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Miquel Salicrú

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MEASURES OF INFORMATION ASSOCIATED WITH CSISZÁR'S DIVERGENCES

MIQUEL SALICRÚ

This paper reviews and completes the relationships between the measures of information associated with divergences, attending, in the multivariate case, to disturbances of the parameter in the directions of the coordinate axis and considering the matrix which defines the metric in direction to tangent space.

1. INTRODUCTION

An interesting problem, which is set by information theory, arises from the need to obtain and to select properly the informative measures. In this way, the different functionals, which have been proposed as a measure for the information, can either come from heuristic considerations, and are therefore subject to being applied and interpreted, or from theoretic considerations, founded on good algebraic and analytical properties.

In the present paper, the information measures associated to Csiszár's divergencies for the univariate case are revised and completed (Section 3), and these are generalized to the multivariate case (Section 5), due to disturbances of the parameter in the directions of the coordinate axis. For the multivariate case, Rényi's information-matrix is retrieved, as an application for the general result.

An alternative way of obtaining informative matrices is also presented through several considerations of the differential metric in the direction of the tangent space, and this for a prefixed distance measure (Section 4).

2. PREVIOUS DEFINITIONS

For a measureable space (χ, Ξ), and for a family of probability distributions

$$\{F_\theta : \theta \in \Theta, \text{ where } \Theta \text{ is an open set of } \mathbb{R}^k\}$$

which is dominated by the σ-finite, μ-measure defined in Ξ, where $f(x,\theta) = \frac{dF_\theta}{d\mu}$ are the densities. The measures based on the following functionals are considered to be distantiation measures

$$D_\phi^C(f_1,f_2) = \int_S f_1\phi\left[\frac{f_2}{f_1}\right]\,\mathrm{d}\mu \qquad \text{(Csiszár divergence)}$$

$$J_\phi(f_1,f_2) = \int_X \{\lambda\phi(f_1)+(1-\lambda)\phi(f_2)-\phi(\lambda f_1+(1-\lambda)f_2)\}\,\mathrm{d}\mu \qquad (J\text{-divergence})$$

$$M_\phi(f_1,f_2) = \int_X \left[\sqrt{\phi(f_1)}-\sqrt{\phi(f_2)}\right]^2\,\mathrm{d}\mu \qquad (M\text{-divergence})$$

where $S=\{x; f_1(x,\theta)>0\}$ and $\phi(x)$ is a real, convex function, which is three times differentiable with continuity, positive for the M-divergences and with $\phi(1)=0$, for Csiszár's divergences.

With respect to Csiszár's divergences, the following measures are considered in this work.

$$D_X^{KL}(f_1,f_2) = \int f_1\log\frac{f_1}{f_2}\,\mathrm{d}\mu \qquad \text{(Kullback–Leibler)}$$

$$D_X^{KLM}(f_1,f_2) = \int f_2\log\frac{f_2}{f_1}\,\mathrm{d}\mu \qquad \text{(Modified Kullback–Leibler)}$$

$$D_X^{J}(f_1,f_2) = \int \left[f_1\log\frac{f_1}{f_2}+f_2\log\frac{f_2}{f_1}\right]\,\mathrm{d}\mu \qquad \text{(Jeffreys)}$$

$$D_X^{M}(f_1,f_2) = \int \left(\sqrt{f_1}-\sqrt{f_2}\right)^2\,\mathrm{d}\mu \qquad \text{(Matusita)}$$

$$D_X^{Ka}(f_1,f_2) = \int \frac{(f_2-f_1)^2}{f_2}\,\mathrm{d}\mu \qquad \text{(Kagan)}$$

$$D_X^{R}(f_1,f_2) = \frac{1}{\alpha-1}\log\int f_1^\alpha f_2^{1-\alpha}\,\mathrm{d}\mu,\ \alpha>0\ \alpha\neq 1 \qquad \text{(Rényi)}$$

$$D_X^{B}(f_1,f_2) = -\log\int f_1^{1/p} f_2^{1/q}\,\mathrm{d}\mu,\ \frac{1}{p}+\frac{1}{q}=1 \qquad \text{(Bhattacharyya)}$$

and as an information measure, we considered Fisher's measure, defined by

$$I_X^F(\theta)=\begin{cases} \mathsf{E}_\theta\left[\dfrac{\partial}{\partial\theta}\log f(x,\theta)\right]^2 & \text{if } \theta \text{ is univariate}\\ \left(\mathsf{E}_\theta\left[\dfrac{\partial}{\partial\theta_i}\log f(x,\theta)\dfrac{\partial}{\partial\theta_j}\log f(x,\theta)\right]\right)_{k\times k} & \text{if } \theta \text{ is } k\text{-variate}\end{cases}$$

For the whole, we also considered the following regularity conditions:

a) $S_\theta=\{x;\ f(x,\theta)>0\}$ is independent of θ.

b) $\dfrac{\partial f}{\partial\theta_i}$, $\dfrac{\partial^2 f}{\partial\theta_i\partial\theta_j}$ and $\dfrac{\partial^3 f}{\partial\theta_i\partial\theta_j\partial\theta_k}$ exist for every $\theta\in\Theta$.

c) $\int f(x,\theta)\,\mathrm{d}\mu$ is derivable at least twice within the integral sign.

3. MEASURES OF INFORMATION IN THE UNIVARIATE CASE

Some information measures in relation to the groups of probability measures have been obtained from the pre-established distance measures, through the following expression

$$I_X^D(\theta) = \liminf_{t\to 0} \frac{1}{t^2} D\left[f(x,\theta), f(x,\theta+t)\right],$$

where D is the prefixed distance. In this sense, if one takes Csiszár's divergence as a distantiation measure, one obtains:

$$I_X^C(\theta) = \liminf_{t\to 0} \frac{1}{t^2} \int f(x,\theta)\phi\left[\frac{f(x,\theta+t)}{f(x,\theta)}\right] \mathrm{d}\mu. \tag{1}$$

Note that if one defines the function

$$g(t) = \int f(x,\theta)\phi\left[\frac{f(x,\theta+t)}{f(x,\theta)}\right] \mathrm{d}\mu \tag{2}$$

and considers its McLaurin's expression, then

$$g(t) = \frac{g''(0)}{2!} t^2 + o(t^3), \tag{3}$$

where

$$g''(0) = \phi''(1) \int \frac{1}{f} \left[\frac{\partial f}{\partial \theta}\right]^2 \mathrm{d}\mu.$$

In this way the following result is proved.

Theorem 1.

$$I_X^C(\theta) = \frac{\phi''(1)}{2} I_X^F(\theta).$$

If one considers the same procedure in functions of Csiszár's measure, one obtains the following results.

Theorem 2.

$$\begin{aligned} I_X^h(\theta) &= \liminf_{t\to 0} \frac{1}{t^2} h\left\{\int [f(x,\theta)]^\alpha \cdot [f(x,\theta+t)]^{1-\alpha} \,\mathrm{d}\mu\right\} = \\ &= \frac{\alpha(\alpha-1)h'(1)}{2} I_X^F(\theta) \end{aligned}$$

for a differentiable function h in the neighborhood of "1", with $h(1) = 0$.

Proof.

$$\begin{aligned} I_X^h(\theta) &= \liminf_{t\to 0} \frac{1}{t^2} h\left\{1 + \int f(x,\theta)\left[\left(\frac{f(x,\theta+t)}{f(x,\theta)}\right)^{1-\alpha} - 1\right] \mathrm{d}\mu\right\} = \\ &= \liminf_{t\to 0} \frac{1}{t^2} h\{1 + g(t)\} \end{aligned}$$

with $g(t)$ in the form (2), and $\phi(x) = x^{1-\alpha} - 1$.
If we consider development (3), while using l'Hôpital's rule, we can see that

$$\begin{aligned} I_X^h(\theta) &= \liminf_{t\to 0} \frac{h\left[1 + \frac{g''(0)}{2}t^2 + 0(t^3)\right]}{t^2} = \\ &= h'(1)\cdot\frac{g''(0)}{2} = h'(1) I_X^C(\theta) = \frac{h'(1)\cdot\phi''(1)}{2} I_X^F(\theta) = \\ &= \frac{\alpha(\alpha-1)h'(1)}{2} I_X^F(\theta). \end{aligned}$$

□

Corollary 1.

a) For $h(x) = \frac{1}{\alpha - 1}\log x$: $I_X^h(\theta) = \frac{\alpha}{2} I_X^F(\theta)$.

b) For $h(x) = -\log x$, $\alpha = \frac{1}{p}$, $\frac{1}{p} + \frac{1}{q} = 1$: $I_X^h(\theta) = \frac{1}{2pq} I_X^F(\theta)$.

c) For $h(x) = \arcsin(x-1)$: $I_X^h(\theta) = \frac{\alpha(\alpha-1)}{2} I_X^F(\theta)$.

d) For $h(x) = \operatorname{arctg}(x-1)$: $I_X^h(\theta) = \frac{\alpha(\alpha-1)}{2} I_X^F(\theta)$.

Table 1.

NAME	DETERMINING FUNCTION	RELATIONSHIP
Kullback–Leibler	$\phi(x) = -\log x$	$I_X^{KL}(\theta) = \frac{1}{2} I_X^F(\theta)$
Modified Kullback–Leibler	$\phi(x) = x\log x$	$I_X^{KLM}(\theta) = \frac{1}{2} I_X^F(\theta)$
Jeffreys invariant	$\phi(x) = (x-1)\log x$	$I_X^J(\theta) = I_X^F(\theta)$
Matusita	$\phi(x) = \left(1 - \sqrt{x}\right)^2$	$I_X^M(\theta) = \frac{1}{4} I_X^F(\theta)$
Kagan	$\phi(x) = (1-x)^2$	$I_X^{Ka}(\theta) = I_X^F(\theta)$
Rényi	$h(x) = \frac{1}{\alpha-1}\log x$	$I_X^R(\theta) = \frac{\alpha}{2} I_X^F(\theta)$
Bhattacharyya	$\begin{cases} h(x) = -\log x \\ \alpha = \frac{1}{p}, \frac{1}{p} + \frac{1}{q} = 1 \end{cases}$	$I_X^B(\theta) = \frac{1}{2pq} I_X^F(\theta)$

From some particular cases of Csiszár's measure, some relationships between information measures, associated to some Csiszár's divergences and also to Fisher's information measure are shown in table 1. In this sense, some results have been obtained by Kagan [8], Vajda [15], Aggarwal [1], Boekee [3], and Ferentinos and Papaioannou [6], for Csiszár divergences and Rényi distance.

4. MEASURES OF INFORMATION IN THE k-VARIATE CASE: FIRST ALTERNATIVE

As a generalization of the anterior method to the multivariate case, it is possible to consider the matrix, which defines the metric in direction to the tangent space, as an information matrix, associated to a predefined distance. In this sense, and for Csiszár's measure, the line element is defined by the following expression

$$ds^2 = \liminf_{t \to 0} \frac{1}{t^2} \int f(x,\theta)\phi\left(\frac{f(x,\theta) + t df(x,\theta)}{f(x,\theta)}\right) \mathrm{d}\mu. \tag{4}$$

From analogous considerations in the univariate case, we obtain

$$ds^2 = \frac{1}{2}\int \phi''(1) \cdot \frac{1}{f}[df]^2 \, \mathrm{d}\mu$$

and taking into account that $df = \sum_{i=1}^{k} \frac{\partial f}{\partial \theta_i} d\theta_i$, the anterior expression can be reduced to

$$ds^2 = \sum_{i,j=1}^{k} \left[\frac{\phi''(1)}{2} \int \frac{1}{f} \frac{\partial f}{\partial \theta_i} \frac{\partial f}{\partial \theta_j} \mathrm{d}\mu\right] \mathrm{d}\theta_i \, \mathrm{d}\theta_j .$$

In this way, the elements of the matrix, which defines the metric, are determined by

$$I_{ij}^{C-D}(\theta) = \frac{\phi''(1)}{2} I_{ij}^{F}(\theta)$$

and the matrix, which defines the metric, can be reduced to

$$I_X^{C-D}(\theta) = \frac{\phi''(1)}{2} I_X^{F}(\theta).$$

Remark 1. The results obtained for the univariate case are also valid for the multivariate case.

When the prefixed distance does not fit to one of Csiszár's divergences, it is interesting to consider some functionals, which may come from other divergences. In this sense, the functionals, which arise from the J and M divergences, become particularly interesting.

a) For the J-divergence

$$ds^2 = \liminf_{t\to 0} \frac{1}{t^2} \int \{\lambda\phi(f) + (1-\lambda)\phi(f+tdf) - \phi[\lambda f + (1-\lambda)(f+tdf)]\} \, \mathrm{d}\mu =$$
$$= \frac{\lambda(1-\lambda)}{2} \int \phi''(f) [df]^2 \, \mathrm{d}\mu = \sum_{i,j=1}^{k} \left[\frac{\lambda(1-\lambda)}{2} \int \phi''(f) \frac{\partial f}{\partial \theta_i} \frac{\partial f}{\partial \theta_j} \mathrm{d}\mu \right] \mathrm{d}\theta_i \, \mathrm{d}\theta_j$$

and the expression of the matrix, which defines the metric, is reduced to

$$I_X^{J-D}(\theta) = \left(\frac{\lambda(1-\lambda)}{2} \int \phi''(f) \frac{\partial f}{\partial \theta_i} \frac{\partial f}{\partial \theta_j} \mathrm{d}\mu \right)_{k\times k}.$$

b) For the M-divergence

$$\begin{aligned} ds^2 &= \liminf_{t\to 0} \frac{1}{t^2} \int \left[\sqrt{\phi(f)} - \sqrt{\phi(f+tdf)} \right]^2 \mathrm{d}\mu = \\ &= \int \left[\left(\sqrt{\phi(f)} \right)' \right]^2 [df]^2 \, \mathrm{d}\mu = \\ &= \sum_{i,j=1}^{k} \left\{ \int \left[\left(\sqrt{\phi(f)} \right)' \right]^2 \frac{\partial f}{\partial \theta_i} \frac{\partial f}{\partial \theta_j} \mathrm{d}\mu \right\} \mathrm{d}\theta_i \, \mathrm{d}\theta_j \end{aligned}$$

and the expression of the matrix, which defines the metric, is reduced to

$$I_X^{M-D}(\theta) = \left(\int \left[\left(\sqrt{\phi(f)} \right)' \right]^2 \frac{\partial f}{\partial \theta_i} \frac{\partial f}{\partial \theta_j} \mathrm{d}\mu \right)_{k\times k}.$$

In this way, and with respect to the different functionals, which define the metric, we have shown.

Theorem 3.

a) $I_X^{C-D}(\theta) = \dfrac{\phi''(1)}{2} I_X^F(\theta)$.

b) $I_X^{J-D}(\theta) = \left(\dfrac{\lambda(1-\lambda)}{2} \displaystyle\int \phi''(f) \frac{\partial f}{\partial \theta_i} \frac{\partial f}{\partial \theta_j} \mathrm{d}\mu \right)_{k\times k}$.

c) $I_X^{M-D}(\theta) = \left(\displaystyle\int \left[\left(\sqrt{\phi(f)} \right)' \right]^2 \frac{\partial f}{\partial \theta_i} \frac{\partial f}{\partial \theta_j} \mathrm{d}\mu \right)_{k\times k}$.

One particularly interesting case for the measure $I_X^{J-D}(\theta)$, obtained for the group of functions

$$\phi_\alpha(t) = \begin{cases} (\alpha-1)^{-1}(t^\alpha - t) & \text{for } \alpha \neq 1 \\ t\log t & \text{for } \alpha = 1, \end{cases}$$

where the value of $I_X^{J-D}(\theta)$ is reduced to

$$I_X^{J-D}(\theta) = \frac{\alpha \cdot \lambda(1-\lambda)}{2} \left(\mathsf{E}_\theta \left[f^{\alpha-1} \frac{\partial \log f}{\partial \theta_i} \frac{\partial \log f}{\partial \theta_j} \right] \right)_{k \times k}$$

That is to say that $I_X^{J-D}(\theta)$ turns out to be a multiple of the α-order informative matrix.

If one particularizes the measure $I_X^{M-D}(\theta)$ to the group of functions $\phi_\alpha(t) = t^\alpha$, one obtains

$$I_X^{M-D}(\theta) = \frac{\alpha^2}{4} \left(\mathsf{E}_\theta \left[f^{\alpha-1} \frac{\partial \log f}{\partial \theta_i} \frac{\partial \log f}{\partial \theta_j} \right] \right)_{k \times k}$$

Some studies in this sense have been carried on, see Burbea and Rao [5], Burbea [4], Salicrú [11, 12] and Rao [9].

Remark 2. For any information matrix

$$G = (g_{ij}) = \left(\int \psi(f) \frac{\partial f}{\partial \theta_i} \frac{\partial f}{\partial \theta_j} \, \mathrm{d}\mu \right)$$

the Levi–Civita connection of the first kind associated to G is defined as

$$[ij\colon k]^G = \frac{1}{2} \left(\frac{\partial g_{ik}}{\partial \theta_j} + \frac{\partial g_{jk}}{\partial \theta_i} - \frac{\partial g_{ij}}{\partial \theta_j} \right)$$

and the generalized connection as

$$\Gamma_{ijk}^\alpha = [ij\colon k]^F - \frac{\alpha}{2} T_{ijk},$$

where

$$T_{ijk}^G = \mathsf{E}_\theta \left(f\, \psi(f) \frac{\partial f}{\partial \theta_i} \frac{\partial f}{\partial \theta_j} \frac{\partial f}{\partial \theta_k} \right)$$

In this context, the α-connection defined by Amari [2] is obtained when $G = I_X^F(\theta)$.

5. MEASURES OF INFORMATION IN THE k-VARIATE CASE: SECOND ALTERNATIVE

An alternative to the above-mentioned method, consists of defining the elements of the information matrix, based on the distance between one given distribution, and the result of disturbing the parameter into two directions. In this sense, and for $e_1 = (1, 0, \dots, 0)$, $e_2 = (0, 1, 0, \dots, 0), \dots, e_k = (0, \dots, 0, 1)$, the information matrix is defined by the expression

$$\begin{aligned} I_{ij}^C(\theta) &= \liminf_{t \to 0} \frac{1}{t^2} D \left[f(x,\theta), \sqrt{f(x,\theta + te_i) f(x,\theta + te_j)} \right] = \\ &= \liminf_{t \to 0} \frac{1}{t^2} \int f(x,\theta) \phi \left(\frac{\sqrt{f(x,\theta + te_i) f(x,\theta + te_j)}}{f(x,\theta)} \right) \mathrm{d}\mu \end{aligned} \tag{5}$$

with $\theta + te_i$, $\theta + te_j \in \Theta$.

Considering McLaurin's development (the same as in the univariate case), and for the function

$$g(t) = \int f(x,\theta)\phi\left(\frac{\sqrt{f(x,\theta+te_i)f(x,\theta+te_j)}}{f(x,\theta)}\right)\,\mathrm{d}\mu \tag{6}$$

one finds

$$\begin{aligned} I_{ij}^C(\theta) &= \frac{\phi''(1)}{8}\int\frac{1}{f}\left[\frac{\partial f}{\partial\theta_i}+\frac{\partial f}{\partial\theta_j}\right]^2\mathrm{d}\mu \\ &+ \frac{\phi'(1)}{4}\int\left\{\frac{\partial^2 f}{\partial\theta_i^2}+\frac{\partial^2 f}{\partial\theta_j^2}-\frac{1}{2f}\left(\frac{\partial f}{\partial\theta_i}-\frac{\partial f}{\partial\theta_j}\right)^2\right\}\mathrm{d}\mu \\ &= \frac{\phi''(1)-\phi'(1)}{8}\left[I_{ii}^F(\theta)+I_{jj}^F(\theta)\right]+\frac{\phi''(1)+\phi'(1)}{4}I_{ij}^F(\theta). \end{aligned}$$

This way, then, we shave shown.

Theorem 4.

$$I_X^C(\theta) = \frac{\phi''(1)-\phi'(1)}{8}\left[J_X(\theta)+J_X^t(\theta)\right]+\frac{\phi''(1)+\phi'(1)}{4}I_X^F(\phi) \tag{7}$$

with

$$J_X(\theta) = \begin{pmatrix} I_{11} & \cdots & I_{11} \\ I_{22} & \cdots & I_{22} \\ \vdots & \ddots & \vdots \\ I_{kk} & \cdots & I_{kk} \end{pmatrix}$$

and $I_{ii} = I_{ii}^F(\theta) = \mathsf{E}\left[\dfrac{\partial\log f(x,\theta)}{\partial\theta_i}\right]^2$.

In an analogous way to the univariate case, considering functions of Csiszár's measure, we obtain.

Theorem 5.

$$\begin{aligned} I_X^h(\theta) &= \liminf_{t\to 0}\frac{1}{t^2}h\left\{\int[f(x,\theta)]^\alpha\,[f(x,\theta+te_i)f(x,\theta+te_j)]^{(1-\alpha)/2}\,\mathrm{d}\mu\right\} = \\ &= h'(1)\left\{\frac{\alpha^2-1}{8}\left[J_X(\theta)+J_X^t(\theta)\right]+\frac{(\alpha-1)^2}{4}I_X^F(\theta)\right\} \end{aligned}$$

for a differentiable function h, with $h(1) = 0$.

Proof.

$$I_{ij}^h(\theta) = \liminf_{t\to 0} \frac{1}{t^2} h\left\{1 + \int f(x,\theta)\left[\left(\frac{\sqrt{f(x,\theta+te_i)f(x,\theta+te_j)}}{f(x,\theta)}\right)^{1-\alpha} - 1\right] \mathrm{d}\mu\right\} =$$

$$= \liminf_{t\to 0} \frac{1}{t^2} h\left[1 + g(t)\right],$$

where $g(t)$ is the function defined in (6) with $\phi(x) = x^{1-\alpha} - 1$. This way, considering McLaurin's development of $g(t)$, we have

$$I_{ij}^h(\theta) = h'(1) I_{ij}^C(\theta)$$

and therefore

$$I_X^h(\theta) = h'(1)\left\{\frac{\alpha^2-1}{8}\left[J_X(\theta) + J_X^t(\theta)\right] + \frac{(\alpha-1)^2}{4} I_X^F(\theta)\right\}.$$

If one particularizes the two results obtained above, one can see in Table 2, some relationships (for the multivariate case), between measures associated to Csiszár's divergences, and Fisher's information matrix.

Table 2.

NAME	RELATIONSHIP
Kullback–Leibler	$I_X^{KL}(\theta) = \frac{1}{4}\left[J_X(\theta) + J_X^t(\theta)\right]$
Modified Kullback–Leibler	$I_X^{KLM}(\theta) = \frac{1}{2} I_X^F(\theta)$
Jeffreis invariant	$I_X^J(\theta) = \frac{1}{4}\left[J_X(\theta) + J_X^t(\theta)\right] + \frac{1}{2} I_X^F(\theta)$
Matusita	$I_X^M(\theta) = \frac{1}{16}\left[J_X(\theta) + J_X^t(\theta)\right] + \frac{1}{8} I_X^F(\theta)$
Kagan	$I_X^{Ka}(\theta) = \frac{1}{4}\left[J_X(\theta) + J_X^t(\theta)\right] + \frac{1}{2} I_X^F(\theta)$
Rényi	$I_X^R(\theta) = \frac{\alpha+1}{8}\left[J_X(\theta) + J_X^t(\theta)\right] + \frac{\alpha-1}{4} I_X^F(\theta)$
Bhattacharyya	$I_X^B(\theta) = \frac{p+1}{8pq}\left[J_X(\theta) + J_X^t(\theta)\right] - \frac{1}{4q^2} I_X^F(\theta)$

With the relationship in Table 2 and the values, which take $h'(1)$, one can deduce immediately.

Corollary 2.

a) $I_X^J(\theta) = I_X^{K\alpha}(\theta) = 4I_X^M(\theta) = I_X^{KL}(\theta) + I_X^{KLM}(\theta)$.

b) $\lim_{\alpha\to 1} I_X^R(\theta) = I_X^{KL}(\theta)$.

c) $I_X^R(\theta) = q I_X^B(\theta)$.

Remark 3. Note, that if $\phi'(1) = \phi''(1)$, then $I_X^C(\theta) = I_X^{C-D}(\theta)$ and also if $\phi'(1) = \phi''(1) = 2$, then $I_X^C(\theta) = I_X^F(\theta)$.

Remark 4. If in definition (5) one takes $\frac{1}{2}[f(x,\theta+te_i) + f(x,\theta+te_j)]$ instead of $[f(x,\theta+te_i)\cdot f(x,\theta+te_j)]^{\frac{1}{2}}$, then the value of $I_X^C(\theta)$ would be

$$\frac{\phi''(1)}{8}\left[J_X(\theta) + J_X^t(\theta)\right] + \frac{\phi''(1)}{4} I_X^F(\theta)$$

which is equal to the (7) if $\phi'(1) = 0$. Some particular results in this sense have been obtained by Ferentinos and Papaioannou [6], for Rényi distance, Salicrú [12] and Salicrú and Sanchez [14] (for J-divergences).

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Dr. Miquel Salicrú, Departament d'Estadistica, Universitat de Barcelona, Av. Diagonal, 645, 08028 Barcelona. Spain.