1)(1 - p_2)$

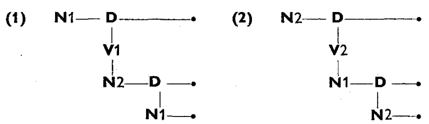


Fig. 5. All possible procedures **P**(1, 2) in the type (iii) for $n = 2$.

It can be shown that in the case the elements are numbered in such a way that $p_1 \leqq p_2 \leqq p_3$, and if the assumption $d \leqq 1 - p_1$ holds, the optimum procedure is **(1)** by the ordering $i = 3, j = 1, k = 2$.

Example 3. The replacement-of-element procedure for $n = 2$. In this case there are only two possible fault-finding procedures shown in Fig. 5. Let us assume that the cost of replacement of re-replacement of an element equals 1, i.e. $\delta = 1$. Then we obtain the costs of both possible procedures as shown in Tab. 4, and the expected costs are given by the formula

$$N_1 = \frac{1}{1 - (1 - p_1)(1 - p_2)} [3(1 - p_1) p_2 + p_1(1 - p_2) + 4p_1p_2 + \\ + 2d(1 - (1 - p_1)(1 - p_2)) - dp_1(1 - p_2)]$$

for the procedure **(1)** and by the formula

$$N_2 = \frac{1}{1 - (1 - p_1)(1 - p_2)} [(1 - p_1) p_2 + 3p_1(1 - p_2) + \\ + 4p_1p_2 + 2d(1 - (1 - p_1)(1 - p_2)) - d(1 - p_1) p_2]$$

for the procedure **(2)**.

Table 4.

Costs of all different procedures in the type (iii) for $n = 2$

ξ_1, ξ_2	Procedures	
	(1)	**(2)**
0 1	$3 + 2d$	$1 + d$
1 0	$1 + d$	$3 + 2d$
1 1	$4 + 2d$	$4 + 2d$

Let the elements be numbered in such a way that $p_1 \leqq p_2$. Then we obtain $N_2 \leqq N_1$ because

$$(N_2 - N_1)[1 - (1 - p_1)(1 - p_2)] = \\ = -2(1 - p_1) p_2 + 2p_1(1 - p_2) - d(1 - p_1) p_2 + dp_1(1 - p_2) = \\ = (2 + d)(p_1 - p_2) \leqq 0\,,$$

i.e. the procedure **(2)** is optimum for the case $p_1 \leqq p_2$.

The above examples illustrate the three discussed types of fault-finding procedures and the technique of determining their optimum. In the following sections the introduced methods will be discussed in more general form.

SIGNAL-MEASUREMENT PROCEDURE

In this section we shall discuss the first type of fault-finding procedures used for the chain-like system (as shown in Fig. 1). Let us assume that by signal measurement on the output of the i-th element we can decide only whether all elements $1, 2, \ldots, i$ are good or whether at least one of them is defective. The general solution of this problem is very tedious and principally it is similar to a special case of well known classical problem of determining randomly chosen number from the set 1, 2, ..., 2^n if permitted questions are of the type: Is the chosen number greater or smaller then 2^k?, only. In authors' opinion this classical problem has not yet been solved.

Our problem could seem to be solved by well known ideas of the information theory, i.e. by measuring η_i for such i for which the conditional entropy $H(\eta_i/\eta_n = 1)$ is maximum. However, in the general case this method does not lead to the optimum, as it will be shown for $n = 6$.

In the following we shall give an algorithm for determining the optimum procedure which is suitable for computers. Let us assume that $p_1 = p_2 = \ldots = p_n = p$ and $\delta = 1$. Then all possible procedures for $\mathbf{P}(1, 2, \ldots, n)$ are given in Fig. 6. The minimum expected costs denoted by $N_{1,1}, N_{2,1}, \ldots, N_{n-1,m_{n-1}}$ for different procedures are given by the following expressions:

$$N_{1,1} = \frac{1}{1-(1-p)^n} \{[1 + N(\mathbf{P}(2, \ldots, n))] (1-p) (1-(1-p)^{n-1}) + \\ + [1 + N(\mathbf{Q}(2, \ldots, n))] p\},$$

$$N_{2,1} = \frac{1}{1-(1-p)^n} \{[1 + N(\mathbf{P}(3, \ldots, n))] (1-p)^2 (1-(1-p)^{n-2}) + \\ + [2 + N(\mathbf{Q}(3, \ldots, n))] p(1-p) + [2 + N(\mathbf{Q}(2, \ldots, n))] p\},$$

$$\vdots$$

$$N_{n-1,m_{n-1}} = \frac{1}{1-(1-p)^n} \{[1 + N(\mathbf{P}(n))] (1-p)^{n-1} (1-(1-p)) + \\ + [2 + N(\mathbf{Q}(n))] p(1-p)^{n-2} + \ldots + [n-1 + N(\mathbf{Q}(3, \ldots, n))] \times \\ \times p(1-p) + [n-1 + N(\mathbf{Q}(2, \ldots, n))] p\}.$$

The optimum procedure is that one for which the expected cost is minimum, i.e. for which the equation

$$N(\mathbf{P}(1, 2, \ldots, n)) = \min_{\substack{1 \leq i \leq n-1 \\ 1 \leq j \leq m_i}} N_{i,j}$$

is valid.

Here, $N(\mathbf{Q}(1, 2, \ldots, k)$ is determined as the minimum expected cost for individual possible procedures $\mathbf{Q}(1, 2, \ldots, k)$ shown in Fig. 7. If we denote the minimum expected cost for these individual possible procedures by $M_{1,1}, M_{2,1}, M_{3,1}, M_{3,2}, \ldots, \ldots, M_{k,m_k}$ then we obtain the following expressions for the minimum costs:

$$M_{1,1} = [1 + N(\mathbf{Q}(2, \ldots, k))] (1-p) + [1 + N(\mathbf{Q}(2, \ldots, k))] p,$$

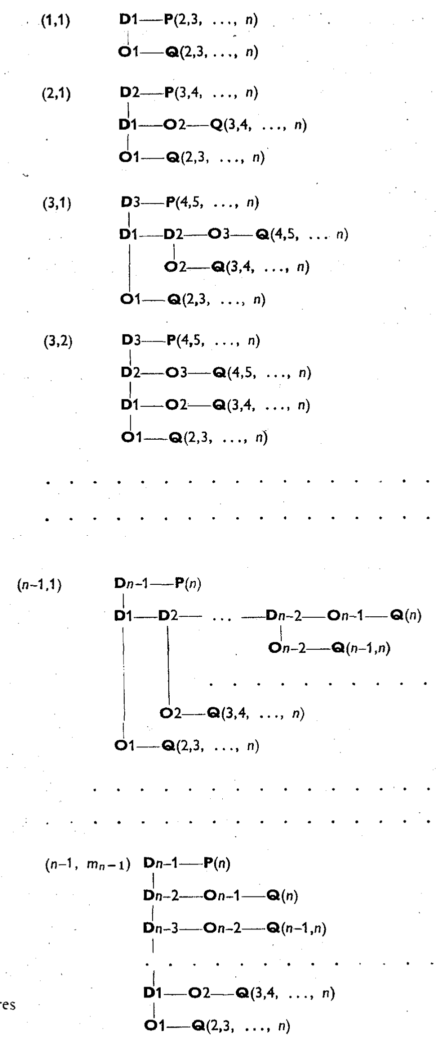


Fig. 6. All possible procedures **P**(1, 2, ..., n) in the type (i).

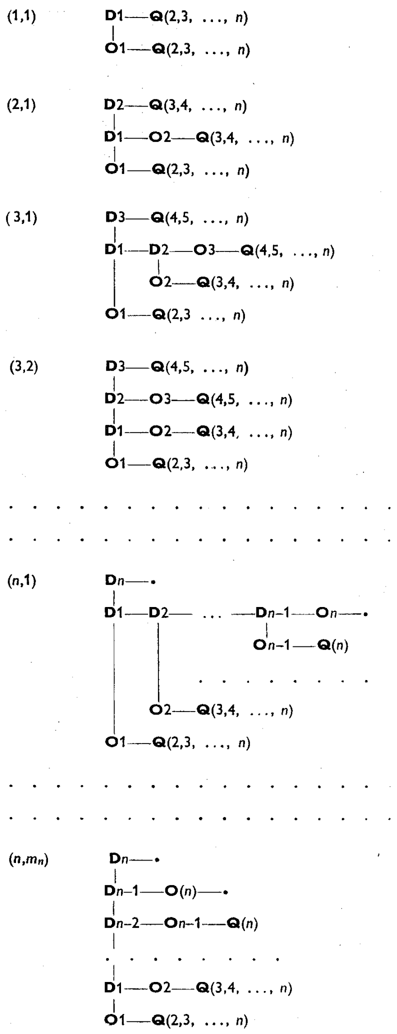


Fig. 7. All possible procedures **Q**(1, 2, ..., *n*) in the type (i).

Table 5.

Optimum procedures for the type (i)

n	Optimum procedure		Notation	Area of p	Expected cost
1	**P**(1)	**O**1—•	**P**(1)	$0 \leqq p \leqq 1$	0
	Q(1)	**D**1—• | **O**1—•	**Q**(1)	$0 \leqq p \leqq 1$	1
2	**P**(1, 2)	**D**1—**P**(2) | **O**1—**Q**(2)	**P**(1, 2)	$0 \leqq p \leqq 1$	$\frac{1}{1-(1-p)^2}[3p-p^2]$
	Q(1, 2)	**D**1—**Q**(2) | **O**1—**Q**(2)	$\mathbf{Q}_1(1, 2)$	$(1-p)^2 - p \leqq 0$	2
		D2—• | **P**(1, 2)	$\mathbf{Q}_2(1, 2)$	$0 \leqq (1-p)^2 - p$	$1 + 3p - p^2$
3	**P**(1, 2, 3)	**D**1—**P**(2, 3) | **O**1—$\mathbf{Q}_1$(2, 3)	$\mathbf{P}_1(1, 2, 3)$	$(1-p)^2 - p \leqq 0$	$\frac{1}{1-(1-p)^3}[8p-7p^2+2p^3]$
		D1—**P**(2, 3) | **O**1—$\mathbf{Q}_2$(2, 3)	$\mathbf{P}_2(1, 2, 3)$	$0 \leqq (1-p)^2 - p$	$\frac{1}{1-(1-p)^3}[7p-4p^2+p^3]$
	Q(1, 2, 3)	**D**1—$\mathbf{Q}_1$(2, 3) | **O**1—$\mathbf{Q}_1$(2, 3)	$\mathbf{Q}_1(1, 2, 3)$	$(1-p)^2 - p \leqq 0$	3

		D2—**Q**(3) | **D**1—**O**2—**Q**(3) | **O**1—$\mathbf{Q}_2(2, 3)$	$\mathbf{Q}_2(1, 2, 3)$	$2(1-p)^2 - 1 \leqq 0 \leqq (1-p)^2 - p$	$2 + 2p + 2p^2 - p^3$
		D3—• | **D**1—**P**(2, 3) | **O**1—$\mathbf{Q}_2(2, 3)$	$\mathbf{Q}_3(1, 2, 3)$	$0 \leqq 2(1-p)^2 - 1$	$1 + 7p - 4p^2 + p^3$
4	**P**(1, 2, 3, 4)	**D**1—$\mathbf{P}_1(2, 3, 4)$ | **O**1—$\mathbf{Q}_1(2, 3, 4)$	$\mathbf{P}_1(1, 2, 3, 4)$	$(1-p)^2 - p \leqq 0$	$\frac{1}{1-(1-p)^4}[15p - 21p^2 + 13p^3 - 3p^4]$
		D1—$\mathbf{P}_2(2, 3, 4)$ | **O**1—$\mathbf{Q}_2(2, 3, 4)$	$\mathbf{P}_2(1, 2, 3, 4)$	$2(1-p)^2 - 1 \leqq 0 \leqq (1-p^2) - p$	$\frac{1}{1-(1-p)^4}[13p - 15p^2 + 11p^3 - 3p^4]$
		D1—$\mathbf{P}_2(2, 3, 4)$ | **O**1—$\mathbf{Q}_3(2, 3, 4)$	$\mathbf{P}_3(1, 2, 3, 4)$	$(1-p)^3 + (1-p)^2 - 1 \leqq 0 \leqq$ $\leqq 2(1-p)^2 - 1$	$\frac{1}{1-(1-p)^4}[12p - 10p^2 + 5p^3 - p^4]$
		D2—**P**(3, 4) | **D**1—**O**2—$\mathbf{Q}_2(3, 4)$ | **O**1 — $\mathbf{Q}_3(2, 3, 4)$	$\mathbf{P}_4(1, 2, 3, 4)$	$0 \leqq (1-p)^3 + (1-p)^2 - 1$	$\frac{1}{1-(1-p)^4}[11p - 5p^2 + p^3]$
	Q(1, 2, 3, 4)	**D**1—$\mathbf{Q}_1(2, 3, 4)$ | **O**1—$\mathbf{Q}_1(2, 3, 4)$	$\mathbf{Q}_1(1, 2, 3, 4)$	$(1-p)^2 - p \leqq 0$	4
		D1—$\mathbf{Q}_2(2, 3, 4)$ | **O**1—$\mathbf{Q}_2(2, 3, 4)$	$\mathbf{Q}_2(1, 2, 3, 4)$	$(1-p)^4 - p^2 \leqq 0 \leqq (1-p)^2 - p$	$3 + 2p + 2p^2 - p^3$

Table 5. (*Continuation*)

n	Optimum procedure		Notation	Area of p	Expected cost
		D2—$\mathbf{Q}_2$(3, 4) | **D**1—**O**2—$\mathbf{Q}_2$(3, 4) | **O**1—$\mathbf{Q}_2$(2, 3, 4)	$\mathbf{Q}_3$(1, 2, 3, 4)	$2(1-p)^2 - 1 \leqq 0 \leqq (1-p)^4 - p^2$	$2 + 6p - 3p^2 + 3p^3 - p^4$
		D2—$\mathbf{Q}_2$(3, 4) | **D**1—**O**2—$\mathbf{Q}_2$(3, 4) | **O**1—$\mathbf{Q}_3$(2, 3, 4)	$\mathbf{Q}_4$(1, 2, 3, 4)	$(1-p)^4 + (1-p)^2 - 1 \leqq 0 \leqq$ $\leqq 2(1-p)^2 - 1$	$2 + 5p + 2p^2 - 3p^3 + p^4$
		D4—• | **D**2—**P**(3, 4) | **D**1—**O**2—$\mathbf{Q}_2$(3, 4) | **O**1—$\mathbf{Q}_3$(2, 3, 4)	$\mathbf{Q}_5$(1, 2, 3, 4)	$0 \leqq (1-p)^4 + (1-p)^2 - 1$	$1 + 11p - 5p^2 + p^3$
5	**P**(1, 2, 3, 4, 5)	**D**1—$\mathbf{P}_1$(2, 3, 4, 5) | **O**1—$\mathbf{Q}_1$(2, 3, 4, 5)	$\mathbf{P}_1$(1, 2, 3, 4, 5)	$(1-p)^2 - p \leqq 0$	$\frac{1}{1-(1-p)^5}[24p - 46p^2 + 44p^3 - 21p^4 + 4p^5]$
		D1—$\mathbf{P}_2$(2, 3, 4, 5) | **O**1—$\mathbf{Q}_2$(2, 3, 4, 5)	$\mathbf{P}_2$(1, 2, 3, 4, 5)	$(1-p)^4 - p^2 \leqq 0 \leqq (1-p^2) - p$	$\frac{1}{1-(1-p)^5}[21p - 36p^2 + 38p^3 - 20p^4 + 4p^5]$

	D1—$\mathbf{P}_2$(2, 3, 4, 5) | **O**1—$\mathbf{Q}_3$(2, 3, 4, 5)	$\mathbf{P}_3$(1, 2, 3, 4, 5)	$(1-p)^4+(1-p)^3+(1-p)^2-$ $-1 \leqq 0 \leqq (1-p)^4-p^2$	$\frac{1}{1-(1-p)^5}[20p-32p^2+$ $+33p^3-16p^4+3p^5]$
	D2—$\mathbf{P}_2$(3, 4, 5) | **D**1—**O**2—$\mathbf{Q}_2$(3, 4, 5) | **O**1—$\mathbf{Q}_3$(2, 3, 4, 5)	$\mathbf{P}_4$(1, 2, 3, 4, 5)	$2(1-p)^2-1 \leqq 0 \leqq (1-p)^4+$ $+(1-p)^3+(1-p)^2-1$	$\frac{1}{1-(1-p)^5}[18p-23p^2+$ $+23p^3-11p^4+2p^5]$
	D2—$\mathbf{P}_2$(3, 4, 5) | **D**1—**O**2—$\mathbf{Q}_3$(3, 4, 5) | **O**1—$\mathbf{Q}_4$(2, 3, 4, 5)	$\mathbf{P}_5$(1, 2, 3, 4, 5)	$(1-p)^4+(1-p)^2-1 \leqq 0 \leqq$ $\leqq 2(1-p)^2-1$	$\frac{1}{1-(1-p)^5}[17p-18p^2+$ $+17p^3-9p^4+2p^5]$
	D2—$\mathbf{P}_2$(3, 4, 5) | **D**1—**O**2—$\mathbf{Q}_3$(3, 4, 5) | **O**1—$\mathbf{Q}_5$(2, 3, 4, 5)	$\mathbf{P}_6$(1, 2, 3, 4, 5)	$0 \leqq (1-p)^4+(1-p)^2-1$	$\frac{1}{1-(1-p)^5}[16p-12p^2+$ $+10p^3-5p^4+p^5]$
Q(1, 2, 3, 4, 5)	**D**1—$\mathbf{Q}_1$(2, 3, 4, 5) | **O**1—$\mathbf{Q}_1$(2, 3, 4, 5)	$\mathbf{Q}_1$(1, 2, 3, 4, 5)	$(1-p)^2-p \leqq 0$	5
	D2—$\mathbf{Q}_2$(3, 4, 5) | **D**1—**O**2—$\mathbf{Q}_2$(3, 4, 5) | **O**1—$\mathbf{Q}_2$(2, 3, 4, 5)	$\mathbf{Q}_2$(1, 2, 3, 4, 5)	$(1-p)^4-p^2 \leqq 0 \leqq (1-p^2)-p$	$3+5p+p^2-p^3$
	D2—$\mathbf{Q}_2$(3, 4, 5) | **D**1—**O**2—$\mathbf{Q}_2$(3, 4, 5) | **O**1—$\mathbf{Q}_3$(2, 3, 4, 5)	$\mathbf{Q}_3$(1, 2, 3, 4, 5)	$2(1-p)^2-1 \leqq 0 \leqq (1-p)^4-p^2$	$3+4p+5p^2-6p^3+4p^4-p^5$

Table 5. (*Continuation*)

n	Optimum procedure		Notation	Area of p	Expected cost
		D2—Q$_3$(3, 4, 5) | D1—O2—Q$_3$(3, 4, 5) | O1—Q$_4$(2, 3, 4, 5)	Q$_4$(1, 2, 3, 4, 5)	$(1-p)^3+(1-p)^2-1 \leqq 0 \leqq$ $\leqq 2(1-p)^2-1$	$2+10p-7p^2+7p^3-4p^4+p^5$
		D3—Q$_2$(4, 5) | D1—D2—O3—Q$_2$(4,5) | | | O2—Q$_3$(3, 4, 5) | O1—Q$_4$(2, 3, 4, 5)	Q$_5$(1, 2, 3, 4, 5)	$(1-p)^4+(1-p)^2-1 \leqq 0 \leqq$ $\leqq (1-p)^3+(1-p)^2-1$	$2+9p-p^2-2p^3+p^4$
		D3—Q$_2$(4, 5) | D1—D2—O3—Q$_2$(4,5) | | | O2—Q$_3$(3, 4, 5) | O1—Q$_5$(2, 3, 4, 5)	Q$_6$(1, 2, 3, 4, 5)	$2(1-p)^5-(1-p)^3-p \leqq 0 \leqq$ $\leqq (1-p)^4+(1-p)^2-1$	$2+8p+5p^2-9p^3+5p^4-p^5$
		D5—• | P$_6$(1, 2, 3, 4, 5)	Q$_7$(1, 2, 3, 4, 5)	$0 \leqq 2(1-p)^5-(1-p)^3-p$	$1+16p-12p^2+10p^3-5p^4+p^5$
6	P(1, 2, 3, 4, 5, 6)	D1—P$_1$(2, 3, 4, 5, 6) | O1—Q$_1$(2, 3, 4, 5, 6)	P$_1$(1, 2, 3, 4, 5, 6)	$(1-p)^2-p \leqq 0$	$\frac{1}{1-(1-p)^6}[35p-85p^2+$ $+110p^3-80p^4+31p^5-5p^6]$

		D1—**P**$_2$(2, 3, 4, 5, 6) | **O**1—**Q**$_2$(2, 3, 4, 5, 6)	**P**$_2$(1, 2, 3, 4, 5, 6)	$(1-p)^4 - p^2 \leqq 0 \leqq (1-p)^2 - p$	$\frac{1}{1-(1-p)^6}[30p - 67p^2 + 95p^3 - 74p^4 + 30p^5 - 5p^6]$
		D1—**P**$_3$(2, 3, 4, 5, 6) | **O**1—**Q**$_3$(2, 3, 4, 5, 6)	**P**$_3$(1, 2, 3, 4, 5, 6)	$(1-p)^5 + (1-p)^4 + (1-p)^3 + (1-p)^2 - 1 \leqq 0 \leqq (1-p)^4 - p^2$	$\frac{1}{1-(1-p)^6}[29p - 63p^2 + 90p^3 - 70p^4 + 29p^5 - 5p^6]$
		D2—**P**$_2$(3, 4, 5, 6) | **D**1—**O**2—**Q**$_3$(3, 4, 5, 6) | **O**1—**Q**$_3$(2, 3, 4, 5, 6)	**P**$_4$(1, 2, 3, 4, 5, 6)	$2(1-p)^2 - 1 \leqq 0 \leqq (1-p)^5 + (1-p)^4 + (1-p)^3 + (1-p)^2 - p$	$\frac{1}{1-(1-p)^6}[26p - 49p^2 + 70p^3 - 55p^4 + 23p^5 - 4p^6]$
		D2—**P**$_3$(3, 4, 5, 6) | **D**1—**O**2—**Q**$_4$(3, 4, 5, 6) | **O**1—**Q**$_4$(2, 3, 4, 5, 6)	**P**$_5$(1, 2, 3, 4, 5, 6)	$(1-p)^3 + (1-p)^2 - 1 \leqq 0 \leqq 2(1-p)^2 - 1$	$\frac{1}{1-(1-p)^6}[24p - 37p^2 + 47p^3 - 34p^4 + 13p^5 - 2p^6]$
		D2—**P**$_4$(3, 4, 5, 6) | **D**1—**O**2—**Q**$_4$(3, 4, 5, 6) | **O**1—**Q**$_5$(2, 3, 4, 5, 6)	**P**$_6$(1, 2, 3, 4, 5, 6)	$(1-p)^4 + (1-p)^2 - 1 \leqq 0 \leqq (1-p)^3 + (1-p)^2 - 1$	$\frac{1}{1-(1-p)^6}[23p - 31p^2 + 38p^3 - 29p^4 + 12p^5 - 2p^6]$
		D2—**P**$_4$(3, 4, 5, 6) | **D**1—**O**2—**Q**$_5$(3, 4, 5, 6) | **O**1—**Q**$_6$(2, 3, 4, 5, 6)	**P**$_7$(1, 2, 3, 4, 5, 6)	$2(1-p)^5 - (1-p)^3 - p \leqq 0 \leqq (1-p)^4 + (1-p)^2 - 1$	$\frac{1}{1-(1-p)^6}[22p - 25p^2 + 31p^3 - 25p^4 + 11p^5 - 2p^6]$
		D2—**P**$_4$(3, 4, 5, 6) | **D**1—**O**2—**Q**$_5$(3, 4, 5, 6) | **O**1—**Q**$_7$(2, 3, 4, 5, 6)	**P**$_8$(1, 2, 3, 4, 5, 6)	$0 \leqq 2(1-p)^5 - (1-p)^3 - p$	$\frac{1}{1-(1-p)^6}[21p - 17p^2 + 14p^3 - 6p^4 + p^5]$

$$M_{2,1} = [1 + N(\mathbf{Q}(3, \ldots, k))]\,(1 - p)^2 + [2 + N(\mathbf{Q}(3, \ldots, k))]\,p(1 - p) + $$
$$+ [2 + N(\mathbf{Q}(2, \ldots, k))]\,p\,,$$
$$\vdots$$
$$M_{k,1} = [(1 - p) + kp]\,(1 - p)^{k-1} + [k + N(\mathbf{Q}(k))]\,p(1 - p)^{k-2} + \ldots +$$
$$+ [3 + N(\mathbf{Q}(3, \ldots, k))]\,p(1 - p) + [2 + N(\mathbf{Q}(2, \ldots, k))]\,p\,,$$
$$\vdots$$
$$M_{k,m_k} = [1 + p]\,(1 - p)^{k-1} + [3 + N(\mathbf{Q}(k))]\,p(1 - p)^{k-2} + \ldots +$$
$$+ \ldots + [k + N(\mathbf{Q}(3, \ldots, k))]\,p(1 - p) +$$
$$+ [k + N(\mathbf{Q}(2, \ldots, k))]\,p\,.$$

These equations imply the formula

$$N(\mathbf{Q}(1, 2, \ldots, k)) = \min_{\substack{1 \leqq i \leqq k \\ 1 \leqq j \leqq m_i}} M_{i,j}\,.$$

The above formulae can be used for determining the optimum procedures giving the minimal costs $N(\mathbf{P}(1, 2, \ldots, n))$ and $N(\mathbf{Q}(1, 2, \ldots, n))$ for any n. The calculation of these optimum procedures is not difficult but is tedious and therefore it is omitted here and final expressions are shown only. The following Tab. 5 presents the optimum procedures for particular n. For these procedures, the range of their optimality in terms of p and corresponding expected costs are tabulated, and presented in graphical form in Fig. 8, 9 and 10.

As we have mentioned at the beginning of this section, it is not possible to obtain the optimum procedure for this type of fault-finding by application information

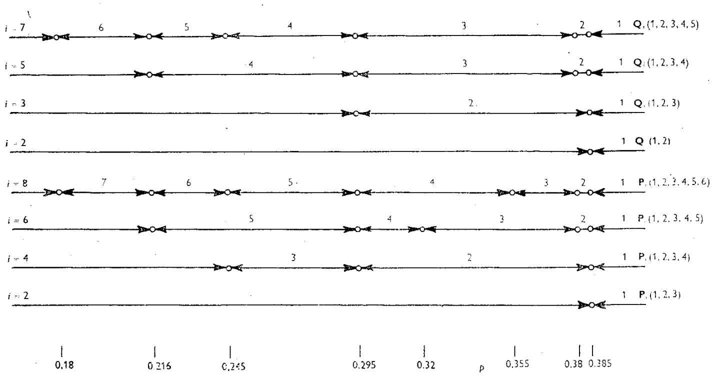


Fig. 8. Optimum procedure areas for different p in the type (i).

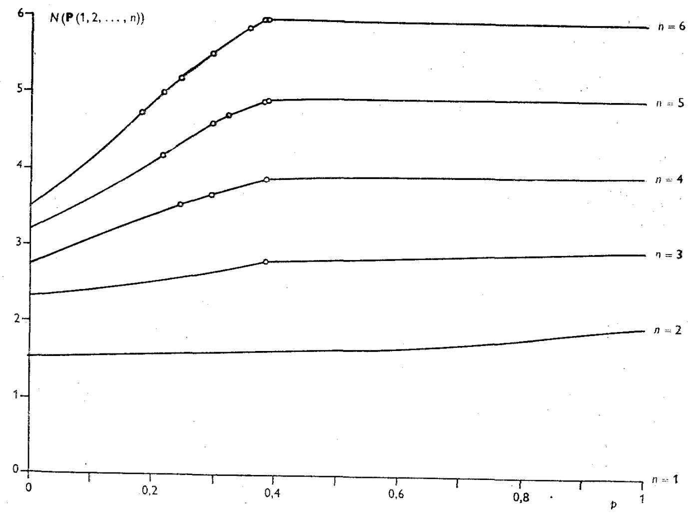


Fig. 9. Minimum expected costs for **P**(1, 2, 3, ..., n) in the type (i).

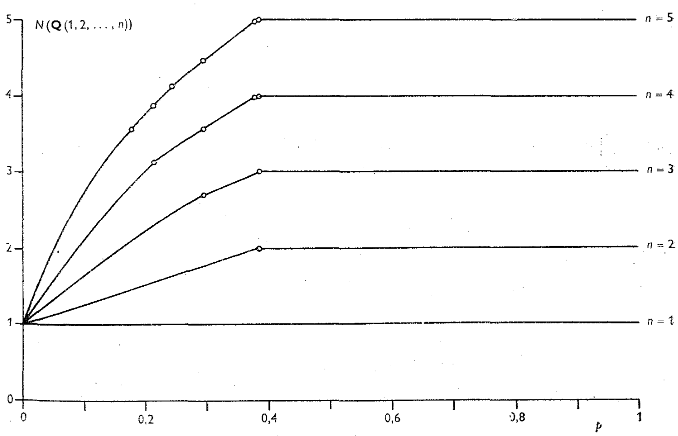


Fig. 10. Minimum expected costs for **Q**(1, 2, 3, ..., n) in the type (i).

theory ideas as the following example shows. Let $n = 6$ and $p = 0{\cdot}1$. The procedure can be started by measurement of one of random variables $\eta_1, \eta_2, \eta_3, \eta_4$ or η_5. We shall choose such a variable η_i for which $H(\eta_i \mid \eta_6 = 1)$ is maximum. It can be shown that the conditional entropy is maximal for such a random variable η_i for which the absolute value of difference $|P(\eta_i = 0 \mid \eta_6 = 1) - P(\eta_i = 1 \mid \eta_6 = 1)|$ is minimum.

Obviously

$$P(\eta_1 = 0 \mid \eta_6 = 1) = \frac{(1-p)[1-(1-p)^5]}{1-(1-p)^6},$$

$$P(\eta_1 = 1 \mid \eta_6 = 1) = \frac{p}{1-(1-p)^6},$$

$$P(\eta_2 = 0 \mid \eta_6 = 1) = \frac{(1-p)^2[1-(1-p)^4]}{1-(1-p)^6},$$

$$P(\eta_2 = 1 \mid \eta_6 = 1) = \frac{1-(1-p)^2}{1-(1-p)^6},$$

$$P(\eta_3 = 0 \mid \eta_6 = 1) = \frac{(1-p)^3[1-(1-p)^3]}{1-(1-p)^6},$$

$$P(\eta_3 = 1 \mid \eta_6 = 1) = \frac{1-(1-p)^3}{1-(1-p)^6},$$

$$P(\eta_4 = 0 \mid \eta_6 = 1) = \frac{(1-p)^4[1-(1-p)^2]}{1-(1-p)^6},$$

$$P(\eta_4 = 1 \mid \eta_6 = 1) = \frac{1-(1-p)^4}{1-(1-p)^6},$$

$$P(\eta_5 = 0 \mid \eta_6 = 1) = \frac{(1-p)^5[1-(1-p)]}{1-(1-p)^6},$$

$$P(\eta_5 = 1 \mid \eta_6 = 1) = \frac{1-(1-p)^5}{1-(1-p)^6},$$

and therefore

$$\Delta_1 = |P(\eta_1 = 0 \mid \eta_6 = 1) - P(\eta_1 = 1 \mid \eta_6 = 1|) = \left| 1 - \frac{2p}{1-(1-p)^6} \right|,$$

$$\Delta_2 = |P(\eta_2 = 0 \mid \eta_6 = 1) - P(\eta_2 = 1 \mid \eta_6 = 1)| = \left| 1 - \frac{2[1-(1-p)^2]}{1-(1-p)^6} \right|,$$

$$\Delta_3 = |P(\eta_3 = 0 \mid \eta_6 = 1) - P(\eta_3 = 1 \mid \eta_6 = 1)| = \left| 1 - \frac{2[1-(1-p)^3]}{1-(1-p)^6} \right|,$$

$$\Delta_4 = |P(\eta_4 = 0 \mid \eta_6 = 1) - P(\eta_4 = 1 \mid \eta_6 = 1)| = \left| 1 - \frac{2[1 - (1 - p)^4]}{1 - (1 - p)^6} \right| ,$$

$$\Delta_5 = |P(\eta_5 = 0 \mid \eta_6 = 1) - P(\eta_5 = 1 \mid \eta_6 = 1)| = \left| 1 - \frac{2[1 - (1 - p)^5]}{1 - (1 - p)^6} \right| .$$

For $p = 0{\cdot}1$ we obtain

$$\Delta_1 = 5{\cdot}73159 ;$$
$$\Delta_2 = 1{\cdot}89003 ;$$
$$\Delta_3 = 1{\cdot}56738 ;$$
$$\Delta_4 = 4{\cdot}67905 ;$$
$$\Delta_5 = 7{\cdot}47955 .$$

This leads to starting the procedure by measurement of η_3, but the optimum procedure starts by measurement of η_2 because $0 < 2 \,.\, 9^5/10^5 - 9^3/10^3 - 1/10$, i.e. the optimum procedure is $\mathbf{P}_8(1, 2, 3, 4, 5, 6)$ as shown in Tab. 5.

ELEMENT-MEASUREMENT PROCEDURE

In this section the general solution of the second type of fault-finding procedures will be derived. This type of procedures is based on the assumption that the measurement of any element of the system determines whether this element is good or defective. The optimum procedure will be derived for the assumption that every element found to be defective is repaire dimmediately, and this step is followed by the checking of the entire system.

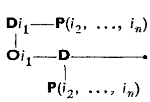


Fig. 11. A possible procedure in the type (ii).

Every procedure is characterized by a sequence of indices $i_1, i_2, \ldots, i_n$ in the formal description of the procedure shown Fig. 11.

Let the elements of the system be numbered by $1, 2, \ldots, n$ in such a way that $p_1 \leqq p_2 \leqq \ldots \leqq p_n$. Let $\delta = 1$, and $d < 1 - p_1$. Then the optimum procedure is characterized by the sequence $n, n - 1, n - 2, \ldots, 3, 1, 2$ as theorem 1 shows.

Let us denote the expected cost of the procedure characterized by the sequence $i_1, i_2, \ldots, i_n$ by $\mathcal{N}(\mathbf{P}(i_1, \ldots, i_n))$.

According to Fig. 8 we can write

$$\mathcal{N}(\mathbf{P}(i_1, \ldots, i_n)) = P(\xi_{i_1} = 0 \mid \eta = 1) [1 + \mathcal{N}(\mathbf{P}(i_2, \ldots, i_n))] +$$
$$+ (d + 1) P(\xi_{i_1} = 1, \; \xi_{i_j} = 0, \;\; j = 2, \ldots, n \mid \eta = 1) +$$
$$+ [1 + d + \mathcal{N}(\mathbf{P}(i_2, \ldots, i_n))] P(\xi_{i_1} = 1, \max_{2 \leqq j \leqq n} \xi_{i_j} = 1 \mid \eta = 1) .$$

 By substituion of terms

$$P(\xi_{i_1}=0\mid\eta=1)=\frac{(1-p_{i_1})\left[1-\prod_{j=2}^{n}(1-p_{i_j})\right]}{1-\prod_{j=1}^{n}(1-p_{i_j})},$$

$$P(\xi_{i_1}=1,\max_{2\leq j\leq n}\xi_{i_j}=0\mid\eta=1)=\frac{p_{i_1}\prod_{j=2}^{n}(1-p_{i_j})}{1-\prod_{j=1}^{n}(1-p_{i_j})},$$

$$P(\xi_{i_1}=1,\max_{2\leq j\leq n}\xi_{i_j}=1\mid\eta=1)=\frac{p_{i_1}\left[1-\prod_{j=2}^{n}(1-p_{i_j})\right]}{1-\prod_{j=1}^{n}(1-p_{i_j})}$$

in the above formula we obtain after simple calculations

$$\mathcal{N}(\mathbf{P}(i_1,\dots,i_n))=\left[1-\prod_{j=2}^{n}(1-p_{i_j})\right]\mathcal{N}(\mathbf{P}(i_2,\dots,i_n))+dp_{i_1}+$$
$$+\left[1-\prod_{j=1}^{n}(1-p_{i_j})\right].$$

Whence

$$\mathcal{N}(\mathbf{P}(i_1,\dots,i_n))=1+\frac{1}{1-\prod_{j=1}^{n}(1-p_{i_j})}\sum_{j=1}^{n-1}\left(dp_{i_j}+1-\prod_{k=j}^{n}(1-p_{i_k})\right)=$$
$$=1+\frac{1}{1-\prod_{j=1}^{n}(1-p_{i_j})}S(i_1,\dots,i_n),$$

where

$$S(i_1,i_2,\dots,i_n)=\sum_{j=1}^{n-1}(dp_{i_j}+1-\prod_{k=j}^{n}(1-p_{i_k})).$$

Now we can formulate the following lemma:

Lemma 1. *If for any two indices i_s and i_r $(s<r,\ r=2,3,\dots,n-1)$ the inequality*

$$p_{i_s}\leqq p_{i_r}$$

holds, then

$$S(i_1,\dots,i_s,i_{s+1},\dots,i_{r-1},i_r,i_{r+1},\dots,i_n)\geqq$$
$$\geqq S(i_1,\dots,i_r,i_{s+1},\dots,i_{r-1},i_s,i_{r+1},\dots,i_n).$$

Proof. According to the definition of the function $S(i_1, \ldots, i_n)$ we obtain

$$S(i_1, \ldots, i_s, i_{s+1}, \ldots, i_{r-1}, i_r, i_{r+1}, \ldots, i_n) -$$
$$- S(i_1, \ldots, i_r, i_{s+1}, \ldots, i_{r-1}, i_s, i_{r+1}, \ldots, i_n) =$$
$$= (p_{i_r} - p_{i_s}) [(1 - p_{i_{s+1}}) \ldots (1 - p_{i_{r-1}}) (1 - p_{i_{r+1}}) \ldots (1 - p_{i_n}) +$$
$$+ (1 - p_{i_{s+2}}) \ldots (1 - p_{i_{r-1}}) (1 - p_{i_{r+1}}) \ldots (1 - p_{i_n}) + \ldots +$$
$$+ (1 - p_{i_{r-1}}) (1 - p_{i_{r+1}}) \ldots (1 - p_{i_n}) + (1 - p_{i_{r+1}}) \ldots (1 - p_{i_n})]$$

and this expression is non-negative because $p_{i_r} \geqq p_{i_s}$.

Theorem 1. *If*

$$p_1 \leqq p_2 \leqq \ldots \leqq p_n$$

then

$$N(\mathbf{P}(1, 2, \ldots, n)) = \mathcal{N}(\mathbf{P}(n, n-1, \ldots, 3, 1, 2)) .$$

Proof. Using the Lemma 1 we obtain for any $j = 1, 2, \ldots, n$ and for any sequence of indices $i_1, i_2, \ldots, i_n$ where $i_n = j$

$$S(n, n-1, \ldots, j+1, j-1, \ldots, 2, 1, j) \leqq S(i_1, i_2, \ldots, i_n)$$

and moreover

$$S(n, n-1, \ldots, 3, 1, 2) \leqq S(n, n-1, \ldots, j+1, j-1, \ldots, 1, j)$$

and therefore $N(\mathbf{P}(1, 2, \ldots, n)) = \mathcal{N}(\mathbf{P}(n, n-1, \ldots, 3, 1, 2))$.

The optimum of the element-measurement procedures $\mathbf{P}(1, 2, \ldots, n)$ is given in Fig. 12.

THE REPLACEMENT-OF-ELEMENT PROCEDURE

The determination of the optimum procedure of the third type of fault-finding procedures, i.e. by replacement of the elements of the system, is very tedious. In both preceding types, any procedure terminates by the step when the system is reoperating, while in this third type of procedures the checking follows, by which we make sure of the necessity of any replacement. The replacement of a good element of the system by a spare part would lead to inadmissible losses.

The general case of the system containing n elements seems to be solved only by the determination of all possible procedures, by the calculation their expected costs for known values of $p_1, p_2, \ldots, p_n$, and by the choice of the optimum according to expected costs. The most tedious step is the determination of all possible methods, because their number grows very rapidly with increasing n. Therefore in this paper we will discuss the case $n = 3$ only, which illustrates quite well the methods.

We suppose the cost of replacement or re-replacement of any element to be $\delta = 1$. In the determination of all possible procedures, the following symbols will be used

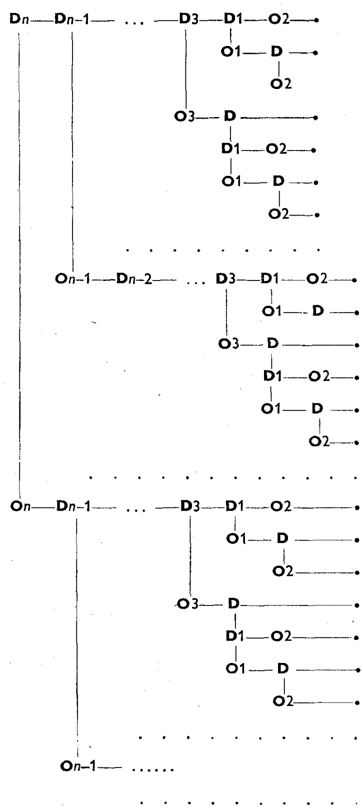


Fig. 12. The optimum procedure in the type (ii).

in the formal descriptions:

$$\mathbf{Q}(i) \equiv \text{Is the } i\text{-th element good?}$$

$$\mathbf{R}(i) \equiv \text{Is only the } i\text{-th element defective?}$$

The positive answer is symbolized by the rightward continuation in the formal description, and the negative answer by the downward continuation. The double dash represents the common continuation after both answers. The full description of symbols $\mathbf{Q}(i)$ and $\mathbf{R}(i)$ is shown in fig. 13.

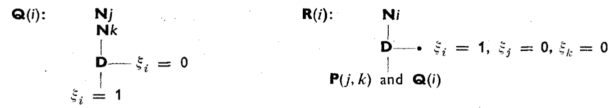


Fig. 13. The full description of symbols **Q**(i) and **R**(i) in the type (iii).

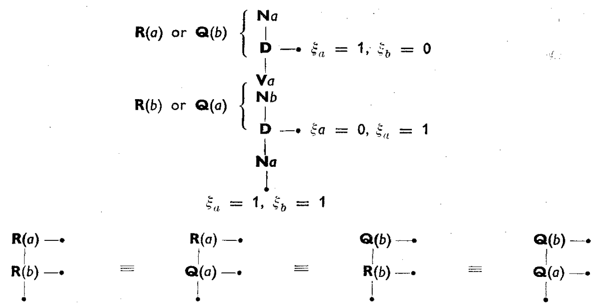


Fig. 14. The full and shortened description of one procedure **P**(a, b) in the type (iii).

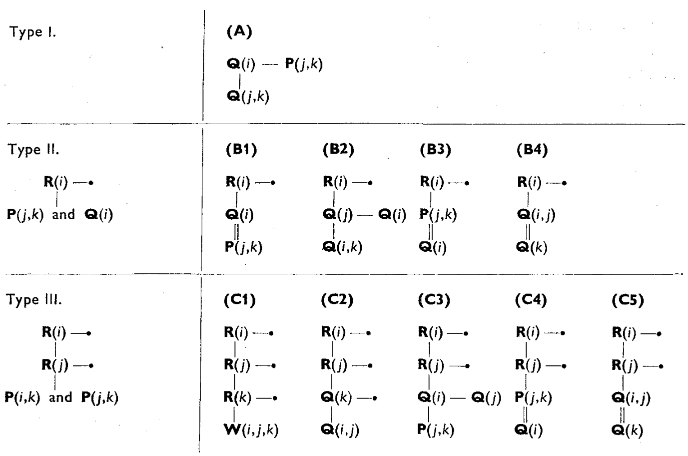


Fig. 15. All possible procedures **P**(i, j, k) in the type (iii). In the first step of the procedure of the type I stays **Q**(i), whereas two other types start with the step **R**(i). The second step is not **R**(j) in the type II, and is **R**(j) in the type III.

The sense of symbols $\mathbf{P}(i, j)$ and $\mathbf{Q}(i, j)$ is the same as in previous sections. Another symbol will be used, too:

$$\mathbf{W}(i, j, k) \equiv \text{Are two and only two of three elements defective?}$$

By use of symbols $\mathbf{Q}(i)$ and $\mathbf{R}(i)$ we can obtain four versions of the description of $\mathbf{P}(a, b)$ procedure, which was discussed already in the introductory section. Its full description and all variants of shortened description are shown in Fig. 14.

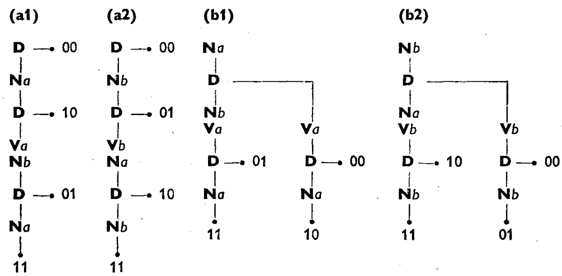


Fig. 16. All possible procedures $\mathbf{Q}(a, b)$ in the type (iii).

All possible procedures for $\mathbf{P}(i, j, k)$ are described in Fig. 15. Since for $\mathbf{P}(a, b)$ there are two possible procedures (see the introductory section), for $\mathbf{Q}(a, b)$ four possible procedures (see Fig. 16; the figures 0 or 1 at the end points of procedures represent realization of random variables ξ_a and ξ_b; the same holds for subsequent schemes, too), and for $\mathbf{W}(i, j, k)$ six possible procedures (all permutations of elements i, j, k according to Fig. 17), and since there are six permutations of three elements i, j, k, the symbolic description in Fig. 15 represents 228 procedures.

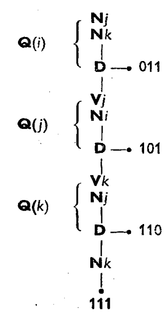


Fig. 17. One possible realization of the procedure $\mathbf{W}(i, j, k)$ in the type (iii).

But for three elements numbered in such a way that $p_1 \leqq p_2 \leqq p_3$ one of two possible procedures for $\mathbf{P}(a, b)$ dominates the other (see the introductory section), and similarly, as shows Tab. 6, one of two procedures for $\mathbf{Q}(a, b)$ of the type **(a)** dominates the other whereas two remaining procedures of the type **(b)** are equivalent. Thus the number of the good procedures diminishes.

Moreover, procedures of type **(B3)** are contained in the type **(B2)**; **(B4)** in **(B1)** and **(B2)** or in the type of **(C)**; **(C3)** in **(C5)**; **(C4)** in **(C2)** and **(C5)**; and the procedures

Table 6.

Costs of all possible procedures $\mathbf{Q}(a, b)$ in the type (iii)

ξ_a, ξ_b	Procedures			
	(a1)	**(a2)**	**(b1)**	**(b2)**
0 0	d	d	$2 + 2d$	$2 + 2d$
0 1	$3 + 3d$	$1 + 2d$	$3 + 2d$	$3 + 2d$
1 0	$1 + 2d$	$3 + 3d$	$3 + 2d$	$3 + 2d$
1 1	$4 + 3d$	$4 + 3d$	$4 + 2d$	$4 + 2d$

of the type **(C5)** using the procedure of $\mathbf{Q}(i, j)$ of the type **(a)** are contained in the type **(C1)**.

Let us denote the procedures using the procedure of $\mathbf{Q}(i, j)$ of the type **(a)** or **(b)** by letter **a** or **b**, respectively. Thus the remaining procedures can be listed as follows:

(Aa), (Ab),

(B1), (B2a), (B2b),

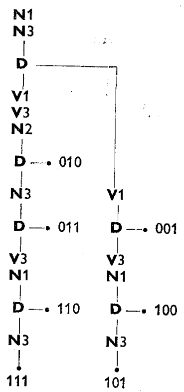
(C1α), (C1β), (C2b), (C5b).

Here **(C1α)** is such a procedure where the third step (say $\mathbf{R}(k)$), i.e the question "Is only this one element defective?", is immediately followed by the fourth step of the type "Is the same element defective?" (i.e. $\mathbf{Q}(k)$), whereas in the procedure **(C1β)** after the same third question as in the previous case the fourth step is "Is another element defective?" (i.e. $\mathbf{Q}(k)$ does not follow immediately after $\mathbf{R}(k)$).

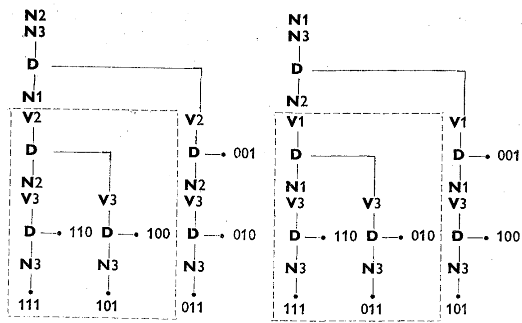
For elements numbered in such a way that $p_1 \leqq p_2 \leqq p_3$ for every general procedure of our list, the optimum permutation of elements i, j, k minimizing the expected cost for this procedure can be found. Thus we obtain 10 good procedures, described in Fig. 18, and the expected costs of which are given by the formula

$$\mathcal{N}(\mathbf{P}(1, 2, 3)) = \frac{1}{1 - (1 - p_1)(1 - p_2)(1 - p_3)} \{k_{001}(1 - p_1)(1 - p_2) p_3 + \\ + k_{010}(1 - p_1) p_2(1 - p_3) + k_{100} p_1(1 - p_2)(1 - p_3) + k_{011}(1 - p_1) p_2 p_3 + \\ + k_{101} p_1(1 - p_2) p_3 + k_{110} p_1 p_2(1 - p_3) + k_{111} p_1 p_2 p_3\},$$

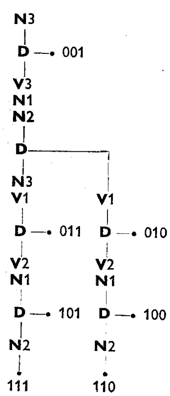
(Aa)
[2,1,3] ≡ [2,3,1]



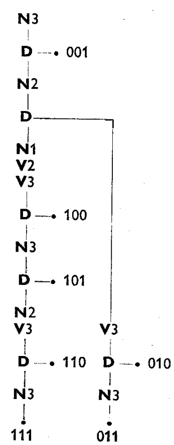
(Ab) (two equivalent procedures)
[1,2,3] ≡ [1,3,2]
[2,1,3] ≡ [2,3,1]



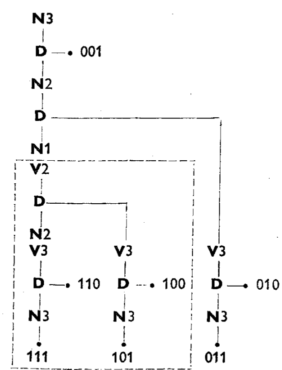
(B1)
[3,1,2] ≡ [3,2,1]



(B2a)
[3,1,2]



(B2b)
[3,1,2]



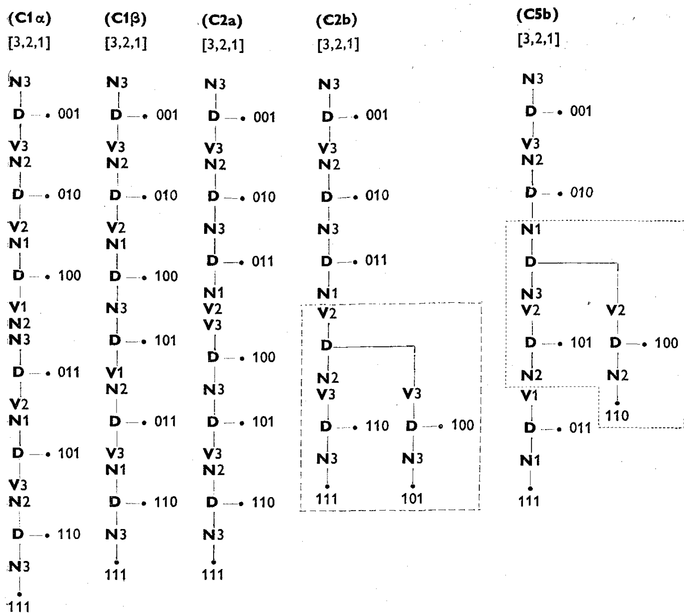


Fig. 18. The good procedures **P**(1, 2, 3) in the type (*iii*) for the case $p_1 = p_2 = p_3$. The dashed boxes contain the procedure **Q**(2, 3) (or **Q**(1, 3)) of the type **(b)**, whose second equivalent form can be obtained by mutual change of figures 2 and 3 (or 1 and 3 respectively) in the dashed box. The dotted box in the procedure **(C5b)** is **Q**(2, 3), too, but in this case the second eventuality would increase the cost of the entire procedure and therefore it is inadmissible. The permutation of the elements i, j, k of the general procedure (see Fig. 15) leading to the good procedure is written is square brackets.

where coefficients $k_{001}, k_{010}, \ldots, k_{111}$ are listed in Tab. 7. The optimum procedure for given p_1, p_2, p_3 and d can be now easily determined. It is that one for which the expected cos $\mathcal{N}(\mathbf{P}(1, 2, 3))$ is minimum of all 10 procedures, i.e. its value is $N(\mathbf{P}(1, 2, 3))$.

This method for determining the optimum procedure can be applied for any number n of elements of the system, but for greater values of n the aid of a computer is necessary.

Let us suppose the case $p_1 = p_2 = p_3 = p$ for the fault-finding procedure by replacement-of-elements. For 10 procedures from Fig. 18 we obtain the expected

Table 7.

Coefficients for the calculation of $\mathcal{N}(\mathbf{P}(1, 2, 3))$ for $p_1 \leqq p_2 \leqq p_3$ in the type (iii)

Procedure	k_{001}	k_{010}	k_{011}	k_{100}	k_{101}	k_{110}	k_{111}
(Aa)	$3 + 2d$	$5 + 2d$	$6 + 3d$	$5 + 3d$	$6 + 3d$	$8 + 4d$	$9 + 4d$
(Ab)	$3 + 2d$	$5 + 3d$	$6 + 3d$	$5 + 3d$	$6 + 3d$	$6 + 3d$	$7 + 3d$
(B1)	$1 + d$	$5 + 3d$	$6 + 3d$	$7 + 4d$	$8 + 4d$	$8 + 4d$	$9 + 4d$
(B2a)	$1 + d$	$3 + 3d$	$4 + 3d$	$5 + 3d$	$6 + 4d$	$8 + 5d$	$9 + 5d$
(B2b)	$1 + d$	$3 + 3d$	$4 + 3d$	$5 + 4d$	$6 + 4d$	$6 + 4d$	$7 + 4d$
(C1α)	$1 + d$	$3 + 2d$	$8 + 4d$	$5 + 3d$	$10 + 5d$	$12 + 6d$	$13 + 6d$
(C1β)	$1 + d$	$3 + 2d$	$8 + 5d$	$5 + 3d$	$6 + 4d$	$10 + 6d$	$11 + 6d$
(C2a)	$1 + d$	$3 + 2d$	$4 + 3d$	$7 + 4d$	$8 + 5d$	$10 + 6d$	$11 + 6d$
(C2b)	$1 + d$	$3 + 2d$	$4 + 3d$	$7 + 5d$	$8 + 5d$	$8 + 5d$	$9 + 5d$
(C5b)	$1 + d$	$3 + 2d$	$8 + 5d$	$5 + 4d$	$6 + 4d$	$6 + 4d$	$9 + 4d$

Table 8.

Coefficients for the calculation of $\mathcal{N}(\mathbf{P}(1, 2, 3))$ for $p_1 = p_2 = p_3$ in the type (iii)

Procedure	k_1	k_2	k_3
(Aa)	$13 + 7d$	$20 + 10d$	$9 + 4d$
(Ab)	$13 + 8d$	$18 + 9d$	$7 + 3d$
(B1)	$13 + 8d$	$22 + 11d$	$9 + 4d$
(B2a)	$9 + 7d$	$18 + 12d$	$9 + 5d$
(B2b)	$9 + 8d$	$18 + 11d$	$7 + 4d$
(C1α)	$9 + 6d$	$30 + 15d$	$13 + 6d$
(C1β)	$9 + 6d$	$24 + 15d$	$11 + 6d$
(C2a)	$11 + 7d$	$22 + 14d$	$11 + 6d$
(C2b)	$11 + 8d$	$20 + 13d$	$9 + 5d$
(C5b)	$9 + 7d$	$20 + 13d$	$9 + 4d$

costs by the formula

$$\mathcal{N}(\mathbf{P}(1, 2, 3)) = \frac{1}{1 - (1 - p)^3} \{k_1(1 - p)^2 p + k_2(1 - p) p^2 + k_3 p^3\},$$

where the coefficients k_1, k_2 and k_3 can be found in Tab. 8 for particular procedures, This table shows that procedure **(B1)** is worse than **(Ab)**, **(C1α)** is worse than **(C1β)**.

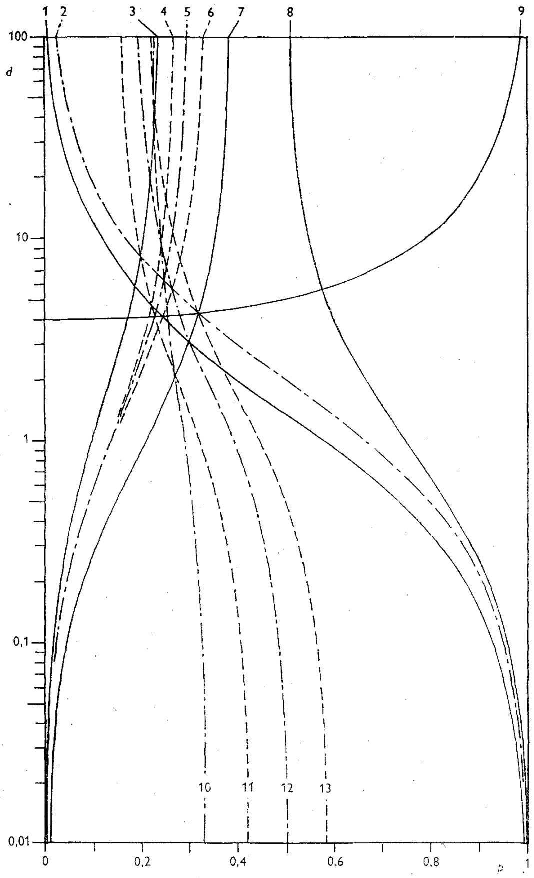


Fig. 19a. Optimum border-lines for every two good procedures **P**(1, 2, 3) in the type (iii). (Border line 1: in the left area the procedure **(C5b)** is better than **(Aa)**; 2: **(B2a)**—**(Aa)** and **(B2b)**—**(Ab)**; 3: **(C1β)**—**(B2a)**; 4: **(C1β)**—**(C5b)**; 5: **(C1β)**—**(B2b)**; 7: **(Aa)**—**(Ab)** and **(B2a)**—**(B2b)**; 8: **(B2a)**—**(C5b)**; 9: **(Aa)**—**(B2b)**; 10: **(C1β)**—**(Ab)**; 11: **(C1β)**—**(Aa)**; 12: **(C5b)**—**(Ab)**; 13: **(B2a)**—**(Ab)**.)

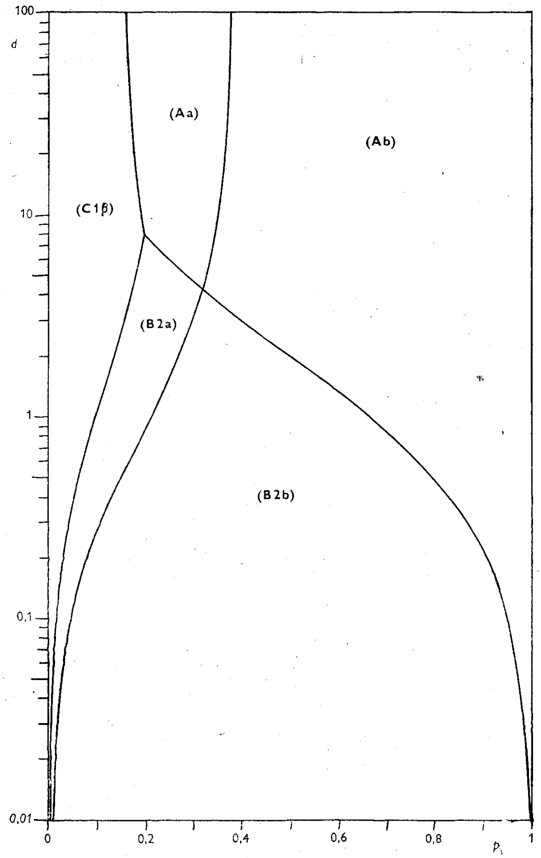


Fig. 19b. Optimum areas for the procedures **P**(1, 2, 3) in the type (iii).

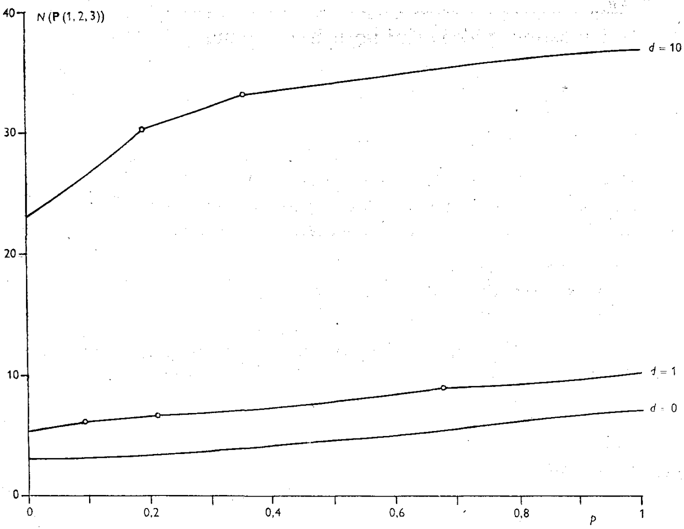


Fig. 20a. Minimum expected costs of procedures **P**(1, 2, 3) in the type (iii).

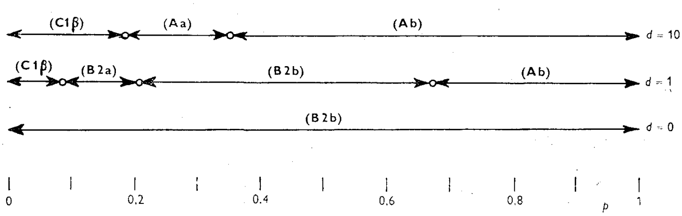


Fig. 20b. Optimum procedure areas in the type (iii).

and **(C2a)** and **(C2b)** are worse than **(C5b)**. The worse procedures can be thus omitted and for remaining six procedures the optimum one for given p and d can be determined from Fig. 19, which shows the areas in the plane (p, d), where individual procedures are optimum. (The procedure **(C5b)** is worse than **(Ab)** for $p \leqq 1/2$ for any d, and worse than **(B2a)** for $p \geqq 1/2$ for any d.) Fig. 20 shows the expected costs for the optimum procedures for several values of d.

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Některé způsoby vyhledávání poruch v systému

Libor Kubát, Milan Ullrich

V článku se probírají metodami teorie pravděpodobnosti tři různé způsoby vyhledávání poruch v systému a stanoví se takové postupy, které v průměru vyžadují minimální náklady. Uvažované způsoby vyhledávání poruch jsou:

(i) metoda měření signálu,
(ii) metoda měření prvků,
(iii) metoda nahrazování prvků.

Při všech metodách se předpokládá, že jsou známy pravděpodobnosti toho, že jednotlivý prvek systému bude vadný, a že náklady jsou způsobeny pouze zjišťováním vadných prvků, nikoliv jejich opravou.

Metoda (i) je řešena pro n sériově uspořádaných stejných prvků a jsou uvedeny tabulky optimálních postupů až pro $n = 8$. Rovněž je ukázán obecný algoritmus pro určení optimálního postupu.

Metoda (ii) je řešena zcela obecně a je dán obecný algoritmus optimálního postupu.

Metoda (iii) je vzhledem k rozsáhlosti řešení demonstrována pouze pro případ $n = 3$ a je ukázán postup, který by bylo možno analogicky použít i pro větší n, přestože vede k velmi zdlouhavým výpočtům.

Na rozdíl od metody (i), která je řešena pro sériové uspořádání prvků, platí odvozené výsledky pro metody (ii) a (iii) pro libovolnou konfiguraci prvků.

Inž. Libor Kubát, CSc., Inž. Milan Ullrich, CSc., Ústav teorie informace a automatizace ČSAV, Vyšehradská 49, Praha 2.