Martin Hála

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METHOD OF RITZ FOR RANDOM EIGENVALUE PROBLEMS

MARTIN HÁLA

Boundary value problems for ordinary differential equations with random coefficients are dealt with. Asymptotic normality of the eigenvalues is derived under proper conditions. The method of Ritz enables to extend the results. Application of the presented theory in dynamics is added.

1. INTRODUCTION, PERTURBATION RESULTS

The paper has been inspired by certain technical applications. We can meet eigenvalue problems for ordinary differential equations, the coefficients of which are random, namely in mechanics and dynamics (see the last section for an example). We can often assume the random parts of these coefficients to be very small and near to white noise, especially when random deviations in the shape, in the quality of the material, etc. are considered.

The notation and assumptions are similar as in Hála [2] but we stress the approximate approach to the problems namely using the method of Ritz. First of all let us make some notes concerning perturbation theory.

Consider the deterministic eigenvalue problem

$$M_0 u + M_1 u = \lambda (N_0 u + N_1 u), \quad U_j[u] = 0, \quad j = 1, 2, \ldots, 2m, \tag{1}$$

where

$$M_k u = \sum_{i=0}^{m} (-1)^i \left[f_{ki} u^{(i)} \right]^{(i)}, \qquad N_k u = \sum_{i=0}^{n} (-1)^i \left[g_{ki} u^{(i)} \right]^{(i)},$$

$$U_j[u] = \sum_{t=0}^{2m-1} \alpha_{jt} u^{(t)}(0) + \beta_{jt} u^{(t)}(L)$$

(f_{ki}, g_{ki} are sufficiently smooth real functions, α_{jt}, β_{jt} are real constants, $k = 0, 1$, $m > n$).

The principle of the perturbation theory is to express the eigenvalues and eigenfunctions of (1) in terms of the perturbations f_{1i}, g_{1i} and various characteristics of the so called unperturbed problem

$$M_0 u = \mu N_0 u; \quad U_j[u] = 0, \quad j = 1, 2, \ldots, 2m. \tag{2}$$

Theorem 1. Assume the operators M_k, N_k in (1) to be positive, let $f_{1m} = 0$. Furthermore, let the equations

$$\sum_{j=0}^{m} \sum_{t=0}^{j-1} (-1)^{j+t} \left[\varphi(x) u^{(j)}\right]^{(j-t-1)} v^{(t)} \Big|_0^L = 0 \tag{3}$$

hold for all admissible functions u, v and for all functions $\varphi = f_{ki}$ or $\varphi = g_{ki}$.

Assume that (2) possesses a discrete spectrum, let μ denote some simple eigenvalue of (2) and $w(x)$ denote the normalized eigenfunction associated with μ.

There exists a constant $\varepsilon > 0$ depending only on the problem (2) such that for every $\delta \in (0, \varepsilon)$ the following statement holds.

If $|f_{1i}(x)| \le \delta$, $|g_{1i}(x)| \le \delta$ for every i and x then there exist terms λ_k, $u_k(x)$, $k = 0, 1, 2 \ldots$ such that the series

$$\lambda = \sum_{k=0}^{\infty} \lambda_k, \qquad u(x) = \sum_{k=0}^{\infty} u_k(x)$$

converge and determine a solution of (1).

In particular,

$$\lambda_0 = \mu, \qquad u_0(x) = w(x),$$

$$\lambda_1 = \sum_{i=0}^{m-1} \int_0^L (u_0^{(i)}(x))^2 (f_{1i}(x) - \mu g_{1i}(x)) \mathrm{d}x, \tag{4}$$

$$|\lambda - (\lambda_0 + \lambda_1)| \le C\delta^2, \tag{5}$$

where we set $g_{1i}(x) = 0$ for $i > n$ and C is a constant depending only on the problem (2).

For the proof see Purkert–Scheidt [3].

The rather unconvenient condition $f_{1m} = 0$ expresses that the order of M_1 should be less than the one of M_0.

Let us finally note that (3) holds for example when the boundary conditions in (1) are

$$u(0) = u'(0) = \ldots = u^{(m-1)}(0) = u(L) = u'(L) = \ldots = u^{(m-1)}(L) = 0.$$

Using the method of Ritz for searching for an approximate solution of the problems like (1) we get matrix eigenvalue problems. Perturbation results for them are presented in the following theorem.

Consider the equations

$$(A + C)u = \lambda(B + D)u, \tag{6}$$
$$Au = \mu Bu, \tag{7}$$

where A, B, C, D are symmetric matrices of the size $n \times n$. Let the matrices B, $B + D$ be positive definite. Then (6) has n real eigenvalues $\lambda_1 \le \ldots \le \lambda_n$ and (7) has n real eigenvalues $\mu_1 \le \cdots \le \mu_n$.

Theorem 2. Let μ_i be a simple eigenvalue of (7) and w_i let be the corresponding B-normalized eigenvector, i.e. $(Bw_i, w_i) = 1$.

Then there exists a constant $\gamma_0 > 0$ depending only on (7) such that if $\sum\sum(c_{ij}^2 + d_{ij}^2) \le \gamma \le \gamma_0$ then λ_i is also simple and the expression

$$\lambda_i = \mu_i + \sum_{k=1}^{\infty} \lambda_{ik} \tag{8}$$

holds, where

$$\lambda_{i1} = (Cw_i, w_i) - \mu_i(Dw_i, w_i), \tag{9}$$
$$\left|\sum_{k=2}^{\infty} \lambda_{ik}\right| \le R\gamma. \tag{10}$$

(R is a constant depending only on (7).)

For the proof see Purkert–Scheidt [3].

Let us introduce the assumptions and notation used in this paper.

We shall consider the problem in the form (1), where M_0, N_0, $U_j[u]$ are the same as in (1), the operators M_1, N_1 are assumed to be random:

$$M_1 u = \sum_{i=0}^{m}(-1)^i \left[X_i u^{(i)}\right]^{(i)}, \quad N_1 u = \sum_{i=0}^{n}(-1)^i \left[X_{m+1+i} u^{(i)}\right]^{(i)},$$

where $\mathbf{X}(x) = (X_0(x), \ldots, X_m(x), X_{m+1}(x), \ldots, X_{m+1+n}(x))'$ is a vectorial stochastic process on $\langle 0, L\rangle$ with sufficiently smooth trajectories that depends on a real parameter.

We suppose the operators M_0, N_0 to be positive, the same must hold for $M_0 + M_1$, $N_0 + N_1$ a.s. Analogous equations as (3) are assumed to hold a.s.

We will now describe in a more detailed way the supposed nature of the process $\mathbf{X}(x)$. Let each of its components be in the form

$$X_i(x) = \sqrt{\varepsilon(a)}\,\varphi_i(x)\,Y_i^{(a)}(x).$$

Here ε is a positive real function of a parameter a, φ_i are real deterministic functions and $\mathbf{Y}_a(x) = (Y_0^{(a)}(x), \ldots Y_{m+1+n}^{(a)}(x))'$ is a real vectorial centralized Gaussian stationary symmetric process with rational spectral density depending on a.

Let $\mathbf{R}_a(x)$ be the matrix correlation function of $\mathbf{Y}_a(x)$, its elements being denoted by $R_a^{ik}(x)$, $i,k = 0,1,\dots,m+1+n$. $\mathbf{f}_a(\lambda)$ denotes the spectral matrix density of $\mathbf{Y}_a(x)$. $\mathbf{f}_a(\lambda)$ is supposed to be rational and so it must be a real symmetric even matrix function of a constant rank r. In particular

$$\mathbf{f}_a(\lambda) = \frac{1}{2\pi}\mathbf{B}_a(\mathrm{i}\lambda)\mathbf{B}_a^T(-\mathrm{i}\lambda), \tag{11}$$

where $\mathbf{B}_a^{(m+n+2)\times r}(s)$ is a rational matrix function analytic in $\{\, s \in \mathbb{C} : \mathrm{Re} \sim \geq \nvdash \,\}$ and real for real s.

Let the matrices $\mathbf{R}_a(x)$ fulfil the conditions

$$\lim_{a\to\infty} \int_{-\delta}^{\delta} \mathbf{R}_a(x)\,\mathrm{d}x = \mathbf{R}, \qquad \lim_{a\to\infty} \left(\int_{-\infty}^{-\delta} + \int_{\delta}^{+\infty} \right) |R_a^{ik}(x)|\,\mathrm{d}x = 0,$$

$$\int_{-\infty}^{\infty} |R_a^{ik}(x)|\,\mathrm{d}x \leq K < +\infty$$

for arbitrary i, k, $\delta > 0$, where $\mathbf{R}$ is a constant matrix and K is a constant independent on a.

Let $b(a)$, $c(a)$ denote the terms of an arbitrary complex partial fraction $b(a)/(s+c(a))^k$ of an arbitrary element of the matrix $\mathbf{B}_a(s)$. Let these terms fulfil for sufficiently large a:

$$\lim_{a\to\infty} \mathrm{Re}\; c(a) = +\infty, \quad \varepsilon(a) \leq \min\left\{ \frac{(\mathrm{Re}\; c(a))^{1-q}}{|b(a)|^2}, \frac{(\mathrm{Re}\; c(a))^{2-q}}{|b(a)|^4} \right\},$$

where $q > 0$ is a constant.

It was stated in Hála [2] that under rather restrictive condition $X_m = 0$ the asymptotic normality of the variable

$$\frac{\lambda(a) - \mu}{\sqrt{\varepsilon(a)}}$$

can be derived and its limit variance can be computed (μ is a simple eigenvalue of (2) and $\lambda(a)$ is the corresponding eigenvalue of (1) that is near to μ).

We will present similar result for the approximate solution of (1) in the following section.

2. METHOD OF RITZ

We will consider firstly the centralized problem (2). Let $\{\psi_1, \psi_2, \dots\}$ be a bazis of the energetic space $\mathcal{H}_{M_0}$ of the operator M_0. We can derive from (2) for fixed $N \in \mathbb{N}$ the matrix eigenvalue problem (7) where $\boldsymbol{A} = \{a_{ij}\}_{i,j=1}^N$, $\boldsymbol{B} = \{b_{ij}\}_{i,j=1}^N$, $a_{ij} = (\psi_i, \psi_j)_{M_0} = \int_0^L M_0\psi_i \cdot \psi_j \,\mathrm{d}x$, $b_{ij} = (\psi_i, \psi_j)_{N_0} = \int_0^L N_0\psi_i \cdot \psi_j \,\mathrm{d}x$.

This problem has under our assumptions real eigenvalues ${}^N\mu_1 \leq \ldots \leq {}^N\mu_N$ with corresponding B-normalized eigenvectors ${}^N\boldsymbol{w}_i = ({}^N\boldsymbol{w}_{i1}, \ldots, {}^N\boldsymbol{w}_{iN})'$, $i = 1, \ldots, N$. This eigenvalues and eigenvectors approximate the eigenvalues and eigenfunctions of (2) in the following sense.

If μ_i is a simple eigenvalue of (2) with corresponding N_0-normalized eigenfunction $w_i(x)$ (we suppose the order $\mu_1 \leq \mu_2 \leq \ldots \leq \mu_{i-1} < \mu_i < \mu_{i+1} \leq \ldots$) and if ${}^N w_i(x) = \sum_{k-1}^N {}^N w_{ik} \cdot \psi_k(x)$, then ${}^N\mu_i \to \mu_i$ and ${}^N w_i \xrightarrow{M_0} w_i$ for $N \to +\infty$.

Consider now the perturbed problem (1), let the notation and assumptions of the previous sections hold. From the problem (1) we can derive the matrix problem (6), where $\boldsymbol{A}$, $\boldsymbol{B}$ were introduced above, $\boldsymbol{C} = \{c_{ij}\}_{i,j=1}^N$, $\boldsymbol{D} = \{d_{ij}\}_{i,j=1}^N$, $c_{ij} = (\psi_i, \psi_j)_{M_1}$, $d_{ij} = (\psi_i, \psi_j)_{N_1}$. The eigenvalues of (6) are denoted ${}^N\lambda_1, \ldots, {}^N\lambda_N$.

While we can expect that ${}^N\lambda_i$ approximates λ_i for increasing N, the limit variance of ${}^N\lambda_i$ (for N fixed and $a \to +\infty$) can be stated exactly:

Lemma 3. The random variable

$$\frac{{}^N\lambda_i(a) - {}^N\mu_i}{\sqrt{\varepsilon(a)}}$$

converges in distribution to a centralized Gaussian random variable with the variance

$$\begin{aligned} {}^N\sigma_i^2 &= \sum\sum_{j,k=0}^m \int_0^L ({}^N w_i^{(j)}(x) {}^N w_i^{(k)}(x))^2 [A^{jk} - {}^N\mu_i(A^{m+1+j,k} + A^{j,m+1+k}) + \\ &\quad + {}^N\mu_i^2 A^{m+1+j,m+1+k}] \, \mathrm{d}x. \end{aligned}$$

where A^{jk} is the abbreviation for the function $R^{jk}\varphi_j(x)\varphi_k(x)$ and we set $R^{jk} = 0$, $\varphi_j(x) = 0$ for $j, k > m + 1 + n$.

The p r o o f is based on the perturbation results summarized in Theorem 2 and it is very similar to the proof of Theorem 3 in [2].

Firstly when we introduce the variable $Y = \sum\sum(c_{ij}^2 + d_{ij}^2)$, then the upper estimate of the probability $P[Y > \delta]$ like (25) in [2] can be derived. From this estimate it immediatelly follows that Y tends to 0 in probability and Theorem 2 is applicable.

We can write using (8)

$$\frac{{}^N\lambda_i(a) - {}^N\mu_i}{\sqrt{\varepsilon(a)}} = \frac{{}^N\lambda_{i1}(a)}{\sqrt{\varepsilon(a)}} + \sum_{k=2}^{\infty} \frac{{}^N\lambda_{ik}(a)}{\sqrt{\varepsilon(a)}}.$$

It can be shown due to (9) that the first term on the right converges in distribution to the centralized normal variable with the variance ${}^N\sigma_i^2$. The second term converges to 0 in probability due to (10).

For the exact proof see [1].

3. APPLICATION – BENDING VIBRATIONS OF A BAR

Consider a horizontal bar of the length L with clamped ends. Let $u(x)$ denotes the vertical deviation. The equation

$$(EIu'')'' = \lambda \rho A u, \qquad u(0) = u(L) = u''(0) = u''(L) = 0 \tag{12}$$

holds, where E is the modulus of elasticity, I denotes the moment of inertia (EI is the bending stiffness), A is the cross-sectional area and ρ denotes the mass per unit length. λ is the square of the eigenfrequency of the vibrations.

When we admit small perturbations of the shape and of the quality of the material then E, I, ρ and A should be considered as random processes. We cannot use Theorem 3 from [2] because the highest coefficient in (12) is random.

Let us use the following notation and simplifications: $L = 1$, $EI = f_2 + X_2$, $\rho F = g_0 + X_3$, where $f_2 = \text{const} > 0$, $g_0 = \text{const} > 0$. The centralized equation has then constant coefficients and its eigenvalues can be easily computed: $\mu_i = (f_2/g_0)(i\pi)^4$, $i = 1, 2, \ldots$

Let $X_i(x) = \sqrt{\varepsilon(a)} Y_i^{(a)}(x)$ (i.e. $\varphi_i(x) = 1$), $i = 2, 3$ and $(Y_2^{(a)}, Y_3^{(a)})'$ let be centralized Gaussian stationary process with the same properties as previously. Let $\boldsymbol{R} = \begin{pmatrix} R_{22} & R_{23} \\ R_{23} & R_{33} \end{pmatrix}$ be the limit matrix from the assumptions stated in the previous section.

Finally we select the basis of $\mathcal{H}_{M_0}$:

$$\begin{aligned} \psi_1(x) &= x - 2x^3 + x^4, \\ \psi_2(x) &= 7x - 10x^3 + 3x^5, \\ \psi_i(x) &= x^i(1-x)^3, \quad i = 3, 4, \ldots . \end{aligned}$$

From Lemma 3 it follows that the limit variance of

$$\frac{{}^N\lambda_i(a) - {}^N\mu_i}{\sqrt{\varepsilon(a)}}$$

is

$$\begin{aligned} {}^N\sigma_i^2 = \int_0^1 \Bigg[& R_{22} \left(\sum_{j=1}^N {}^N w_{ij} \psi_j''(x) \right)^4 - 2\,{}^N\mu_i R_{23} \left(\sum_{j=1}^N {}^N w_{ij} \psi_j(x) \right)^2 \times \\ & \times \left(\sum_{j=1}^N {}^N w_{ij} \psi_j''(x) \right)^2 + {}^N\mu_i^2 R_{33} \left(\sum_{j=1}^N {}^N w_{ij} \psi_j(x) \right)^4 \Bigg] \mathrm{d}x. \end{aligned}$$

We can compare the exact values of μ_i, the approximate values of ${}^N\mu_i$ and ${}^N\boldsymbol{w}_i$ for $N = 4$:

i	$(g_0/f_2)\mu_i$	$(g_0/f_2)({}^4\mu_i)$	$\sqrt{g_0}({}^4w_i)$
1	97.4091	97.4091	$(4.4423, 0, 1.6606, 0)$
2	1558.5454	1558.6401	$(132.9890, -17.7319, 35.1891, -70.5783)$
3	7890.1363	8337.9414	$(-16.0195, 0.0011, 403.9913, -0.0528)$
4	24936.7270	28151.1516	$(364.6832, -48.6246, -1142.1741, 2284.4111)$

The limit variance for $i = 1$ is:

$$ {}^4\sigma_1^2 = 11612.7(R_{22}/g_0^2) - 28465.5(R_{23}f_0/g_0^3) + 14232.7(R_{33}f_0^2/g_0^4). $$

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Dr. Martin Hála, Department of Mathematics, Faculty of Civil Engineering – Czech Technical University, Thákurova 7, 166 29 Praha 6. Czech Republic.